Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 6, Issue 3
Published online: January 25, 2016|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Pseudo C-M-Injective and Pseudo C-Quasi Principally Injective Systems Over Monoids

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ABSTRACT

In this paper , the concepts of pseudo C-quasi injective and pseudo C-quasi principally injective systems over monoids are introduced which represents generalizations of quasi injective and quasi principally injective respectively . Characterization of these concepts are given. At the same time some properties of pseudo C-quasi principally injective systems are studied in terms of their endomorphism monoid. Relate the concepts of Hopfian, co-Hopfian and directly finite with pseudo C-quasi injective systems over monoids. Conditions are investigated under which subsystems are inherit pseudo C-quasi principally injective and pseudo C-quasi injective. The relationship among the class of pseudo C-quasi injective and pseudo C-quasi principally injective systems with some generalizations of injectivity (such as extending and continuous systems) are considered .

Key words: CS systems; Pseudo closed quasi injective systems; Pseudo closed quasi principally injective systems; Continuous systems.

1-INTRODUCTION

Throughout this paper , by a monoid S we always means monoid with zero element 0 and every right S-system M is a unitary with zero element Θ which denoted by M_s. A right S-system M_s with zero is a non-empty set with a function

$f : M\times S\rightarrow M$ such that $f(m,s) \longmapsto ms$ such that the following properties hold :

(1) $m\cdot 1= m$ (2) $m(st)= (ms)t$ for all $m \in M$ and $s , t \in S$, where 1 is the identity element of S.

Let A_s , M_s be two S-systems . A_s is called M-injective if given an S-monomorphism $\alpha : N \rightarrow M_s$ where N is a subsystem of M_s and every S-homomorphism $\beta : N\rightarrow A_s$, can be extended to an S-homomorphism $\sigma : M_s \rightarrow A_s$ [16].An

S-system A_s is injective if and only if it is M-injective for all S-systems M_s . The concept of injectivity was generalized to quasi injective S-system by A.M.Lopez [10] , such that an S-system A_s is quasi injective if and only if it is A-injective . More generally, Yan gave generalized quasi injective S-system to pseudo injective, such that an S-system M_s is called pseudo injective if each S-monomorphism of a subsystem of M_s into M_s extends to S-endomorphism of M_s [15] . Then , Yan studied the properties of linear equation S-system on the class of pseudo injective [15]. But, the author studied the general case [4] . Lastly , the concept of quasi injectivity was generalized to some concepts for example , principally quasi injective (PQ) , quasi , principally injective (QP), pseudo principally quasi injective (PPQ) and pseudo quasi principally injective (PQP) system by the author [1] ,[2] , [4] . Note that we will use terminology and notations from [2] and [4] freely .

Let M_s , N_s be two S-systems . An S-system N_s is called M-principally injective if for every S-homomorphism from M-cyclic subsystem of M_s into N_s can be extended to an S-homomorphism from M_s into N_s (for short N_s is M-P-injective)[2] .

An S-system M_s is called quasi-principally injective if it is M-P-injective , that is every S-homomorphism from M-cyclic subsystem of M_s to M_s can be extended to S-endomorphism of M_s , then we write M_s is QP-injective [2] .

A subsystem N of S-system M_s is called closed if it has no proper ∩-large in M_s that is the only solution of N $\hookrightarrow^{\cap l}$ L $_{\neq}\hookrightarrow$ M_s is N = L [13] .

An S-system M_s is said to be satisfy C_1-condition if every closed subsystem of M_s is a retract of M_s . An S-system M_s is said to be satisfy C_2-condition if every subsystem of M_s which is isomorphic to retract of M_s is itself retract of M .

An S-system M_s is called a CS-system or extending system if it is satisfy C_1-condition . An S-system M_s is called continuous if it is satisfy C_1 and C_2-conditions.

This paper is essentially generalization of part of my doctoral dissertation written under the direction of professor Dr. M.S. ABBAS, such that we adopt another generalization of pseudo injective and generalization of pseudo quasi principally injective system and which are pseudo C-quasi injective and pseudo C-quasi principally injective systems .

2- PSEUDO C-M-INJECTIVE SYSTEMS OVER MONOIDS :

Definition(2.1): Let M_s and N_s be two S-systems . A right S-system N_s is called pseudoclosed M-injective (for short pseudo C-M-injective) if for anymonomorphismfrom a closed subsystem of M_s to N_s can be extended to homomorphism from M_s to N_s . A right S-system N_s is called pseudoC-quasi injective if N_s is pseudo C-N-injective .

Remarks and Examples (2.2) :

(1) Every pseudo injective is pseudo closed M-injective , but the converse is not true in general , for example Z as Z-system is pseudo closed injective which is not pseudo injective since only closed subsystem of Z are Z and one zero subsystem (this means one element subsystem {Θ}) .

