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Weakly Primary Submodules over Non-commutative Rings

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Abstract

Let R be an associative ring with nonzero identity and let M be a unitary left $R-$module. In this paper, we introduce the concept of weakly primary submodules of M and give some basic properties of these classes of submodules. Several results on weakly primary submodules over non-commutative rings are proved. We show that N is a weakly primary submodule of a left $R-$module M iff for every ideal P of R and for every submodule D of M with $0 \neq PD \subseteq N$, either $P \subseteq \sqrt{(N:M)}$ or $D \subseteq N$. We also introduce the definitions of weakly primary compactly packed and maximal compactly packed modules. Then we study the relation between these modules and investigate the condition on a left $R-$module M that makes the concepts of primary compactly packed modules and weakly primary compactly packed modules equivalent. We also introduce the concept of weakly primary radical submodules and show that every Bezout module that satisfies the ascending chain condition on weakly primary radical submodules is weakly primary compactly packed module.

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1. Introduction

Throughout this paper, all rings are assumed to be associative not necessarily commutative with non-zero identities, and all modules are unitary left modules. By "an ideal" we mean a 2-sided ideal.

Recently, extensive researches have been done on prime and primary ideals and submodules. The study of prime submodules is one interesting topic in module theory. In particular, a number of papers concerning prime submodules have been studied by various authors, see for example [9], [12], [15]. Weakly prime ideals in a commutative ring with nonzero identity have been introduced and studied by Anderson and Smith in [2]. They defined a weakly prime ideal P over a commutative ring R with identity as a proper ideal with the property that if whenever $a, b \in R$ with $0 \neq ab \in P$, then either $a \in P$ or $b \in P$. The structure of weakly primary ideals in a commutative ring has been studied by Atani and

Farzalipour in [6]. They defined a weakly primary ideal P over a commutative ring R with identity as a proper ideal with the property that if $0 \neq ab \in P$, where $a, b \in R$, then $a \in P$ or $b^n \in P$ for some positive integer n .The structure of weakly prime ideals over non-commutative rings has been studied by Hirano, Poon, and Tsutsui in [10]. They defined a weakly prime ideal P over an associative ring R with identity as a proper ideal with the property that if $0 \neq AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$ for any ideals A, B of R. Recently, Ashour and Hamoda have been extended the concept of weakly primary ideals over a commutative ring to non-commutative rings in [5]. They defined a right weakly primary ideal P over an associative ring R with identity as a proper ideal with the property that if whenever A, B are ideals of R such that $0 \neq AB \subseteq P$, then $A \subseteq P$ or $B^n = \{b^n : b \in B\} \subseteq P$ for some $n \in \mathbb{N}$. A proper ideal P of R is called left weakly primary if whenever A, B are ideals of R such that $0 \neq AB \subseteq P$, then $B \subseteq P$ or $A^n = \{a^n : a \in A\} \subseteq P$ for some $n \in \mathbb{N}$. The ideal P is called weakly primary if it is both right and left weakly primary.

The studying of prime submodules is extended in many ways, such as weakly prime submodules, primary submodules, graded prime submodules, and n−absorbing submodules, see [7], [8], [17], [18].The motivation of this paper is to continue the studying of the family of primary submodules, also to extend the results of Atani and Frazalipour [7] and Smith [18] to the weakly primary submodules over non-commutative rings. In fact, a number of results concerning weakly primary submodules over non commutative rings are given.

