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Forming a mixed Quadrature rule using an anti-Lobatto four point Quadrature rule

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Abstract

A mixed quadrature rule of higher precision for approximate evaluation of real definite integrals has been constructed using an anti-Lobatto rule. The analytical convergence of the rule has been studied. The relative effciencies of the mixed quadrature rule has been shown with the help of suitable test integrals. The error bound has been determined asymptotically.

Keywords: Lobatto two point rule; anti-Lobatto three point rule; Fejer three point second rule; mixed quadrature rule.

2000 Mathematics Subject Classification: 65D30, 65D32.

1 Introduction

The concept of mixed quadrature was first coined by R.N Das and G.Pradhan [5].The method of mixing quadrature rule is based on forming a mixed quadrature rule of higher precision by taking linear/convex combination of two quadrature rules of lower precision. Though in literature we find precision enhancement through Richardson Extrapolation [3] and Kronord extension [3, 10, 11] taking respectively trapezoidal rule and Gaussian quadrature as base rules, these methods are quite cumbersome. On the other hand, the precision enhancement through mixed quadrature method is very simple and easy to handle. Many authors [5, 12-16] have developed mixed quadrature rules for numerical evaluation of real definite integrals. Authors [4,6-9] have also developed mixed quadrature rules for approximate evaluation of the integrals of analytic functions following F. Lether [2]. So far this is one of the few papers in which an anti-Lobatto quadrature has been used to construct a mixed quadrature rule.

Dirk P. Laurie [1] is first to coin the idea of anti-Gaussian quadrature formula. An anti-Gaussian quadrature formula is an (n+1) point formula of degree (2n-1) which integrates all polynominals of degree upto (2n+1) with an error equal in magnitude but opposite in sign to that of n-point Gaussian formula. If $H^{(n+1)} = \sum_{i=1}^{n+1} \lambda_i f(\xi_i)$ be (n + 1) point anti-Gaussian formula and $G^{(n)}(p)$ be n point Gaussian formula, then by hypothesis $I(p) - H^{(n+1)}(p) = -(I(p) - G^{(n)}(p)), p \in P_{2n+1}$ where p is a polynominal of degree $\leq 2n + 1$.

In this paper, we incorporate the idea of anti-Gaussian quadrature rule to design an anti-Lobatto quadrature rule following LAURIE. We mix this anti-Lobatto rule with Fejers three point second rule to form a mixed quadrature rule. The relative efficiencies of the mixed rule has been shown by numerically evaluating some test integrals.

2 Construction of anti-Lobatto four point rule from Lobatto three point rule

We choose the Lobatto three point rule,

$$L_w^3(f) = \frac{1}{3}[f(-1) + 4f(0) + f(1)] \tag{1}$$

and develop a four point anti-Lobatto rule $RH_w^4(f)$ from three point Lobatto rule $L_w^3(f)$.

Using the principle $I(p) - H^{(n+1)}(p) = -(I(p) - G^{(n)}(p))$ as adopted in Dirk P. Laurie [1], we obtain

$$RH_w^4(f) = 2\int_{-1}^{1} f(x)dx - L_w^3(f) \tag{2}$$

$$\alpha_1 f(-1) + \alpha_2 f(\xi_1) + \alpha_3 f(\xi_2) + \alpha_4 f(1) = 2\int_{-1}^{1} f(x)dx - L_w^3(f) \tag{3}$$

where $RH_w^4(f) = \alpha_1 f(-1) + \alpha_2 f(\xi_1) + \alpha_3 f(\xi_2) + \alpha_4 f(1)$.

In order to obtain the unknown weights and nodes, we assume that (i) the rule is exact for all polynomials of degree≤ 3 . (ii) the rule integrates all polynomials of degree up to five with an error equal in magnitude and opposite in sign to that of Lobatto rule. Thus we obtain following system of six equations having six unknowns namely $\alpha_i, (i = 1,2,3,4), \xi_i, (i = 1,2)$for $f(x) = x^i$, i =0, 1, 2, 3, 4, 5,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2 \tag{4}$$

$$-\alpha_1 + \alpha_2\xi_1 + \alpha_3\xi_2 + \alpha_4 = 0 \tag{5}$$

$$\alpha_1 + \alpha_2\xi_1^2 + \alpha_3\xi_2^2 + \alpha_4 = \frac{2}{3} \tag{6}$$

$$-\alpha_1 + \alpha_2\xi_1^3 + \alpha_3\xi_2^3 + \alpha_4 = 0 \tag{7}$$

$$\alpha_1 + \alpha_2\xi_1^4 + \alpha_3\xi_2^4 + \alpha_4 = \frac{2}{15} \tag{8}$$

$$-\alpha_1 + \alpha_2\xi_1^5 + \alpha_3\xi_2^5 + \alpha_4 = 0 \tag{9}$$

Solving the above system of equation we get, $\alpha_1 = -\frac{1}{9} = \alpha_4, \alpha_2 = \frac{10}{9} = \alpha_3,\ \xi_1 = \sqrt{\frac{2}{5}}$ $, \xi_2 = -\sqrt{\frac{2}{5}}$.