(2) Isomorphic system to pseudo C-M-injective is pseudo C-M-injective for any S-system M .

(3) Let N_1 and N_2 be two S-systems such that $N_1 \cong N_2$. If M_s is pseudo C-N_1-injective , then M_s is pseudo C-N_2-injective.

Proof : Assume that M_s is pseudo C- N_2-injective system where N_2 be S-system . Let N_1 be S-system and f : $N_1 \rightarrow N_2$ be S-isomorphism . Let A be closed subsystem of N_1 , so f(A) is closed subsystem of N_2 . Suppose that α :A $\rightarrow M_s$ be any monomorphism , then since $f^{-1} : N_2 \rightarrow N_1$ is isomorphism such that $ff^{-1} = I$, thus (αf^{-1}) be S-monomorphism from f(N) to M_s . Since M_s is pseudo C- N_2-injective , therefore (αf^{-1}) extended to S-homomorphism g : $N_2 \rightarrow M_s$. It is clear that there exists S-homomorphism gf(=h) from N_1 to M_s which is extends α . Hence M_s is pseudo C-N_1-injective system

(4) Every retract of S-system is closed .

Proof : Let A be a retract of an S-system M_s and A is $\cap$-large subsystem of closed subsystem B in M_s . As A is a retract of M_s , and every retract is direct summand by remarks and examples(2.2)(3)[1] , so there exists a subsystem C of M_s such that M_s= A $\dot{\cup}$ C . This means that M_s= A$\cup$C and A $\cap$ C= Θ . In fact Θ = A $\cap$ C = A $\cap$ B $\cap$ C = A $\cap$ (B $\cap$ C) , which implies that B $\cap$ C is subsystem of B in M_s and since A is $\cap$-large subsystem of B , so B $\cap$ C = Θ , but B is maximal essential extension of A , thus B = A and A is closed .

The following proposition is generalization to proposition 2.3 in [4] :

Proposition(2.3): Let M_s , N_s be two S-system . If N_s is pseudo C-M-injective , then any monomorphism from closed subsystem N of M_s into M_s split .

Proof : Let α : N $\rightarrow M_s$ be monomorphism such that N is closed subsystem of M_s . Since N is C-M-injective , so there exists S-homomorphism $\beta : M_s \rightarrow N$ such that $\beta o \alpha = I_N$. This means that N is a retract of M_s , since N $\cong \alpha(N)$, so $\alpha(N)$ is a retract of M_s and f is split .

The following proposition is generalization to lemma 2.1 in [12] :

Proposition(2.4): Let M_s and N_s be two S-systems . If N_s is pseudo C-M-injective system , A is a retract of N_s and B_s is a closed subsystem of M_s , then :

(1) A is a pseudo C- B-injective system ,

(2) A is a pseudo C- M-injective system ,

(3) N_s is a pseudo C- B-injective system .

Proof :

(1) Let K be a closed subsystem of B and α is monomorphism from K to A . Since A is a retract of N_s , so there exists a subsystem W of N_s such that N_s = A $\dot{\cup}$ W . Let i_k , i_B be the inclusion maps of K into B and B into M_s respectively . Let j_A , π_A be the injection and projection map of A into N_s(and N_s onto A respectively) . Since N_s is pseudo C-M-injective system , so there exists $\beta : M_s \rightarrow N_s$ such that $\beta o i_B o i_K = j_A o \alpha$. Put $\bar{\beta} = \pi_A o \beta o i_B$ from B into A . Then, this implies that $\bar{\beta} o i_K = \pi_A o \beta o i_B o i_K = \pi_A o j_A o \alpha = \alpha$. Hence A is pseudo C- B-injective system .

(2) Let X be closed subsystem of S-system M_s and f be S-monomorphism from X into A . Since N is pseudo C- M-injective system , so there exists S-homomorphism g from M into N_s such that $goi_X = j_A of$, where j_A is the injection map of A into N_s. Put

$h = \pi_A og$, then $hoi_X = \pi_A ogoi_X = \pi_A oj_A of = f$ and A is pseudo C- M-injective .

(3) Let X be closed subsystem of B in M_s, and f be S-monomorphism from X into N_s . Since N_s is pseudo C- M-injective , so there exists S-homomorphism g from M_s into N_s such that $goi_B oi_X = f$, where i_X , i_B be the inclusion map of X into B and B into M_s respectively . Put $h= goi_B$, then $hoi_X = goi_B oi_X = f$. Thus N_s is pseudo C- B-injective system .

