Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION

Volume 7, Issue 2

Published online: April 14, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Oscillation of Second Order Quasilinear Differential Equations with Several Neutral Terms

R.Arul and V.S.Shobha

Department of Mathematics, Kandaswami Kandar's College, Velur-638182, Namakkal Dt., Tamil Nadu, India.

Abstract. This paper deals with the oscillatory behavior of solutions of second order neutral type differential equations. The results obtained here extend some of the existing results. Examples are provided to illustrate the main results.

2010 AMS Subject Classification: 34C10, 34K11

Keywords and Phrases: Second order; quasilinear; neutral differential equation.

1. Introduction

In this paper, we investigate the oscillatory behavior of solutions of second order quasilinear neutral differential equation of the form

(1.1) $$\left(r(t)\left(z'(t)\right)^{\alpha}\right)'+q(t)f\left(x\left(\sigma(t)\right)\right)=0,\quad t\ge t_0$$

where $z(t)=x(t)+\sum_{i=1}^{m}p_i(t)x(\tau_i(t)),\ m>0$ is an integer. We assume that the following conditions hold without further mention:

(C_1) $q\in C[t_0,\infty),\ p_i,\ \tau_i,\ \sigma\in C'[t_0,\infty);$

(C_2) $r(t)>0,\ q(t)>0,\ 0\le p_i\left(t\right)\le b_i<\infty$ for $i=1,\,2,\,\ldots,\,m;$

(C_3) $\lim\limits_{t\to\infty}\sigma\left(t\right)=\infty,\ \tau_i\circ\sigma=\sigma\circ\tau_i,\ \tau_i'(t)\ge\lambda_i>0$ for $i=1,\,2,\,\ldots,\,m;$

(C_4) $f\in C\left(\mathbb{R},\mathbb{R}\right)$ is nondecreasing with $uf(u)>0$ and $\dfrac{f(u)}{u^{\beta}}\ge M>0;$

(C_5) α and β are ration of odd positive integers;

(C_6) $R(t)=\int_{t_0}^{t}\dfrac{ds}{r^{1/\alpha}(s)}\to\infty$ as $t\to\infty$.

By a solution of equation (1.1), we mean a real valued function $x \in C[T_x, \infty),\ T_x \geq t_0$ which has the property $r(t)\left(z'(t)\right)^{\alpha} \in C'[T_x, \infty),$ and satisfies equation (1.1) on the interval $[T_x, \infty)$. We consider only those solutions x of equation (1.1) which satisfy the condition $\sup\{|x(t)|: t \geq T\} > 0$ for all $T \geq T_x$. We assume that equation (1.1) posses such solutions. A solution of equation (1.1) is called oscillatory if it has infinitely many zeros on $[T_x, \infty)$; otherwise it is said to be nonoscillatory.

Neutral differential equations arise in modeling several real world problems occurring in physics, engineering, biology, etc., see for example [2, 4, 5], and the references citied therein. Determining oscillation criteria for natural type differential equations received great interest in recent years, see for example [1, 3, 6, 8, 9], and the references contained therein.

In [8], the author has established some oscillation criteria for the following equation

(1.2) $$\left(r(t)\left(x'(t)\right)^{\alpha}\right)' + q(t)x^{\beta}\left(\sigma(t)\right) = 0,\ t \geq t_0$$

under the condition $\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} dt = \infty.$

In [9], the authors considered the equation of the form

(1.3) $$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}\left(\sigma(t)\right) = 0,\ t \geq t_0$$

where $z(t) = x(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)),\ m > 0$ and $\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} dt = \infty.$

Motivated by the results concerning equations (1.2) and (1.3), we study the oscillatory behavior of equation (1.1). The results obtained here reduce to those presented in [1] for the case when m=1 and $\alpha = \beta = 1$, and in [9] for the case $\alpha = \beta$.

In Section 2, we present sufficient conditions for the oscillation of all solutions of equation (1.1) and in Section 3, we provide some examples to illustrate the main results.