Hence, the anti-Lobatto four point rule becomes,

$$RH_w^4(f) = \frac{10}{9}[f(\sqrt{\frac{2}{5}}) + f(-\sqrt{\frac{2}{5}})] - \frac{1}{9}[f(1) + f(-1)] \tag{10}$$

The error associated with the rule is computed as

$$EH_w^4(f) = \int_{-1}^{1} f(x)dx - RH_w^4(f) = \frac{4f^{iv}(0)}{5! \times 3} + \frac{8f^{vi}(0)}{7! \times 3} + ... \tag{11}$$

3 Construction of mixed quadrature by using anti-Lobatto four point rule with Fejer three point second rule

We have the anti-Lobatto four point rule,

$$RH_w^4(f) = \frac{10}{9}[f(\sqrt{\frac{2}{5}}) + f(-\sqrt{\frac{2}{5}})] - \frac{1}{9}[f(1) + f(-1)] \tag{12}$$

and Fejer three point second rule taken from [17]:

$$Rfj_2(f) = \frac{2}{3}[f(\frac{1}{\sqrt{2}}) + f(-\frac{1}{\sqrt{2}}) + f(0)] \tag{13}$$

Each of the rules $RH_w^4(f)$ and $Rfj_2(f)$ is of precision three. Let $EH_w^4(f)$ and $Efj_2(f)$ denote the errors in approximating the integrals $I(f)$ by the rules $RH_w^4(f)$ and $Rfj_2(f)$ respectively. Now

$$I(f) = RH_w^4(f) + EH_w^4(f) \tag{14}$$

$$I(f) = Rfj_2(f) + Efj_2(f) \tag{15}$$

Using Maclaurines expansion of function in equation (12) and (13). we have,

$$EH_w^4(f) = \frac{4f^{iv}(0)}{5! \times 3} + \frac{8f^{vi}(0)}{7! \times 3} + \ldots \tag{16}$$

$$Efj_2(f) = \frac{f^{iv}(0)}{360} + \frac{f^{vi}(0)}{6048} + \ldots \tag{17}$$

Eliminating $f^{iv}(0)$ from equation (16) and (17) we have

$$I(f) = \frac{1}{3}[4Rfj_2(f) - RH_w^4(f)] + \frac{1}{3}[4Efj_2(f) - EH_w^4(f] \tag{18}$$

or

$$I(f) = RH_w^4 fj_2(f) + EH_w^4 fj_2(f) \tag{19}$$

Where

$$EH_w^4 fj_2(f) = \frac{1}{3}[4Efj_2(f) - EH_w^4(f)] \tag{20}$$

and

$$RH_w^4 fj_2(f) = \frac{1}{3}[4Rfj_2(f) - RH_w^4(f)] \tag{21}$$

$$RH_w^3 fj_2(f) = \frac{8}{9}[f(\frac{1}{\sqrt{2}}) + f(-\frac{1}{\sqrt{2}}) + f(0)] - \frac{10}{27}[f(\sqrt{\frac{2}{5}}) + f(-\sqrt{\frac{2}{5}})] + \frac{1}{27}[f(1) + f(-1)] \tag{22}$$

This is the desired mixed quadrature rule of precision five. The truncation error generated in this approximation is given by

$$EH_w^4 fj_2(f) = \frac{1}{3}[4Efj_2(f) - EH_w^4(f)] \tag{23}$$

or

$$EH_w^4 fj_2(f) = \frac{1}{22680} f^{vi}(0) + \ldots \tag{24}$$

$$|EH_w^4 fj_2(f)| = \frac{1}{22680}|f^{vi}(\eta)|, -1 < \eta < 1. \tag{25}$$

4 Error analysis:

An asymptotic error estimate and an error bound of the rule (20) are as under.

Theorem 4.1.

Let $f(x)$ be sufficiently differentiable function in the closed interval [-1,1].Then the error $EH_w^4fj_2(f)$ associated with the rule $RH_w^4fj_2(f)$ is given by

$$|EH_w^4fj_2(f)| \approx \frac{1}{22680}|f^{vi}(\eta)|, -1 < \eta < 1 \tag{26}$$

Proof : The theorem follows from (20) and (21) we have

$$RH_w^4fj_2(f) = \frac{1}{3}[4Rfj_2(f) - H_w^4(f)] \tag{27}$$

And the truncation error generated in this approximation is given by,

$$EH_w^4fj_2(f) = \frac{1}{3}[4Efj_2(f) - EH_w^4(f)] \tag{28}$$

Hence we have,

$$|EH_w^4fj_2(f)| \approx \frac{1}{22680}|f^{vi}(\eta)|, -1 < \eta < 1 \tag{29}$$

Theorem 4.2.