Corollary(2.5) : Let M_s and N_s be two S-systems. Then , N_s is pseudo C- M-injective system if and only if N_s is pseudo C- X-injective system for every closed subsystem X of M_s .

Proof : Suppose that N_s is pseudo C- M-injective system , by proposition(2.4)(3) , we have N_s is pseudoC- X-injective for every closed subsystem X of M_s. Conversely, since M is closed subsystem of M_s and by assumption, we have N_s is pseudo C- M-injective system .

Proposition(2.6): Let M_s be S-system and N be closed subsystem of M_s . If N is pseudo C- M-injective , then N is a retract of M_s .

The following proposition give a condition under which subsystem of pseudo C-M-injective inherit this property :

Proposition(2.7): Let M_s be pseudo C-M-injective system . Then every fully invariant closed subsystem of M_s is pseudo C-quasi injective .

Proof : Let N be fully invariant closed subsystem of M_s and let K be closed subsystem of N and let $\alpha : K \to N$ be S-monomorphism. Since N is closed subsystem of M_s, it follows that K is closed subsystem of M_s . Then , since M_s is pseudo C-M-injective , so there exists $\beta : M_s \to M_s$ that extends α . Since $\beta(N) \subseteq N$ by hypothesis , so restrict of β from N into N extends α and then N is pseudo C-quasi injective .

Recall that an S-systems M_i , $i \in I$ are called relatively pseudo C-injective systems if M_i is pseudo C-M_j-injective for all distinct i, j$\in$ I , where I is the index set .

Lemma(2.8): Let M_1 and M_2 be two S-systems and $M_s = M_1 \oplus M_2$. If M_s is quasi pseudo C-quasi injective system , then M_1 and M_2 are both pseudo C- quasi injective system and they are relatively pseudo C-injective system .

Proof : By proposition(2.4)(2) and corollary(2.5) , we have M_1 and M_2 are pseudo C-quasi injective systems . It is also that M_1 and M_2 are relatively pseudo C-injective system .

Theorem(2.9): Let M_s and N_s be two S-systems . If $M_s \oplus N_s$ is pseudo C-quasi injective , then N_s is C-M-injective .

Proof : Let A be closed subsystem of S-system M_s , and $f : A \to N_s$ be S-homomorphism . Define $g : A \to M_s \oplus N_s$ by $g(a) = (a,f(a))$ for all $a \in A$, then it is easy to show that g is S-monomorphism . Since$M_s \oplus N_s$ is pseudo C-quasi injective system , then by corollary(2.5) , we have $M_s \oplus N_s$ is pseudo C-M-injective , where M_s is closed subsystem of $M_s \oplus N_s$.

Thus g extends to S-homomorphism $g^{/}:M_s \to M_s \oplus N_s$. Let $\pi_N : M_s \oplus N_s \to N_s$ be natural projection , then $\pi_N \, o \, g^{/} : M_s \to N_s$ is extension of f and N_s is C- M-injective system .

Definition(2.10) : Let M_s be S-system . Then :

C_1-condition : An S-system is said to satisfy C_1-condition , if everyclosed subsystem of M_s is a retract of M_s .

C_2-condition : An S-system M_s is said to satisfy C_2-condition , if every subsystem of M_s isomorphic to a retract of M_s is itself a retract of M_s .

Proposition(2.11): Every pseudo C- quasi injective system satisfies C_2-condition .

Proof : Assume that M_s is pseudo C- quasi injective system (this means pseudo C- M-injective) . Let A , B be subsystems of M_s with $B \cong A$ and A is a retract of M_s Since A is closed , so B is closed . Since M_s is pseudo C-M-injective , so A is pseudo C-M-injective by proposition(2.4)(2) , but $A \cong B$, thus by remarks and examples(2.2)(2) , B is pseudo C-M-injective. Then , by proposition(2.6) , B is a retract of M_s . Hence M_s satisfies C_2-condition .

Corollary(2.12): Let M_s and N_s be two S-systems . N_s satisfies C_2-condition if and only if , for every subsystem K of M_s which is isomorphic to a retract of N_s , every monomorphism $\alpha : K \to N_s$ can be lifted to homomorphism $\beta : M_s \to N_s$.

Proof : The necessity is immediate . Conversely , let A be a retract of N_s and S-isomorphism $f : K \to A$. Let j_A , π_A be the injection and projection of A into N_s and N_s onto A respectively , then $j_A of : K \to N_s$ where f(K)=A is a retract of N_s . By hypothesis , f can be lifted to S-homomorphism $g : M_s \to N_s$. Then β $(=\pi_A og):M_s \to A$. Note that $\beta(K) = (\pi_A og)(K) = \pi_A(f(K)) = f(K)$, $\forall \, k \in K$. Thus , $(\beta oi_K)(K) = f(K)$, this means $\beta oi_K = I_{f(K)}$ and since $f(K) \cong K$, so this implies that every monomorphism

$i_K : K \to M_s$split and K is a retract of M_s .