We begin by reviewing the relevant definitions that are used in the sequel of this paper in Section 2.In Section 3, we construct main results and theorems concerning weakly primary submodules over non-commutative rings. We show that if N is a weakly primary submodule of a left R−module M with $\sqrt{(N:M)}N \neq 0$, then N is a primary submodule of M, (see Theorem 3.1.). The first main result of this section is (Theorem3.3.). We show that N is a weakly primary submodule of M iff for every ideal Pof R and for every submodule D of M with $0 \neq PD \subseteq N$, either $P \subseteq \sqrt{(N:M)}$ or $D \subseteq N$. One important part of this section is the second main result(Theorem3.5.).We show that N is a weakly primary submodule of M if for m∈M –N, $(N:Rm) \subseteq \sqrt{(N:M)} \bigcup (0:Rm)$. Finally, in Section 4, we introduce the concepts of weakly primary compactly packed and maximal compactly packed modules and investigate the relation between these concepts. Thus we show (Theorem 4.3.) that if M is a left R−module with $T(M) = \{m \in M : rRm = 0 \text{ for some } 0 \neq r \in R\} = 0$. Then M is primary compactly packed module if and only if M is weakly primary compactly packed module. Also, we give the main result of this section (Theorem 4.7.). We show that if M is a Bezout module that satisfies the ascending chain condition on weakly primary radical submodules, then M is a weakly primary compactly packed module.

2. Preliminaries

We start by the following definition:

Definition 2.1. [14] Let R be an associative ring with identity, M be a left R−module, and N be a submodule of M. The set (N : M) = {r ∈R : rM⊆N} is a left ideal of R called the left residual of N by M.

In particular, if m ∈M, then $(0, m) = \{r \in R : rm = 0\}$ is called the left annihilator of m.

Similarly the right analogous for right residual and right annihilator can be defined for right R−modules.

Note that $(0, m)$ need not be a two-sided ideal of R. However if $(0, m)$ is a two-sided ideal of R, then $(0, m) = (0, Rm)$.

Definition 2.2. [11] Let R be an associative ring with identity, let M be a left R−module.

Then the set (0 : M) is a two sided ideal of R called the left annihilator of M.

Similarly the right analogous for the right annihilator can be defined for right R−modules.

In [9] Dauns defined the prime submodule over an associative ring with identity as follows:

Definition 2.3. Let R be an associative ring with identity and M be a left R−module. A

proper submodule N of M is called a prime submodule of M if rRm⊆N (r ∈R, m ∈M), implies that either m ∈N or r ∈(N : M).

Definition 2.4. [16] An associative ring R with identity is called a semi commutative ring

if ab = 0 implies aRb = 0 ∀a, b ∈R.

Definition 2.5. [13] An associative ring R with identity is called a local ring if it has a unique maximal left (or right) ideal I of R denoted by (R,I).

In a commutative case we have the following definition:

Definition 2.6. [7] A proper submodule N of a module M over a commutative ring R is said to be weakly prime submodule if whenever $0 \neq rm \in N$, for some r ∈R, m ∈M, then m ∈N or rM⊆N.

3. Weakly primary submodules

Our starting point is the following definitions:

Definition 3.1. Let R be an associative ring with identity, M be a left R−module, and N be a submodule of M; then $\sqrt{(N:M)} = \{r \in R : r^n M \subseteq N$ for some positive integer n $\}$ is called the radical of a submodule N over the ring R.

Definition 3.2. Let M be a left R−module. A proper submodule N of M is called a primary submodule of M if whenever r ∈R and m ∈M with rRm⊆N, then either m ∈N or $r \in \sqrt{(N:M)}$.

Definition 3.3. Let M be a left R−module. A proper submodule N of M is called a weakly prime submodule of M if whenever r ∈R and m ∈M with $0 \neq rRm \subseteq N$, then either m ∈N or r ∈(N : M).

Definition 3.4. Let M be a left R−module. A proper submodule N of M is called a weakly primary submodule of M if whenever r ∈R and m ∈M with $0 \neq rRm \subseteq N$, then either m ∈N or $r \in \sqrt{(N:M)}$.

Remarks 3.1.

(i) It’s clear that every primary submodule of a left R−module is a weakly primary submodule. However, since 0 is always a weakly primary submodule (by definition), a weakly primary submodule does not need to be primary.

(ii) We can prove directly from the definitions that every weakly prime submodule of a left R−module is a weakly primary. However the converse is not true in general, since for example ifR = Z, the set of integers, M = Z × Z and N = (4, 0)Z + (0, 1)Z, then N is a weakly primary submodule of M, however it is not weakly prime submodule of M since $0 \neq 2(2,1) \in N$. But neither 2M ⊆N nor (2, 1) ∈N.

