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Optimal Control Problem Governed By Elliptic Variational Inequalities For Infinite Order

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Abstract.

In this paper we find, the necessary conditions for optimality of a system governed by elliptic variational inequalities of infinite order with obstacle, where the cost function is quadratic associated with the state y (u). When there is no constraint on the control variable, we give the first order necessary conditions of the optimality systems.

Keywords: Control problem; variational inequalities; bilinear form; cost function; conical derivative.

Introduction

I.M.Gali et al [8] presented a set of inequalities defining an optimal control of a system governed by self-adjoint operators with an infinite number of variables.

Subsequently, J.L.Lions suggested a problem related to an optimal control, but in different direction by considering the case of operators of Infinite order with finite dimension [10,11]. This problem which was solved by Gali , have been published in [7] .Moreover, I.M. Gali et al [8,9] presented some control problems generating both elliptic and hyperbolic linear operator of infinite order with finite number of variables.

In a previous paper El -zahaby [5,6] proved the existence of an optimal control of a system governed by variational inequalities with an infinite number of variables.

First order necessary conditions of the optimality system have been obtained by using the theory of Barbu [1,2].

In the present paper we use the conical derivative, using the theory of Mignot [12]and Mignot and Puel [13], to obtain the optimality conditions for a system governed by variational inequalities for infinite order of obstacle type

Some function spaces

The embedding problems for non-trivial sobolev spaces are investigated in [3,4] , and an embedding criterion was established in terms of the characteristic functions of these spaces

In this case

$$W^{\infty}\{a_{\alpha},2\} \subseteq L_2\left(R^n\right) \subseteq W^{-\infty}\{a_{\alpha},2\} \quad (1)$$

Where

$$W^{\infty}\{a_{\alpha},2\} = \left\{u\left(x\right) \in C^{\infty}\left(R^n\right). \sum_{|\alpha|=0}^{\infty} a_{\alpha} \left\|D^{\alpha}u\right\|_2^2 < \infty\right\}$$

are Sobolev spaces of infinite order of periodic function defined on all of R^n and $W^{\infty}\{a_{\alpha},2\}$ denotes their topological dual with respect of $L_2\left(R^n\right)$

Analogous to the above chain we have

$$W_0^{\infty}\{a_{\alpha},2\} \subseteq L_2\left(R^n\right) \subseteq W_0^{-\infty}\{a_{\alpha},2\}$$

Here, $W_0^{\infty}\{a_{\alpha},2\}$ is the set of all functions of $W^{\infty}\{a_{\alpha},2\}$ which vanish on the boundary Γ of R^n.

We recall that $\alpha = \left(\alpha_1,...,\alpha_n\right), \alpha_n \in N$, we use the notations:

$$|\alpha| = \alpha_1 + ... + \alpha_n \text{, and} \quad D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

further $a_{\alpha} \geq 0$ is a numerical sequence ,and $\| \; \|_2$ is the norm in $L_2\left(R^n\right)$(all functional are assumed to be real valued).

Let us consider an elliptic operator of infinite order with finite dimension

$$Bu = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_{\alpha} D^{2\alpha} u \quad (2)$$

This operator has a self-adjoint closure.

We introduce a continuous bilinear form on $W^{\infty}\{a_{\alpha},2\}$

$$\Pi(u.v) = (Bu, v)$$

$$= \sum_{|\alpha|=1}^{\infty} \left((-1)^{|\alpha|} a_{\alpha} D^{2\alpha} u(x), v(x)\right)_{L^2(R^n)} + (q(x)u(x), v(x))_{L_2(R^n)} \quad (3)$$

Where q(x) is a real valued function from $L_2\left(R^n\right)$ such that:

$q(x) \geq \beta, \quad 1 \geq \beta \succ 0$ .Then

$$\Pi(u,v) = \sum_{I\alpha I=1}^{\infty} \int_{R^n} a_\alpha (D^\alpha u)(x)(D^\alpha v)(x)dx + \int_{R^n} q(x)u(x)v(x)dx$$

Lemma 1:

Consider the continuous bilinear form on $W_0^{\infty}\{a_\alpha, 2\}$, then:

$\Pi(u,v)$ is coercive, that is

$$\Pi(u,u) \geq \beta \|u\|^2 \quad , \quad u \in W_0^{\infty}\{a_\alpha, 2\}, \tag{4}$$

Proof:

The ellipticity of B is sufficient for the coerciveness of $\Pi(u,v)$ on

$W_0^{\infty}\{a_\alpha, 2\}$,

In fact, $\Pi(u,u)$ =

$$\sum_{I\alpha I=1}^{\infty} \int_{R^n} a_\alpha (D^\alpha u)(x)(D^\alpha u)(x)dx + \int_{R^n} q(x)u(x)u(x)dx$$

$$\geq \sum_{I\alpha I=1}^{\infty} a_\alpha (D^\alpha u(x), (D^\alpha u)(x))_{L_2(R^n)} + \beta(u(x), u(x))_{L_2(R^n)}$$

$$= \sum_{I\alpha I=1}^{\infty} a_\alpha \|D^\alpha u\|^2_{L_2(R^n)} + \beta \|u(x)\|^2_{L_2(R^n)}$$

$$= \sum_{I\alpha I=1}^{\infty} a_\alpha \|D^\alpha u\|^2_{L_2(R^n)} + \beta \sum_{I\alpha I=1}^{\infty} a_\alpha \|D^\alpha u\|^2_{L_2(R^n)}$$

$$-\beta \Sigma \|D^\alpha u\|^2_{L_2(R^n)} + \beta \|u\|^2_{L_2(R^n)}$$

$$= \beta \|u\|^2_{W^\infty\{a_\alpha, 2\}} + (1-\beta)\Sigma a_\alpha \|D^\alpha u\|^2_{L_2(R^n)}$$

Then, $\pi(u,u) \geq \beta \|u\|^2_{w_0^\infty\{a_\alpha, 2\}}$.

which proves the coerciveness of the bilinear form.

If (.,.) is the duality between $W^\infty\{a_\alpha, 2\}$ and $W^{-\infty}\{a_\alpha, 2\}$ we have

$a(\phi, \varphi) = (B\phi, \varphi)$ for all $\phi, \varphi \in W^{-\infty}\{a_\alpha, 2\}$

where $B \in L(W^{\infty}\{a_\alpha, 2\}, W^{-\infty}\{a_\alpha, 2\})$

Now we define:

$$K = \{u, u \in W^\infty\{a_\alpha, 2\}, \ u \geq o \ \text{in} \Omega\} \tag{5}$$

The set K is closed, convex and nonempty in $W^{\infty}\{a_{\alpha},2\}$

If $f \in L^2(R^n)$, $y = y(f)$ is the solution of the variational inequality

$$a(y, \theta - y) \geq \int_{\Omega} (f, \theta - y)\, dx, \ \forall \theta \in K, \ y \in K. \tag{6}$$

Under assumptions (4) and for all $f \in W^{-\infty}\{a_{\alpha},2\}$ the variational inequality (6) has a unique solution $y(f) \in W^{\infty}\{a_{\alpha},2\}$.

Control problem:

We are now able to define correctly the control problem:

Let f be given in $W^{-\infty}\{a_{\alpha},2\}$, let U_{ad} be a closed convex subset of

$$U \subset L_2(R^n).$$

For each $v \in U_{ad}$ we define $y = y(v)$, the state of the system, as the solution of the variational inequality

$$a(y, \varphi - y) \geq \langle f+v, \varphi - y \rangle, \quad \forall \varphi \in K, \quad y \in K \tag{7}$$

Now, for $z_d \in L^2(\Omega)$ and N > O, we define the cost function J by:

$$J(v) = \frac{1}{2}\int_{\Omega} y(v) - z_d)^2 dx + N/2\int_{\Omega} v^2\, dx \tag{8}$$

we look for u (optimal control) such that

$$u \in U_{ad}, \ J(u) = \inf_{v \in u_{ad}} J(v) \tag{9}$$

Theorem 1.

There exists an optimal control $u \in U_{ad}$ (and, in general, there is no uniqueness)

Proof:

We know that $J(v) \geq o, \forall v \in U_{ad}$ let J be the infimum value of J (v) for $v \in U_{ad}$ and let (v_n) $\in N$ be a minimizing sequence. We then have:

$$\lim_{x \to \infty} J(v_n) = j = \inf_{v \in u_{ad}} J(v)$$

as N is strictly positive, $(v_n)_{n \in N}$ is a bounded sequence in $U_{ad} \subset L^2(\Omega)$ and we can extract a weakly convergent subsequence $(v_{n_k})_{k \in N}$ such that

$$v_{n_k} \to u \ \text{ in } L^2(\Omega) \text{ Weakly, as } k \to \infty$$

then $u \in U_{ad}$ because U_{ad} is a closed and convex as Ω is bounded, the injection from $L_2(\Omega) \to W^{-\infty}\{a_\alpha, 2\}$ is compact and so $v_{n_k} \to u$ in $W^{-\infty}\{a_\alpha, 2\}$ strongly, as $k \to \infty$ using the lower semicontinuity for the weak topology of $L^2(\Omega)$ of $v \to \int_\Omega v^2 dx$, we get

$$j = \lim_{K\to\infty} \inf J\left(v_{n_k}\right) \geq J(u) \quad \text{and} \quad J(u) = \min J(v).$$