2. Oscillation Results

In this section, we establish some new oscillation criteria for the equation (1.1). Without loss of generality, we can only deal with positive solutions of equation (1.1). We begin with the following lemma.

Lemma 2.1. Let x be a positive solution of equation (1.1). Then the corresponding function z satisifies

$$z(t) > 0,\quad z'(t) > 0 \text{ and } \left(r(t)\left(z'(t)\right)^{\alpha}\right)' < 0 \text{ eventually.}$$

Lemma 2.2. Assume that $x_i \geq 0$ for $i = 1, 2, 3, \ldots, m,$ and let $\gamma \geq 1$. Thus

$$\sum_{i=1}^{m} x_i^{\gamma} \geq \frac{1}{m^{\gamma-1}} \left(\sum_{i=1}^{m} x_i\right)^{\alpha}.$$

The proof of Lemmas 2.1 and 2.2 can be found in [9].

In the sequel, we use the following notations:

$$Q(t) = \min\{q(t), q(\tau_1(t)), q(\tau_2(t)), \ldots, q(\tau_m(t))\},$$

and $$\bar{Q}(t,t_1) = Q(t)\left(R(\sigma(t)) - R(t_1)\right)^{\beta}$$

where $t \ge t_1 \ge t_0$ is sufficiently large.

Theorem 2.1. Assume that $1 \le \alpha \le \beta$ and $b_i \le 1$ for all $i = 1, 2, 3, \ldots, m$. If the first order neutral differential inequality

(2.1) $$\left(y(t) + \sum_{i=1}^{m} \frac{b_i^{\alpha}}{\lambda_i} y\left(\tau_i(t)\right) \right)' + M \frac{\bar{Q}(t,t_1)}{(m+1)^{\beta-1}} y^{\beta/\alpha}(\sigma(t)) \le 0$$

has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof: Let x be an eventually positive solution of equation (1.1), then we

(2.2) $$x^{\beta}(\sigma(t)) + \sum_{i=1}^{m} b_i^{\beta} x^{\beta}\left(\tau_i(\sigma(t))\right) \ge \frac{1}{(m+1)^{\beta-1}} \left(x(\sigma(t)) + \sum_{i=1}^{m} b_i\, x\left(\tau_i(\sigma(t))\right) \right)^{\beta} \ge \frac{1}{(m+1)^{\beta-1}} z^{\beta}(\sigma(t))$$

where we have used Lemma 2.2, and condition $p_i(t) \le b_i$, $i = 1, 2, \ldots, m$. From (1.1) and (C_4), we have

(2.3) $$\left(r(t)(z'(t))^{\alpha}\right)' + Mq(t)x^{\beta}(\sigma(t)) \le 0,$$

which yields

$$0 \ge \sum_{i=1}^{m} \frac{b_i^{\alpha}}{\tau_i'(t)} \left(r(\tau_i(t))\right)\left(z'(\tau_i(t))\right)^{\alpha} + M \sum_{i=1}^{m} b_i^{\alpha} q(\tau_i(t)) x^{\beta}\left(\sigma(\tau_i(t))\right)$$

or

(2.4) $$0 \ge \sum_{i=1}^{m} \frac{b_i^{\alpha}}{\lambda_i} \left(r(\tau_i(t))\right)\left(z'(\tau_i(t))\right)^{\alpha} + M \sum_{i=1}^{m} b_i^{\alpha} q(\tau_i(t)) x^{\beta}\left(\sigma(\tau_i(t))\right).$$

Inview of (2.3) and (2.4), we obtain

$$\left(r(t)(z'(t))^{\alpha}\right)' + \sum_{i=1}^{m} \frac{b_i^{\alpha}}{\lambda_i} \left(r(\tau_i(t))\right)\left(z'(\tau_i(t))\right)^{\alpha} + M \sum_{i=1}^{m} b_i^{\alpha} q(\tau_i(t)) x^{\beta}\left(\sigma(\tau_i(t))\right) + Mq(t)x^{\beta}(\sigma(t)) \le 0.$$