(iii) If N is a weakly primary submodule of a left R−module M, then (N : M) is not in general a weakly primary ideal of R. For example, let M be the cyclic left Z−moduleZ/6Z. The zero module is a weakly primary submodule of M, but (0 : M) = 6Z is not a weakly primary ideal of Z.

(iv) If R is a commutative ring with identity. A proper submodule N is a weakly primary submodule of a left R−module M iff whenever r ∈R and m ∈M with $0 \neq rm \in N,$ then m ∈N or $r \in \sqrt{(N:M)}.$ This is clear by the equivalence rm ∈N iff Rrm ⊆N.

As in the previous remark, we see that a weakly primary submodules need not be primary.

The following theorem gives the condition that makes the weakly primary submodule primary.

Theorem 3.1. Let N be a weakly primary submodule of a left R−module M. If $\sqrt{(N:M)}N \neq 0,$ then N is a primary submodule of M.

Proof. Let r ∈R and m ∈M with rRm⊆N. If $rRm \neq 0,$ then N is weakly primary submodule gives m ∈N or $r \in \sqrt{(N:M)}.$ So assume that rRm = 0. If $0 \neq rN,$ then ∃x ∈N such that $rx \neq 0.$ Now $0 \neq rRx = rR(m+x) \subseteq N,$ so N is weakly primary submodule gives (m + x) ∈N or $r \in \sqrt{(N:M)}.$ Thus m ∈N or $r \in \sqrt{(N:M)}.$ Now we assume that rN = 0. If $\sqrt{(N:M)}m \neq 0,$

then $\exists k \in \sqrt{(N:M)}$ such that $km \neq 0.$ So $0 \neq kRm = (r+k)Rm \subseteq N.$ So m ∈N or $(r+k) \in \sqrt{(N:M)}.$ Since $k \in \sqrt{(N:M)},$ then we have m ∈N or $r \in \sqrt{(N:M)}.$ So we can assume that $\sqrt{(N:M)}m = 0.$ Since $\sqrt{(N:M)}N \neq 0,$ $\exists f \in \sqrt{(N:M)}$ and d ∈N such that $fd \neq 0.$ Then $0 \neq fRd = (r+f)R(m+d) \subseteq N,$ so (m+ d) ∈N or $(r+f) \in \sqrt{(N:M)}.$ Thus m ∈N or $(r+f) \in \sqrt{(N:M)}$, so m ∈N or $r \in \sqrt{(N:M)}.$

Now, the following result follows immediately from Theorem 3.1.

Corollary 3.2. Let N be a weakly primary submodule of a left R−module M. If N is not primary submodule of M, then for any ideal P of R such that $P \subseteq \sqrt{(N:M)}$ we have PN = 0. In particular $\sqrt{(N:M)}N = 0.$

In a similar manner we can prove the following result:

Remark 3.2. If N is a weakly prime submodule of a left R−module M that is not prime, then forany ideal P of R such that P ⊆(N : M), then PN = 0. In particular(N : M)N = 0.

Theorem 3.3. Let N be a proper submodule of a left R−module M. Then the following are equivalent:

(i) N is a weakly primary submodule of M.

(ii) For every ideal P of R and for every submodule D of M with $0 \neq PD \subset N,$ either $P \subseteq \sqrt{(N:M)}$ or D ⊆N.