The bound of the truncation error $EH_w^4fj_2(f) = I(f) - RH_w^4fj_2(f)$ is given by

$|EH_w^4fj_2(f)| \leq \frac{M}{270}|\eta_2 - \eta_1|$,$\eta_1, \eta_2 \in [-1,1]$ where $M = \max_{-1\leq x\leq 1} |f^v(x)|$.

Proof : We have

$$EH_w^4(f) = \frac{1}{90}f^{iv}(\eta_1), -1 < \eta_1 < 1 \tag{30}$$

and

$$Efj_2(f) = \frac{1}{360}f^{iv}(\eta_2), -1 < \eta_2 < 1 \tag{31}$$

$$EH_w^4fj_2(f) = \frac{1}{3}[4Efj_2(f) - EH_w^4(f)] \tag{32}$$

$$|EH_w^4fj_2(f)| \leq \frac{1}{270}|f^{iv}(\eta_2) - f^{iv}(\eta_1)| \tag{33}$$

$= \frac{1}{270}\int_{\eta_1}^{\eta_2} f^v(x)dx \leq \frac{M}{270}|\eta_2 - \eta_1|.$

where $M = \max_{-1\leq x\leq 1} |f^v(x)|.$

Which gives a theoretical error bound as η_1,η_2 are unknown points in [-1, 1]. From this theorem it is clear that the error in approximation will be less if points η_1,η_2 are closer to each other.

Corollary-1. The error bound for the truncation error $EH_w^4fj_2(f)$ is given by,

$$|EH_w^4fj_2(f)| \leq \frac{2M}{270} \tag{34}$$

Proof : The proof follows from theorem (4.2) and $|\eta_1 - \eta_2| \leq 2.$

5 Numerical verification table and graphs

Using the results of the table and the notations for the absolute errors of different methods four bar graphs namely A,B,C and D have been constructed in fig-1.The bar graphs A,B,C,D from correspond to $I_1 = \int_{-1}^{1} e^x dx$, $I_2 = \int_0^1 e^{-x^2}dx$, $I_3 = \int_0^1 e^{x^2}dx$, $I_4 = \int_1^3 \frac{sin^2 x}{x}dx$ spectively.

In the bar graphs, error names E_1, E_2, E_3 and E_4 have been embedded along X-axis. Along Y-axis the respective values of the errors depicting heights of the bars are taken.

Where $E_1 = |I(f) - RL_3(f)|$, $E_2 = |I(f) - RH_w^4(f)|$,$E_3 = |I(f) - Rfj_2(f)|$, $E_4 = |I(f) - RH_w^4fj_2(f)|.$

Also we have chosen the unit along Y-axis as

$1 = -\log 10^{-1}$,$2 = -\log 10^{-2}$,$3 = -\log 10^{-3}$,$4 = -\log 10^{-4}$,$5 = -\log 10^{-5}$,$6 = -\log 10^{-6}$.

Therefore in each the bar graph, the larger the height of the bar the smaller is the error.

Table 1: Comparision of mixed quadrature rule with other rules in some test integrals.

Integrals	Exactvalue I(f)	$RL_3(f)$ E_1	$RH_w^4(f)$ E_2	$Rfj_2(f)$ E_3	$RH_w^4fj_2(f)$ E_4
$\int_{-1}^{1} e^x dx$	2.35040238	2.3620537 0.01165	2.35811374 0.0077	2.3474557 0.00294	2.35034983 0.0000525
$\int_0^1 e^{-x^2} dx$	0.746825	0.74718 0.00035	0.747054 0.00022	0.7467297 0.000095	0.7468162 0.0000088
$\int_0^1 e^{x^2} dx$	1.4627	1.4757305 0.01303	1.471156 0.0084	1.45926153 0.0034	1.4624597 0.00024
$\int_1^3 \frac{sin^2 x}{x} dx$	0.794825	0.7894351 0.0053	0.7911007 0.0037	0.7960751 0.0012	0.7946843 0.00014
$\int_0^1 \sqrt{x} dx$	0.666666	0.6380711 0.0285	0.6598341 0.0068	0.6712232 0.0045	0.6667902 0.000124
$\int_0^1 \sqrt{x} sinx dx$	0.364221	0.3662485 0.00202	0.36523635 0.00101	0.3637008 0.00052	0.364116 0.000105
$\int_0^1 \sqrt{x} sinx dx$	0.364221	0.3662485 0.00202	0.36523635 0.00101	0.3637008 0.00052	0.364116 0.0001
$\int_0^1 sin\sqrt{\pi x} dx$	0.849726	0.6798206 0.169905	0.6746972 0.175029	0.67728655 0.17243	0.67635138 0.17337
$\int_0^1 x^{16} cosx^{16} dx$	0.0491217	0.090605 0.6306	0.00847 0.04065	0.06710871 0.01798	0.04554602 0.0035