Proposition(2.13): The following statements are equivalent for S-system M_s :

(1) N_s satisfies C_2-condition ,

(2) For any retract A of N_s , every S-monomorphism $\alpha : A \to M_s$, where $\alpha(A)$ is closed subsystem of M_s , there is S-homomorphism $\beta : M_s \to N_s$ such that $\beta o \alpha = j_A$, where $j_A : A \to N_s$ is injection mapping ,

(3) For every subsystem K of M_s which is S-isomorphism to a retract of N_s , every S-monomorphism $f : K \to N_s$ can be lifted to S-homomorphism $g : M_s \to N_s$.

Proof : ($1 \to 2$) Let A be a retract of N_s and $f : A \to M_s$ be S-monomorphism . This implies that $f(A) \cong A$, then by (1) f(A) is a retract of M_s . Thus f(A) is closed subsystem of M_s .since$f(A) \cong A$,this means thatthere exists $\beta : f(A) \to A$ is S-isomorphism .Then for f $(=i_{f(A)} o \beta^{-1}) : A \to M_s$, there exists S-monomorphism $h(j_A o \beta) : f(A) \to N_s$, where $j_A : A \to N_s$. Since f(A) is a retract ofM_s by the proof, so there exists projection map $\pi_{f(A)}$ from M_s onto f(A) . Now, put $g(j_A o \beta o \pi_{f(A)}):M_s \to N_s$ such that $j_A o \beta = goi_{f(A)}$. Thus $goi_{f(A)} o \beta^{-1} = j_A$ and then $gof = j_A$.

(2→1) Let A be a subsystem of M_s and isomorphic to a retract K of N_s , this means there exists S-isomorphism f : A → K. Since K is closed subsystem of N_s in M_s , so A is closed subsystem of M_s . Then , $i_A of^{-1}$: K → M_s , by (2) there exists S-homomorphism g : $M_s \to N_s$ such that $goi_A of^{-1} = j_K$, where $j_K : K \to N_s$ is the injection mapping of K into N_s . Let π_K : N_s → f(A)=K be the projection map of N_s onto K . Then β (= $\pi_K$og) : M_s → f(A) = K . Note that β(A) = ($\pi_K$og)(A) = π_K(f(A)) = f(A) , for all a ∈ A . Thus , (βoi_A)(A) = f(A) , this means βoi_A = $I_{f(A)}$ and since f(A) ≅ A , so this implies that every monomorphism $i_A : A \to M_s$ split and A is a retract of M_s .

(1⟷ 3) By corollary(2.12) , the proof is immediately .

Proposition(2.14) : Let M_s be an S-system and { N_i │i ∈ I } be a family of S-systems . Then $\prod_{i\in I} N_i$ is pseudo C-M-injective if and only if N_i is pseudo C-M-injective for every i ∈ I .

Proof :⟹) Assume that $N_s = \prod_{i\in I} N_i$ is pseudo C-M-injective , where M_s is S-system . Let X be closed subsystem of M_s and f be S-monomorphism from X to N_i . Since N_s is pseudo C-M-injective system then there exists S-homomorphism g : $M_s \to N_s$ such that goi = $j_i$of , where i is the inclusion map of X into M_s and j_i , π_i be the injection and projection maps of N_i into N_s and N_s onto N_i respectively . Define h : $M_s \to N_i$ such that h = $\pi_i$og , then hoi = $\pi_i$ogoi = π_io$j_i$of = f . That is for all x ∈ X , h(x) = h(i(x)) = π_i(g(x)) = π_i(g(i(x))) = π_i(j_i(f(x))) = (π_ioj_i)(f(x)) = f(x) .

⟸) Assume that N_i is pseudo C-M-injective for each i ∈ I , where M_s is S-system . Let f be S-monomorphism from $N_s = \prod_{i\in I} N_i$ to M_s . Since N_i is pseudo C-M-injective system , then there exists S-homomorphism $\beta_i : M_s \to N_i$, such that $\beta_i$oi = $\pi_i$of , so there exists S-homomorphism β : $M_s \to N_s$ such that $\beta_i = \pi_i$oβ . We claim that βoi = f . Since $\beta_i$oi = π_ioβoi , then $\pi_i$of = π_ioβoi , so we obtain f = βoi . Therefore N_s is pseudo C-M-injective .

Corollary(2.15): Let M_s and N_i be S-systems , where i ∈ I and I is a finite index set . Then , for every i , N_i is pseudo C-M-injective if and only if $\oplus_{i=1}^{n} N_i$ is pseudo C-M-injective .