From the last inequality we have

(2.5) $$\left(r(t)(z'(t))^{\alpha}\right)' + \sum_{i=1}^{m} \frac{b_i^{\alpha}}{\lambda_i} \left(r(\tau_i(t))\right)\left(z'(\tau_i(t))\right)^{\alpha} + MQ(t)\left(\sum_{i=1}^{m} b_i^{\alpha} x^{\beta}\left(\sigma(\tau_i(t))\right) + x^{\beta}(\sigma(t)) \right) \le 0.$$

Using (2.2) and $\tau_i \circ \sigma = \sigma \circ \tau_i$ in (2.5), we obtain

(2.6) $$\left(r(t)(z'(t))^{\alpha}\right)' + \sum_{i=1}^{m} \frac{b_i^{\alpha}}{\lambda_i} \left(r(\tau_i(t))\right)\left(z'(\tau_i(t))\right)^{\alpha} + M \frac{Q(t)}{(m+1)^{\beta-1}} z^{\beta}(\sigma(t)) \le 0.$$

Since $y(t) = r(t)(z'(t))^{\alpha} > 0$ is decreasing, and so

(2.7) $$z(t) \geq \int_{t_1}^{t} \frac{\left(r(s)(z'(s))^{\alpha}\right)^{1/\alpha}}{r^{1/\alpha}(s)} ds \geq y^{1/\alpha}(t)\left(R(t)-R(t_1)\right).$$

Therefore from (2.6) and (2.7) we have $y(t)$ is a positive solution of the equality

$$\left(y(t)+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}y\left(\tau_i(t)\right)\right)'+M\frac{Q(t)}{(m+1)^{\beta-1}}\left(R(\sigma(t))-R(\sigma_1(t))\right)^{\beta}y^{\beta/\alpha}(\sigma(t))\leq 0,$$

which is a contradiction. This completes the proof.

Theorem 2.2. Assume that $1\leq\beta\leq\alpha$ and $1\leq p_i(t)\leq b_i$ for all $i=1,2,3,\ldots,m$. If the first order neutral differential inequality

(2.8) $$\left(y(t)+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}y\left(\tau_i(t)\right)\right)'+M\frac{\bar{Q}(t,t_1)}{(m+1)^{\beta-1}}y^{\beta/\alpha}(\sigma(t))\leq 0$$

has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof: The proof is similar to that of Theorem 2.1, and hence the details are omitted.

Remark 2.1. If $\alpha=\beta$, then Theorems 2.1 and 2.2 reduces to Theorem 1 of [9], and to Theorem 1 of [1] when $m=1$ and $\alpha=\beta=1$.

Theorem 2.3. Assume that $1\leq\alpha\leq\beta,\ b_i\leq 1$ and $\tau_i(t)\geq t$ for all $i=1,2,3,\ldots,m$. If the first order neutral differential inequality

(2.9) $$w'(t)+M\frac{\bar{Q}(t,t_1)}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}\right)^{\beta/\alpha}}w^{\beta/\alpha}(\sigma(t))\leq 0$$

has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof: Let x be an eventually positive solution of equation (1.1). Then it follows from the proof of Theorem 2.1 that $y(t)=r(t)(z'(t))^{\alpha}>0$ is decreasing, and satisfies the inequality (2.1). Define

(2.10) $$w(t)=y(t)+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}y\left(\tau_i(t)\right).$$

Since $\tau_i(t)\geq t$, we have

$$w(t)\leq y(t)\left(1+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}\right).$$

Using the last inequality into (2.1), we see that w is a positive solution of equation (2.10). This contradiction completes the proof.

Theorem 2.4. Assume that $1\leq\beta\leq\alpha$ and $1\leq p_i(t)\leq b_i$ for all $i=1,2,3,\ldots,m$. If the first order neutral differential inequality

(2.11) $$w'(t)+M\frac{\bar{Q}(t,t_1)}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}\right)^{\beta/\alpha}}w^{\beta/\alpha}(\sigma(t))\le 0$$

has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof: The proof is similar to that of Theorem 2.3, and hence the details are omitted.