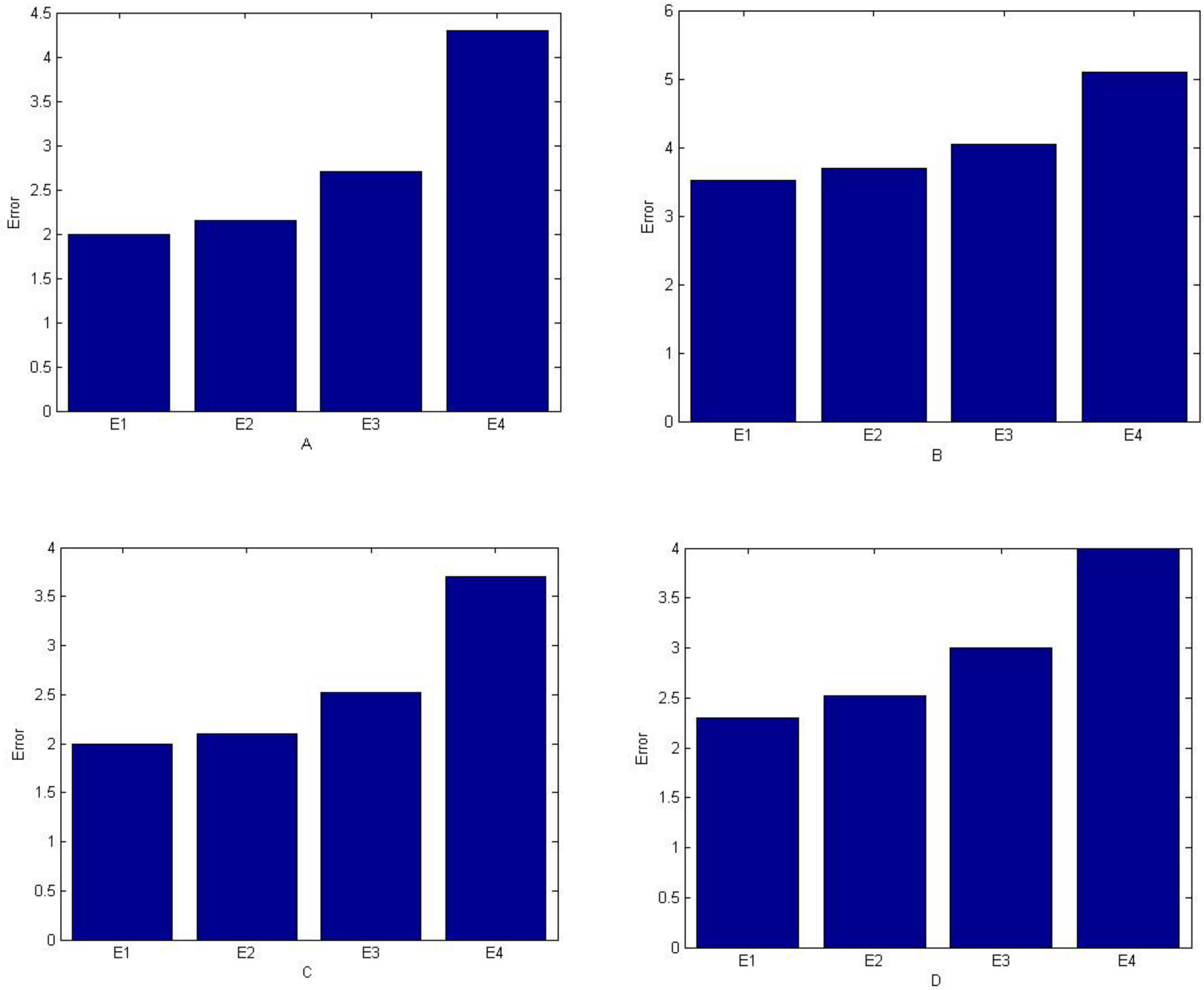

Figure 1

6 Observation

From the table as well as from the bar graphs it is observed that the absolute error corresponding to the mixed quadrature rule $RH_w^4 fj_2(f)$, is lesser than those corresponding to its constituent rules $RL_3(f)$, $RH_w^4(f)$ and $Rfj_2(f)$, when the test integrals are evaluated.

7 Conclusion

After observation one can smartly draw conclusion over the efficiency of the rule formed in this paper as follows.

Mixed rule $RH_w^4 fj_2(f)$ is more efficient than its constituent rules $RL_3(f)$, $RH_w^4(f)$ and $Rfj_2(f)$.

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