Proof. (i)⇒(ii) Suppose that N is a weakly primary submodule of M. If N is primary, then the result is trivial. So assume that N is a weakly primary submodule of M that is not primary. Let $0 \neq PD \subseteq N$ with x ∈D−N. We want to show that $P \subseteq \sqrt{(N:M)}.$ Let r ∈P. If $0 \neq rRx,$ since rRx⊆N and N is weakly primary, so $r \in \sqrt{(N:M)}.$ So assume that 0 = rRx. Now assume that $rD \neq 0,$ say $rt \neq 0$ for some t ∈D. Now $0 \neq rRt \subseteq N,$ then $r \in \sqrt{(N:M)}.$ If $t \notin N,$ then $r \in \sqrt{(N:M)}.$ If t ∈N, then $0 \neq rRt = rR(t+x) \subseteq N,$ so (t + x) ∈N or $r \in \sqrt{(N:M)}.$ Since $x \notin N,$ then $r \in \sqrt{(N:M)}.$ So we can assume that rD = 0.Suppose that $Px \neq 0$, say $ax \neq 0$ where a ∈P. Now $0 \neq aRx \subseteq N$. Then N is weakly primary submodule gives $a \in \sqrt{(N:M)}$. As $0 \neq aRx = (r+a)Rx \subseteq N$, we get $r \in \sqrt{(N:M)}$. Therefore we can assume that Px = 0. Since $PD \neq 0, \exists p \in P$ and $t_1 \in D$ such that $pt_1 \neq 0$. Now $0 \neq pRt_1 \subseteq N$. As $\sqrt{(N:M)}N = 0$. (by Corollary3.2.) and $0 \neq pRt_1 = pR(t_1 + x) \subseteq N$, we have two cases:

Case(I). $p \in \sqrt{(N:M)}$ and $(t_1 + x) \notin N$.

Since $0 \neq (r+p)R(t_1 + x) = pRt_1 \subseteq N$, we obtain $(r+p) \in \sqrt{(N:M)}$, so $r \in \sqrt{(N:M)}$.

Case (II). $p \notin \sqrt{(N:M)}$ and $(t_1 + x) \in N$.

As $0 \neq pRt_1 \subseteq N$. We have $t_1 \in N$, so x ∈N which is a contradiction.

Therefore $r \in \sqrt{(N:M)}$. Thus $P \subseteq \sqrt{(N:M)}$.

(ii)⇒(i) Assume that $0 \neq sRm \subseteq N$ where s ∈R and m ∈M. Take P = Rs and D = Rm. $0 \neq PD = RsRm \subseteq N$, so either $P \subseteq \sqrt{(N:M)}$ or D ⊆N. Thus $s \in \sqrt{(N:M)}$ or m ∈N.

Theorem 3.4. Let N be a proper submodule of a left R−module M. Then the following are equivalent:

(i) For ideal P of R and submodule D of M with 0≠PD ⊆N, either P ⊆(N : M) or D ⊆N.

(ii) N is a weakly prime submodule of M.

(iii) For m∈M –N, (N : Rm) = (N : M)∪(0 : Rm).

(iv) For m∈M –N, (N : Rm) = (N : M) or (N : Rm) = (0 : Rm).

Proof. (i)⇒(ii) Suppose that $0 \neq sRm \subseteq N$ where s ∈R and m ∈M. Take P = Rs and D = Rm. Then $0 \neq PD \subseteq N$, so either P ⊆(N : M) or D ⊆N, hence either s ∈(N : M) or m ∈N. Thus N is weakly prime submodule of M.

(ii)⇒(i) Suppose that N is a weakly prime submodule of M. If N is prime then the result is clear. So we can assume that N is a weakly prime submodule of M that is not prime.

Let $0 \neq PD \subseteq N$ with x ∈D−N. We need to show that $P \subseteq (N:M)$. Let r ∈P. If $0 \neq rRx$, since rRx⊆N and N is weakly prime, so r ∈(N : M). So assume that 0 = rRx. Now assume that $rD \neq 0$, say $rt \neq 0$ for some t ∈D. Now $0 \neq rRt \subseteq N$. If $t \notin N$, then r ∈(N : M). If t ∈N, then $0 \neq rRt = rR(t+x) \subseteq N$, so (t + x) ∈N or r ∈(N : M). Since $x \notin N$, then r ∈(N : M). So we can assume that rD = 0. Suppose that $Px \neq 0$, say $ax \neq 0$ where a ∈P. Now $0 \neq aRx \subseteq N$. ThenN is weakly prime submodule gives $a \in (N:M)$. As $0 \neq aRx = (r+a)Rx \subseteq N$, we get r ∈(N : M), so P ⊆(N : M). Therefore we can assume that Px = 0.Since $PD \neq 0, \exists p \in P$ and $t_1 \in D$ such that $pt_1 \neq 0$. As(N : M)N = 0 (by Remark 3.2.) and $0 \neq pRt_1 = pR(t_1 + x) \subseteq N$, we have two cases:

Case(I). p ∈(N : M) and $(t_1 + x) \notin N$.

Since $0 \neq (r+p)R(t_1 + x) = pRt_1 \subseteq N$, we obtain (r + p) ∈(N : M),so r ∈(N : M).

Case (II). $p \notin (N:M)$ and $(t_1 + x) \in N$.

As $0 \neq pRt_1 \subseteq N$. We have $t_1 \in N$, so x ∈N which is a contradiction.

Therefore r ∈(N : M). Thus P ⊆(N : M).

(ii)⇒(iii) If m ∈M –N, then it is clear that K = (N : M)∪(0 : Rm) ⊆(N : Rm). Let

x ∈(N : Rm). Then xRm⊆N. If $0 \neq xRm$, then x ∈(N : M) since N is weakly prime submodule. If xRm = 0, then x ∈(0 : Rm).

(iii)⇒(iv) Let m ∈M−N, so that (N : Rm) = (N : M)∪(0 : Rm).Now (N : Rm), (N :M) and

(0 : Rm) are all ideals of R, that means (N : M)∪(0 : Rm) is an ideal of R and since the union of two ideals of a ring is an ideal iff one of them is contained in the other, so we have (N : M) ⊆

(0 : Rm) or (0 : Rm) ⊆(N : M), from which we get (N : Rm) = (N : M) or (N : Rm) = (0 : Rm).

(iv)⇒(ii) Suppose that $0 \neq rRm \subseteq N$ where r ∈R and m ∈M−N. Then r ∈(N : Rm) and $r \notin (0:Rm)$. It follows that r ∈(N : M).

Theorem 3.5. Let N be a proper submodule of a left R−module M. Then the following are equivalent:

(i) N is a weakly primary submodule of M.

(ii) For m∈M –N, $(N:Rm) \subseteq \sqrt{(N:M)} \cup (0:Rm)$.

Proof. (i)⇒(ii) Assume that N is a weakly primary submodule of M and let r ∈(N : Rm)

where m ∈M –N. Thus rRm⊆N. If $rRm \neq 0$, then N is weakly primary submodule gives $r \in \sqrt{(N:M)}$, and hence $r \in \sqrt{(N:M)} \cup (0:Rm)$. If rRm = 0, then r ∈(0 : Rm) and hence $r \in \sqrt{(N:M)} \cup (0:Rm)$.

(ii)⇒(i) Suppose that $0 \neq rRm \subseteq N$ with r ∈R and m ∈M –N. Then r ∈(N : Rm) and $r \notin (0:Rm)$. Now $(N:Rm) \subseteq \sqrt{(N:M)} \cup (0:Rm)$ implies $r \in \sqrt{(N:M)}$. Thus, N is weakly primary.

Proposition 3.6. Let N be a proper submodule of a left R−module M. Then the following are equivalent:

(i) For every ideal P of R and for every submodule D of M with PD ⊆N, either $P \subseteq \sqrt{(N:M)}$ or D ⊆N.

(ii) N is a primary submodule of M.

(iii) For every left (or right) ideal P of R and for every submodule D of M with PD ⊆N, either $P \subseteq \sqrt{(N:M)}$ or D ⊆N.

Proof. (i)⇒(ii) Let r ∈R and m ∈M such that rRm⊆N. It follows, since N is a submodule, that (RrR)(Rm) ⊆N. Now, by (i), we get $RrR \subseteq \sqrt{(N:M)}$ or Rm⊆N. If $RrR \subseteq \sqrt{(N:M)}$, then $r \in \sqrt{(N:M)}$. If Rm⊆N, then m ∈N. Therefore N is primary.