Remark 2.2. Theorems 2.3 and 2.4 reduces to Theorem 2 of [9] for $\alpha=\beta$.

Corollary 2.1. Assume that $1\le\alpha=\beta$ and $\tau_i(t)\ge t$ for all $i=1,2,3,\ldots,m$. If $\sigma(t)<t$, and

(2.12) $$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}Q(s)R^{\alpha}(\sigma(s))ds>\frac{(m+1)^{\alpha-1}\left(1+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}\right)}{Me},$$

then every solution of equation (1.1) is oscillatory.

Proof: Since $\alpha=\beta$, from Theorem 2.1.1 of [5] and condition (2.12), we see that the inequality (2.9) has no positive solution. The result now follows from Theorems 2.3 and 2.4.

Corollary 2.2. Assume that $1\le\alpha<\beta,\ b_i(t)\le 1,$ and $\tau_i(t)\ge t$ for all $i=1,2,3,\ldots,m$ and $\sigma(t)$ is continuously differentiable with $\sigma'(t)\ge 0$. Further suppose that there exists a continuously differentiable function $\phi(t)$ such that

(2.13) $$\phi'(t)>0 \quad and \quad \lim_{t\to\infty}\phi(t)=\infty,$$

(2.14) $$\limsup_{n\to\infty}\ \alpha\frac{\phi'(\sigma(t))\sigma'(t)}{\phi'(t)}<1,$$

and

(2.15) $$\liminf_{n\to\infty}\left(\bar{Q}(t,t_1)\frac{e^{-\phi(t)}}{\phi'(t)}\right)>0,$$

then every solution of equation (1.1) is oscillatory.

Proof: Since $\frac{\beta}{\alpha}>1$ by Theorem 1 of [7], conditions (2.13)-(2.15) ensure that the inequality (2.9) has no positive solution. The result now follows from Theorem 2.3.

Corollary 2.3. Assume that $1\le\beta<\alpha,\ 1\le p_i(t)\le b_i,$ and $\tau_i(t)\ge t$ for all $i=1,2,3,\ldots,m$. If $\sigma(t)<t,$ and

(2.16) $$\int_{t_0}^{\infty}\bar{Q}(t,t_1)dt=\infty,$$

then every solution of equation (1.1) is oscillatory.

Proof: Since $\frac{\beta}{\alpha}<1$ by a result in [3, 5] and condition (2.16) , we see that the inequality (2.9) has no positive solution. The result now follows from Theorem 2.4.

In the sequel, we use the notation $\tau(t)=\min\{\tau_i(t): i=1,2,3,\ldots,m\}$, and τ^{-1} stands for the inverse of the function τ.

Theorem 2.5. Assume that $1\le\alpha\le\beta$, $\tau(t)\le t$ and $b_i\le 1$ for all $i=1,2,3,\ldots,m$. If the first order differential inequality

$$w'(t)+\frac{\bar{Q}(t,t_1)}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}\right)^{\beta/\alpha}}w^{\beta/\alpha}(\tau^{-1}(\sigma(t)))\le 0 \tag{2.17}$$

has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof: Let x be an eventually positive solution of equation (1.1). Then it follows from the proof of Theorem 2.1 that $y(t)=r(t)(z'(t))^{\alpha}>0$ is decreasing and satisfies the inequality (2.1). Now define w by (2.10). Since $\tau(t)\le t$, we have

$$w(t)\le y(\tau(t))\left(1+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}\right).$$

Using the last inequality into (2.1), we obtain that w is a positive solution of equation (2.17). This contradiction completes the proof.