Proposition(2.16): M_s^n is pseudo C-quasi injective system for any finite integer n , if and only if M_s is pseudo C-M-injective (this means M_s is pseudo C-quasi injective) .

Proof : Let M_s^n is pseudo C-quasi injective system . Since M_s is retract of M_s^n and then it is closed by remarks and examples(2.2)(4) . By corollary(2.5) , M_s^n is pseudo C-M-injective system . Again since M_s is retract of M_s^n , so by proposition(2.4)(2) M_s is pseudo C-M-injective . Conversely, let M_s is pseudo C-M-injective , so by proposition(2.14) , M_s^n is pseudo C-M-injective system .

3- THE RELATIONSHIP BETWEEN PSEUDO M-C-INJECTIVE WITH CLASS OF INJECTIVITY :

Recall that an S-system M_s is Noetherian if every subsystem of M_s is finitely generated. A monoid S is a right Noetherian if S_s is Noetherian. Equivalently , S is a right Noetherian if and only if S satisfies the ascending chain condition for right ideals (definition1.1.30) [13, p.21] .

The following theorem is generalization to theorem 1.1 in [14] :

Theorem (3.1): The following conditions are equivalent for an S-system M_s , where S is Noetherian monoid :

(1) The direct sum of every two pseudo C-quasi injective S-systems are pseudo C-quasi injective systems .

(2) Every pseudo C-quasi injective system is injective .

Proof :(1⟹2) Assume that M_s is pseudo C-quasi injective system (this means that M_s is pseudo C-M-injective) and E(M) is injective envelope of M_s . Then , since E(M) is injective , so it is pseudo C-quasi injective and by assumption $N_s = M_s \oplus E(M)$ is pseudo C-quasi injective . Consider the injection maps $j_1: M_s \rightarrow E(M)$,$j_2: E(M) \rightarrow M_s \oplus E(M)$, $j_3: M_s \rightarrow M_s \oplus E(M)$ and $I_M : M_s \rightarrow M_s$ is the identity map of M_s . Let $\pi_M : M_s \oplus E(M) \rightarrow M_s$ be the projection map such that $\pi_M o j_3 = I_M$. Now , $M_s \oplus E(M)$ is pseudo C-quasi injective , so this implies there exists S-homomorphism $g : M_s \oplus E(M) \rightarrow M_s \oplus E(M)$ such that $goj_2oj_1 = j_3oI_M$, then $\pi_M ogoj_2oj_1 = \pi_M oj_3oI_M$. Thus $I_M = \pi_M ogoj_2oj_1$, so that $f = \pi_M ogoj_2$ and then $I_M = foj_1$. Therefore M_s is a retract of E(M) and then it is injective .

(2⟹1) Let M_s and N_s be two pseudo C-quasi injective S-system . By (2) M_s and N_s are injective which implies that the direct sum of any two injective S-systems is injective whence S is Noetherian monoid [6 ,theorem 1] and then every injective is pseudo C-quasi injective. Therefore, the direct sum of two pseudo C-quasi injective is pseudo C-quasi injective .

Proposition(3.2) : An S-system M_s satisfies C_1-condition(equivalently , CS system) if and only if any S-system is pseudo C- M-injective .

Proof : ⟹)It is obvious .

⟸) Let N be a closed subsystem of S-system M_s . By hypothesis N_s is pseudo C- M-injective , so by proposition(2.6) , N_s is a retract of M_s . It follows that M_s is CS-system .

Proposition(3.3): The following statements are equivalents for S-system M_s :

(1) M_s is extending system ,

(2) Every S-system is pseudo C-M-injective ,

(3) Every closed subsystem of M_s is pseudo C- M-injective .

Proof: (1→2) Let N be closed subsystem of S-system M_s and f be S-monomorphism from N into M_s . Since M_s is extending system , so N is a retract of M_s . This means that there exists j_N , π_N which are injection and projection map , then define $g(fo\pi_N): M_s \rightarrow N$, so $goj_N = fo\pi_N oj_N = f$. Thus M_s is pseudo C- M-injective.

(2→3) Let N be closed subsystem of S-system M_s and I identity map of N , so by (2) , there exists S-homomorphism g : $M_s \rightarrow N$ such that $goi_N = I_N$, so inclusion map i_N has left inverse and this means that i_N splits and N is a retract of M_s .

(3→1) Let N be closed subsystem of S-system M_s . By (3) N is pseudo C-M-injective and then by proposition(2.6) , N is a retract of M_s , so M_s is extending system .

Definition(3.4): An S-system M_s is called continuous system if it is satisfy C_1 and C_2 condition .