(ii)⇒(iii) Assume that PD ⊆N, for left (or right) ideal P ⊆R and submodule D ⊆M. If $D \nsubseteq N$,then there exists x ∈D − N. For every t ∈P, we have (tR)x = t(Rx) ⊆PD ⊆N which gives, by(ii), $t \in \sqrt{(N:M)}$.

(iii)⇒(i) Obvious.

Now we introduce the following definition:

Definition 3.5. Let M be a left R−module, the subset T(M) of M is defined by

$$T(M) = \{m \in M : rRm = 0 \text{ for some } 0 \neq r \in R\}$$

Note that if R is an integral domain, then it is easy to see that T(M) is a submodule of M.

Theorem 3.7. Let M be a left R−module with T(M) = 0. Then every weakly primary submodule of M is primary.

Proof. Let N be a weakly primary submodule of M. Suppose that rRm⊆N where r∈R, m ∈M. If $0 \neq rRm \subseteq N$, then N is weakly primary submodule gives m ∈Nor $r \in \sqrt{(N:M)}$. If rRm = 0, then r = 0 or m = 0 (since T(M) = 0). Thus N is primary.

Proposition 3.8. Let M be a left module over a semi commutative local ring R with unique maximal left ideal P. If PM = 0, then every proper submodule of M is weakly prime.

Proof. Let N be a proper submodule of M and let $0 \neq rRm \subseteq N$ where r ∈R andm ∈M. Then $0 \neq rm \in N$ (because R is semi commutative).If r is a unit, then m ∈N. If r is not a unit, then

rm∈PM = 0, a contradiction. Hence N is weakly prime.

Theorem 3.9. Let $M_1$and M_2 be left R−modules, M = $M_1 \oplus M_2$ be a direct sum ofM_1 and $M_2$and let N ⊆$M_1 \oplus M_2$. Then the following are satisfied:

(i) If N = Q⊕M_2 is a weakly primary submodule of M for some submodule Q of M_1, then Q is aweakly primary submodule of M_1.

(ii) If N = M_1⊕Q is a weakly primary submodule of M for some submodule Q of M_2,then Q is a weakly primary submodule of M_2.

Proof. (i) Let N = Q⊕M_2 be a weakly primary submodule of M = $M_1 \oplus M_2$. Let

$0 \neq rRq \subseteq Q$ where r ∈R, q ∈M_1 such that q ∉Q, then (q, 0)∉ Q⊕M_2.

$0 \neq rR(q,0) \subseteq Q \oplus M_2$. Since N = Q⊕$M_2$ is a weakly primary submodule of M, ∃a positive integer n such that $r^n(M_1 \oplus M_2)$ ⊆Q⊕M_2.Hence $r^n M_1$⊆Q for some positive integer n. So $r \in \sqrt{(Q:M_1)}$. Therefore Q is weakly primary submodule of M_1.

(ii) Proceed similar as in (i).

4. Weakly primary compactly packed modules

Primary compactly packed and primary finitely compactly packed modules have been introduced and studied by Ashour in [3], [4]. In this section we study the concepts of weakly primary compactly packed and maximal compactly packed modules.

Recall that a proper submodule N of a left R−module M is said to be maximal if there is no submodule K of M such that $N \nsubseteq K \nsubseteq M$.

Definition 4.1. [3] Let M be a left R−module. A submodule N of M is called primary compactly packed submodule of M denoted by pcp−submodule of M if for each family{Pi : i ∈ I} of primary submodules of M with $N \subseteq \cup_{i\in I} P_i$, $N \subseteq P_j$ for some j ∈ I.

M is called primary compactly packed module denoted by pcp−module if every submodule of M is a pcp−submodule.