Theorem 2.6. Assume that $1\le\beta\le\alpha$, $\tau(t)\le t$ and $1\le p_i(t)\le b_i$ for all $i=1,2,3,\ldots,m$. If the first order differential inequality

$$w'(t)+\frac{\bar{Q}(t,t_1)}{(m+1)^{\beta-1}\left(1+\sum_{i=1}^{m}\frac{b_i^{\alpha}}{\lambda_i}\right)^{\beta/\alpha}}w^{\beta/\alpha}(\tau^{-1}(\sigma(t)))\le 0 \tag{2.18}$$

has no positive solution, then every solution of equation (1.1) is oscillatory.

Proof: The proof is similar to that of Theorem 2.5, and hence the details are omitted.

Remark 2.3. Theorems 2.5 and 2.6 reduces to Theorem 3 of [9] in the case $\alpha=\beta$, and involved Theorem 3 of [1] when $\alpha=\beta=1$ and $m=1$.

If $\alpha=\beta$, then one can remove the restrictions on b_i for $i=1,2,3,\ldots,m$ imposed in Theorems 2.5 and 2.6 and obtained the Corollary 2 of [9]. For $\alpha\ne\beta$, one can obtain the following corollaries from Theorems 2.5 ad 2.6 respectively.

Corollary 2.4. Assume that $1\le\alpha<\beta$, $\sigma(t)\le\tau(t)\le t$, and $b_i(t)\le 1$, for all $i=1,2,3,\ldots,m$. If $\sigma(t)$ is continuously differentiable with $\sigma'(t)\ge 0$ and there exists a continuously differentiable function $\phi(t)$ such that (2.13)-(2.15) are satisfied, then every solution of equation (1.1) is oscillatory.

Corollary 2.5. Assume that $1\le\beta<\alpha$, $\sigma(t)\le\tau(t)\le t$, and $1\le p_i(t)\le b_i$ for all $i=1,2,3,\ldots,m$. If condition (2.16) is satisfied, then every solution of equation (1.1) is oscillatory.

3.Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider a second order neutral differential equation

$$(3.1)\qquad \left(\left(\left(x(t)+\frac{1}{2}x(t+2\pi)+\frac{1}{3}x(t+4\pi)\right)'\right)^3\right)'+e^{e^{2t}}x^5(t-2\pi)=0,\ t\geq 1.$$

It is easy to see that all conditions of Corollary 2.2 are satisfied and hence every solution of equation (3.1) is oscillatory.

Example 3.2. Consider a second order neutral differential equation

$$(3.2)\qquad \left(\left(\left(x(t)+x(t-\pi)+2x(t-4\pi)\right)'\right)^3\right)'+4x(t-6\pi)=0,\ t\geq 1.$$

It is easy to see that all conditions of Corollary 2.5 are satisfied and hence every solution of equation (3.2) is oscillatory.

References

[1] Baculikova.B and Dzurina.J (2011). Oscillation theorems for second order neutral differential equations, *Comput. Math. Appl.*, 61, 94-99.

[2] Driver.R (1984). A mixed neutral systems, *Nonlinear Anal.*, 8, 155-158.

[3] Erbe.L.H, Kong.Q and Zhang.B.G (1995). Oscillation Theory for Functional Differential Equations, *Dekker, New York.*

[4] Hale.J.K (1977). Theory of Functional Differential Equations, *Springer, New York.*

[5] Ladde.G.S, Lakshmikanthan.V and Zhang.B.G (1987). Oscillation Theory of Differential Equations with Deviating Arguments, *Dekker, New York.*

[6] Sun.S, Li.T, Han.Z and Zhang.C (2013). On oscillation of second order nonlinear neutral functional differential equations, *Bull. Malays Math. Sci. Soc.*, 36, 541-554.

[7] Tang.X.H (2002). Oscillation for first order superlinear delay differential equations, *J. London Math. Soc.*, 65, 115-122.

[8] Tiryaki.A (2009). Oscillation criteria for a certain second order nonlinear differential equations with deviating arguments, *Elect. J. Qual. Theory Differ. Equ.*, 61, 1-11.

[9] Zhang.C, Mehmet.T.S and Li.T (2014). Oscillation of second order half linear differential equations with several neutral terms, *J. Appl. Math. Comput.*, 44, 511-518.