The following proposition is immediately from proposition(2.11) and proposition(3.2) :

Proposition(3.5): If every S-system is pseudo C- quasi injective S-system , then S-system is continuous .

Proposition(3.6)[4]: Let M_s be a cog-reversible non-singular system with $\ell_M(s) = \Theta$, $\forall$ s $\in$ S . Then M_s is pseudo injective if and only if M_s is quasi injective .

Theorem(3.7): Let M_s be S-system . If M_s is pseudo C- quasi injective which cog-reversible non-singular system with $\ell_M(s) = \Theta$, $\forall$ s $\in$ S , then M_s is C- quasi injective system (this means that M_s is C- M-injective) .

Proof : Let N be closed subsystem of S-system M_s and f be S-homomorphism from N into M_s . If f is monomorphism , then there is nothing to prove . But , if not and since M_s is cog-reversible non-singular system with $\ell_M(s) = \Theta$, $\forall$ s $\in$ S , so by using the proof of proposition(3.6) , we get the required . Thus M_s is C-quasi injective .

The following proposition is generalization to proposition 2.17 in[7] :

Proposition(3.8): A pseudo C- quasi injective system is directly finite if and only if it is co-Hopfian .

Proof : Let f be an injective endomorphism of M_s and I be an identity homomorpism from M_s to M_s . Since M_s is pseudo C-M-injective system , so there exists a homomorphism g : $M_s \rightarrow M_s$ such that gof = I , since M_s is directly finite , so fog = I which implies that f is isomorphism . Hence M_s is co-Hopfian . Conversely , assume that M_s is co-Hopfian . Let f,g$\in$ T=End(M_s) such that fog = I , then g is injective homomorphism . Since M_s is co-Hopfian , so there exists g^{-1} . Thus f = $fgg^{-1}=Ig^{-1} = g^{-1}$, so gf = gg^{-1} = I which implies that M_s is directly finite .

The following proposition give a condition under which the concepts Hopfian , co-Hopfian and directly finite are equivalent :

Proposition(3.9): Let M_s be pseudo C-M-injective system , then the following concepts are equivalent :

(1) M_s is Hopfian ,

(2) M_s is co-Hopfian ,

(3) M_s is directly finite .

Proof : (1$\rightarrow$2) As every Hopfian is directly finite (For this if for any α , $\beta \in$ End(M_s) and $\alpha o \beta$ = I , then this means that α is surjective . Since M_s is Hopfian , then α is isomorphism and β is inverse of α . Thus $\beta o \alpha$ = I which implies that M_s is directly finite system) , so by proposition(3.8) , M_s is co-Hopfian .

(2 $\leftrightarrow$3) By proposition(3.8) .

(3$\rightarrow$1) Let f be surjective endomorphism of M_s , then the inclusion map i : f(M) $\rightarrow M_s$ is isomorphism (since every directly finite pseudo C-M-injective system is co-Hopfian) . Thus foi = $I_{f(M)}$. Again since M_s is directly finite , so iof = I_M (since f(M) $\cong M_s$) . Thus f is isomorphism and M_s is Hopfian.

Recall that a subsystem N of S-system M_s is said to be pseudo-stable , if $\mu(N) \subseteq N$ for each S-monomorphism μ : N $\rightarrow M_s$. M_s is called fully pseudo-stable system if each subsystem of M_s is pseudo-stable . A monoid S is called fully pseudo-stable if it is fully pseudo-stable S-system[5] .

In case , each closed subsystem of M_s is pseudo stable , an S-system M_s is called fully closed pseudo stable . Every fully pseudo stable S-system is fully closed pseudo stable , but the converse is not true in general , for example Z as Z-system .

The following proposition is generalization to proposition 2.15 in [3] :

Proposition(3.10): Let M_s be multiplication S-system . Then M_s is pseudo C-quasi injective(this means M_s is pseudo C-M-injective) , if and only if M_s is fully closed pseudo stable .

Proof : Let N be closed subsystem of S-system M_s and f be S-monomorphism from N into M_s . Since M_s is pseudo C-M-injective , so f extends to S-homomorphism g from M_s into M_s . Since M_s is multiplication system , so N = MI for some right ideal I of S . Thus , f(N) = g(N) = g(MI) = g(M)I ⊆ MI = N . Hence , N is stable and M_s is fully closed pseudo stable . Conversely , let N be closed subsystem of M_s and f be S-monomorphism from N into M_s . Note that for each N and f , there exists s belong S such that f(n) = ns , ∀ n ∈ N . Define $g : M_s \to M_s$ by g(m) = ms . Thus , in tihis case f(n) = g(n) , ∀ n ∈ N . This means that g extends f and M_s is pseudo C-M-injective system .