Now, we give the following definitions:

Definition 4.2. Let M be a left R−module.A submodule N of M is called weakly primary compactly packed submodule of M denoted by wpcp−submodule of M if for each family

{Pi : i ∈ I} of weakly primary submodules of M with $N \subseteq \cup_{i\in I} P_i$, $N \subseteq P_j$ for some j ∈ I.

M is called weakly primary compactly packed module denoted by wpcp−module if every submodule of M is a wpcp−submodule.

Definition 4.3. Let M be a left R−module. A submodule N of M is called maximal compactly packed submodule of M denoted by mcp−submodule of M if for each family{Pi : i ∈ I} of maximal submodules of M with $N \subseteq \cup_{i\in I} P_i$, $N \subseteq P_j$ for some j ∈ I.

M is called maximal compactly packed module denoted by mcp−module if every submodule of M is a mcp−submodule.

Now, we need the following Lemma:

Lemma 4.1. Every maximal submodule in a left R−module M is prime submodule.

Proof. Let K be a maximal submodule of M. Assume that r ∈ R and m ∈ M such that rRm⊆K. Suppose that $m \notin K$. Then m + K is a nonzero element in M/K, which means that M/K is cyclicgenerated by m + K. Hence for every x ∈ M, there exists t ∈ R such that x + K = t(m + K). It follows that x − tm ∈ K and therefore

rx− rtm∈ K. However, by the assumption rtm∈ K and we conclude rx∈ K and consequently r∈ (K : M).

Remark 4.1. Clearly, every wpcp−module is pcp−module, and every pcp−module ismcp−module.

Theorem 4.2. Let M be a finitely generated left R−module. Then M is a mcp−module if and only if every submodule N in M satisfies $N + P_i \neq M$ for some i ∈ I where $N \subseteq \cup_{i\in I} P_i$, Pi′s are weakly primary submodules of M.

Proof. Let M be a finitely generated left R−module, and suppose that N is a submodule of M such that $N \subseteq \cup_{i\in I} P_i$, Pi′s are weakly primary submodules of M. Foreach P_i , there exists a maximal submodule M_i containing P_i . Then $N \subseteq \cup_{i\in I} M_i$ and so $N \subseteq M_i$ for some i ∈ I by hypothesis. Since $P_i \subseteq M_i$, we have $N + P_i \subseteq M_i \neq M$.

Conversely; let N be a submodule of M such that $N \subseteq \cup_{i\in I} M_i$ where each M_i is a maximal submodule of M. Since every prime submodule is weakly prime submodule and every weakly prime submodule is weakly primary submodule, so every maximal submoduleis weakly primary (by Lemma 4.1.), then $N + M_i \neq M$ for some i ∈ I. Therefore, since $M_i \subseteq N + M_i \subsetneq M$, then $N+M_i = M_i$, so $N \subseteq M_i$ for some i ∈ I.

The following theorem follows immediately from Theorem 3.7. and Remark 4.1.

Theorem 4.3. Let M be a left R−module with T(M) = 0. Then M is a pcp−module if and only if M is a wpcp−module.

Now, we give the following definition:

Definition 4.4. Let N be a submodule of a left R−module M. The intersection of all weakly primary submodules containing N is called the weakly primary radical of N and is denoted by wprad(N). If there is no weakly primary submodule containing N, then wprad(N) = M. In particular wprad(M) = M. We say that a submodule N is a weakly primary radical submodule if wprad(N) = N.

The following result can be easily proved:

Proposition 4.4. Let N and L be submodules of a left R−module M. Then the following are hold:

(i) $N \subseteq$ wprad(N).

(ii) wprad(wprad(N))= wprad(N), that is the weakly primary radical of N is a weakly primary radical submodule.

(iii) If $N \subseteq L$, then wprad(N) $\subseteq$ wprad(L).

(iv) wprad($N \cap L$) $\subseteq$ wprad(N)$\cap$wprad(L).

Theorem 4.5. Let M be a left R−module. The following statements are equivalent:

(i) M is a wpcp−module.

(ii) For each proper submodule N of M, there exists $m \in N$ such that wprad(N) = wprad(Rm).