In order to get optimality conditions of first order, we shall assume that $U_{ad} \subset L^2(\Omega)$

If y is a solution of (6) we can define:

$$Z_y = \{x, x \in \Omega, y(x) = o\} \tag{10}$$

defined up to a set of zero capacity.

$$S^y = \{\phi / \phi \in W^\infty\{a_{\alpha,} 2\}, \phi \geq o \text{ on } Z_y, a(y,\phi) = \int f\phi\, dx\} \tag{11}$$

By Mignot [12] ,we know that:

the mapping $y \to y(v)$ possess at each point v a conical derivative $Dy_v(w)$

such that ,for all $w \in W^{-\infty}\{a_\alpha, 2\}$, we have : $\quad Dy_v(w) \in S_{y(v)}$

$$\forall \phi \in S_{y(v)}, a\left(Dy_v(w), \phi - Dy_v(w)\right) \geq \left\langle w, \phi - Dy_v(w)\right\rangle \tag{12}$$

where $S_{y(v)}$ is defined by (11).

Therefore; the mapping $v \to J(w)$ possesses at each v a conical derivative $w \to DJ_V(w)$ defined by

$$DJ_V(w) = \int_\Omega (y(u) - z_d) DJ_V(w) dx + N \int_\Omega v - w dx \tag{13}$$

Lemma 2:

If u is an optimal control, we have

$$\forall v \in w^{-\infty}\{a_\alpha, 2\}, \; DJ_u(W) \geq 0 \tag{14}$$

Proof:

It is evident that $\forall w \in L^2(\Omega)$, $DJ_u(w) \geq 0$ then it is easy to prove (10) , because $L^2(\Omega)$ is dense in the space $w^{-\infty}\{a_\alpha, 2\}$ and because

$w \to D J_V(w)$ is continuous from $W^{-\infty}\{a_\alpha, 2\}$ into R

Remark:

The condition $DJ_u(w) \geq 0$ means at the point u in each half direction w , the functional J(v) does not decrease strictly, up to the first order. So it seems to be a 'good' optimality.

Theorem 2.

If $u \in w^\infty\{a_\alpha, 2\}$, the optimality condition (10) holds at the point u if and only if there exists P such that

$$P \in S_{y(u)},\ P - A^{*^{-1}}\ (y(u) - z_d) \in (S_{y(u)}{}^{a^*}(u))^0$$

$$P_{(y)} + Nu \geq\ 0$$

Where $(S_y^{a^*})^0$ is a polar cone of S_y with respect to the adjoint form

a^{-*} defined by:

$$(S_{y(u)}^{a^*})^0\ = \{\phi / \phi \in W^\infty\{a_\alpha, 2\}, \forall \varphi \in S_{y(u)}, a(\varphi, \phi) \leq 0\},$$

And p is the adjoint state.

Under the given consideration and form theorem 2 and lemma 1, we may apply the result of Mignot $[11]$ we have the following immediately theorem:

Theorem 3

There exists a triple unique solution (u,y,p) $\in (L_2\left(R^n\right) \times W^\infty\{a_\alpha, 2\} \times W^\infty\{a_\alpha, 2\})$

Such that:

$$a(y, \theta - y) \geq \int_{R^n} (f\ + u, \theta - y\ )dx, \forall \theta \in K \tag{15}$$

$$a(\theta, p) \leq \int_{R^n} (y\ - z_d, \theta)dx, \forall \theta \in S_y, p \in S_y$$

$$\int_{R^n} (p + Nu, v\ - u\ )dx\ \geq 0, \forall v\ \in U_{ad} \tag{16}$$

u is the optimal control of (15) , and (16) is the optimality condition.

Conclusion

In this paper we use the conical derivative to find the optimality condition, now we try to study the optimal control problem governed by elliptic variational inequalities for infinite order of obstacle type.

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