Lemma(3.11): Let M_s be fully closed pseudo stable S-system . Then , every two distinct closed subsystems of M_s are not isomorphic .

Proof : Assume that M_s is fully closed pseudo stable S-system and M_s has two distinct closed subsystems N_1 and N_2 of M_s such that

$N_1 \cong N_2$. Without loss of generality if we assumed that $N_2 \nsubseteq N_1$, then there exists $x \in N_2$ and $x \notin N_1$. Since $N_1 \cong N_2$, so this means that there exists S-isomorphism $f : N_1 \to N_2$. Since f is isomorphism , so $f^{-1} : N_2 \to N_1$ is also S-isomorphism . Now , for the inclusion map $i_{N_1} : N_1 \to M_s$ of N_1 , we have $i_{N_1} o f^{-1} : N_2 \to M_s$. Since N_2 is stable , so $(i_{N_1} o f^{-1})(N_2) \subseteq N_2$, (also for $N_1 (i_{N_2} o f \ )(N_1) \subseteq N_1$, where i_{N_2} is the inclusion map of N_2 into M_s). Let $f^{-1}(x) = y \in N_1$, so $ff^{-1}(x) = f(y)$, then x = f(y) and $x = (i_{N_2} o f \ )(y) \subseteq N_1$ which implies that $x \in N_1$ and this is a contradiction . Thus $N_1 \ncong N_2$.

4- PSEUDO C-QUASI PRINCIPALLY INJECTIVE SYSTEM OVER MONOIDS :

Definition(4.1): An S-system N_s is called pseudo closed M- principally injective (for short pseudo C-M-P-injective) if for every S-monomorphism from closed M_s-cyclic subsystem of M_s to N_s extends to S-homomorphism from M_s to N_s . An S-system M_s is called pseudo closed quasi principally injective (for short pseudo C-QP-injective) if it is pseudo closed M-principally injective .

Remarks and Examples(4.2):

(1) Every QP-injective is pseudo C-QP-injective , but the converse is not true in general , for example , Z as Z-system is pseudo C-QP-injective which is not quasi principally injective system .

(2) Isomorphic system to pseudo C-quasi principally injective is pseudo C-quasi principally injective S-system .

(3) If an S-system M_s is pseudo C-X-P-injective with $X \cong Y$, then M_s is pseudo C-Y-P-injective system .

Lemma(4.3): Retract of pseudo C-QP-injective S-system is pseudo C-M-P-injective .

Proof : Let M_s be pseudo C-QP-injective S-system and N be a retract subsystem of M_s . Let A be closed M_s-cyclic subsystem of M_s and f:A→N be S-monomorphism . Define $\alpha : A \to M_s$ by $\alpha = j_N of$, where j_N be the injection map of N into M_s , then α is S-monomorphism . Since M_s is pseudo C-QP-injective , so there exists S-homomorphism $\beta : M_s \to M_s$ such that $\beta o i_A = \alpha$, where i_A be the inclusion map of A into M_s . Now, let π_N be the projection map ofM_s onto N . Then , define $\sigma(=\pi_N \beta) : M_s \to N$. Thus for each a ∈ A we have that $\sigma o i_A(a) = (\pi_N o \beta o i_A)(a) = \pi_N(\alpha(a)) = \pi_N(j_N o f(a)) = \pi_N(f(a)) = f(a)$. Therefore , an S-homomophism σ is extends f . Thus , N is pseudo C-M-P-injective system .

The following proposition give a condition , under which subsystem of pseudo C-QP-injective inherit this property :

Proposition(4.4): Let M_s be pseudo C-M-P-injective system . Then every fully invariant closed subsystem of M_s is pseudo C-quasi principally injective .

Proof : Let N be fully invariant closed subsystem of M_s and let K be closed subsystem of N and let $\alpha : K \to N$ be S-monomorphism . Since N is closed subsystem of M_s , it follows that K is closed subsystem of M_s . Then , since M_s is pseudo C-M-P-injective so there exists $\beta : M_s \to M_s$ that extends α . Since $\beta(N) \subseteq N$ by hypothesis , so this means there exists σ from N into N which is the restrict of β and extends α . Thus N is pseudo C-quasi principally injective .

The following proposition is generalization to proposition 2.6 in [9] :

Proposition(4.5): Let M_s be S-system . M_s is pseudo C-QP-injective if and only if M_s is pseudo C-N-P-injective for every closed M_s-cyclic subsystem N of M_s . In particular , if B is a retract of N , then M_s is pseudo closed B-P-injective S-system .