(iii) For each proper submodule N of M, if $\{N_i : i \in I\}$ is a family of submodules of Mand

$N \subseteq \cup_{i\in I} N_i$, then $N \subseteq$ wprad(N_j) for some $j \in I$.

(iv) For each proper submodule N of M, if $\{N_i : i \in I\}$ is a family of weakly primaryradical submodules of M and $N \subseteq \cup_{i\in I} N_i$, then $N \subseteq N_j$ for some $j \in I$.

Proof. (i)⇒(ii) Assume that M is a wpcp−module and let N be a proper submodule of M. It is clear that wprad(Rm) $\subseteq$ wprad(N) for each $m \in N$. For the other inclusion, suppose that wprad(N) $\nsubseteq$ wprad(Rm) for each $m \in N$. Then for each $m \in N$, there exists a weakly primary submodule P_m for which $Rm \subseteq P_m$ and $N \nsubseteq P_m$. But$N = \cup_{m\in N} Rm \subseteq \cup_{m\in N} P_m$, that is M is nota wpcp−module, which is a contradiction.

(ii)⇒(iii) Let N be a proper submodule of M, and let $\{N_i : i \in I\}$ be a family of submodules of Msuch that $N \subseteq \cup_{i\in I} N_i$ .By (ii), there exists $m \in N$ such that wprad(N) = wprad(Rm).

Then $m \in \cup_{i\in I} N_i$ and hence $m \in N_j$ for some $j \in I$. Hence $N \subseteq$ wprad(N) = wprad(Rm) $\subseteq$ wprad(N_j) for some $j \in I$.

(iii)⇒(iv) Let N be a proper submodule of M, and let $\{N_i : i \in I\}$ be a family of weakly primary radical submodules of M such that $N \subseteq \cup_{i\in I} N_i$. By (iii), there exists $j \in I$ such that $N \subseteq$ wprad(Nj). Since N_j is weakly primary radical submodule of M, then$N \subseteq N_j$.

(iv)⇒(i) Let N be a proper submodule of M, and suppose that $\{N_i : i \in I\}$ is a family of weakly primary submodules of M such that $N \subseteq \cup_{i\in I} N_i$ .Since N_i is weakly primary submodule of M for each $i \in I$, N_i = wprad(N_i) for each $i \in I$. Thus $N \subseteq \cup_{i\in I} N_i = \cup_{i\in I}$wprad($N_i$). By (iv), there exists $j \in I$ such that $N \subseteq$ wprad(N_j) = N_j . Thus M is a wpcp−module.

Now we give the definition of the Bezout module over non commutative ring which is a generalization of the definition of the Bezout module over commutative ring in [1].

Definition 4.5. A left R−module M is said to be a Bezout module, if every finitely generated submodule of M is cyclic.

Consider the following Lemma:

Lemma 4.6. Let M be a left R−module. If M satisfies the ascending chain condition on weakly primary radical submodules, then any weakly primary radical submodule is the weakly primary radical of a finitely generated submodule.

Proof. Assume that there exists a weakly primary radical submodule N which is not weakly primary radical of a finitely generated submodules. Let $m_1 \in N$ and N_1 = wprad(m_1R).Then $N_1 \subseteq N$. So there exists $m_2 \in N - N_1$ .Let N_2 = wprad(m_1R+m_2R). Then $N_1 \subseteq N_2 \subseteq N$. So that there exists $m_3 \in N - N_2$. Continuing in this process, we will have an ascending chain of weakly primary radical submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ which is a contradiction.

Now, we are ready to prove the main result of this section.

Theorem 4.7. Let M be a Bezout module. If M satisfies the ascending chain condition on weakly primary radical submodules, then M is wpcp−module.

Proof. Let N be a proper submodule of M. By Lemma 4.6., there exists a finitely generated submodule K of M such that wprad(N) = wprad(K) and hence K is cyclic submoduleof M, because M is Bezout. It follows by Theorem 4.5. that M is a wpcp−module.

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