Proof: Let N be closed M_s-cyclic subsystem of S-system M_s . Assume that A be closed N_s-cyclic subsystem of N . Let f be S-monomorphism from A into M_s and i_1(i_2) be the inclusion map of A (N) into N (M_s) . Since M_s is pseudo C-QP-injective ,so there exists S-homomorphism $g : M_s \to M_s$ such that $g o i_2 o i_1 = f$, this means g is extension of f . Define an S-homomorphism $g_1(=g o i_2) : N \to M_s$, then $g_1 o i_1 = g o i_2 o i_1 = f$. Thus , g_1 is extension of f and M_s is pseudo C-N-P-injective system . Conversely , by taking M_s is closed M_s-cyclic subsystem of M_s

Corollary(4.6): Let M_s be S-system and N be closed M_s-cyclic subsystem of M_s . If N is pseudo C-M-P-injective , then N is a retract of M_s .

Proof : Let N be closed M_s-cyclic subsystem of S-system M_s and I be the identity map of N . Let i be the inclusion map of N into M_s . Since N is pseudo C-M-P-injective , then there exists S-homomorphism $g : M_s \to N$ such that I = goi , hence i has left inverse and i(N) is a retract of M_s , but N = i(N) , so N is a retract of M_s .

Corollary(4.7): Let M_s be S-system and N_s be pseudo C-M-P-injective system , then N is a retract of M_s if and only if N is closed M_s-cyclic subsystem of M_s .

Proof : As every retract of an S-system M_s is closed by remarks and examples (2.2)(4) and also every retract is M_s-cyclic subsystem of M_s by remarks and examples (2.3)(2) in [2]. Conversely ,by taking f is the identity map of N in the proof of proposition (4.5) .

The proof of the following proposition is similar to proof of proposition(2.14) by replacing X from closed subsystem to closed M_s-cyclic subsystem of M_s :

Proposition(4.8): Let M_s be an S-system and $\{ N_i \mid i \in I \}$ be a family of S-systems . Then $\prod_{i\in I} N_i$ is pseudo C-M-P-injective if and only if N_i is pseudo C-M-P-injective for every $i \in I$.

Corollary(4.9): Let M_s and N_i be S-systems , where $i \in I$ and I is a finite index set. Then , for every i , N_i is pseudo C-M-P-injective if and only if $\oplus_{i=1}^{n} N_i$ is pseudo C-M-P-injective .

Theorem(4.10): Let M_1 and M_2 be two S-systems . If $M_1 \dot{\cup} M_2$ is pseudo C-QP-injective, then M_1 is C-M_2-principally injective.

Proof: Let $M_1 \dot{\cup} M_2$ be pseudo C-QP-injective. Let A be closed M_2-cyclic subsystem of M_2, and f be S-homomorphism from A into M_1 . let j_1 and π_1 be the injection and projection map of M_1 into $M_1 \dot{\cup} M_2$ and $M_1 \dot{\cup} M_2$ onto M_1 respectively . Define

$\alpha : A \to M_1 \dot{\cup} M_2$ by $\alpha(a) = (f(a),a)$, $\forall a \in A$. It is clear that α is S-monomorphism. Since $M_1 \dot{\cup} M_2$ is pseudo C-QP-injective ,so by proposition(4.5) , $M_1 \dot{\cup} M_2$ is pseudo C-M_2-P-injective . Hence, there exists S-homomorphism g from M_2 into $M_1 \dot{\cup} M_2$ such that $goi = \alpha$, where i be the inclusion map of A into M_2 . Now, put $h = \pi_1 og$ from M_2 into M_1 .Thus $\forall a \in A$, we have $h(a) = \pi_1 og(a) = \pi_1 o\alpha(a) = \pi_1(\alpha(a)) = \pi_1(f(a),a) = f(a)$. This means M_1 is C-M_2-P-injective

Corollary(4.11): Let $\{M_i\}_{i\in I}$ be a family of S-systems and I be finite set. If $\dot{\cup}_{i\in I} M_i$ is pseudo C-M_K-P-injective , then M_j is C-M_K-P-injective system for all distinct $j,k \in I$.

Proposition(4.12): For any integer $n \geq 2$, M_s^n is pseudo C-QP-injective if and only if M_s is C-QP-injective .

Proof: If M_s^n is pseudo C-QP-injective, thenM_s^n is pseudo C-M-P-injective for closed subsystem M_s of M_s^n . Now , by theorem (4.10) , we have M_s is C-M-P-injective .This means M_s is C-QP-injective . Conversely, assume that M_s is C-QP-injective system , this means M_s is C-M-P-injective system. By proposition(4.8), M_s^n is C-QP-injective and hence M_s^n is pseudo C-QP-injective S-system.

Definition(4.13): An endomorphism $f \in End(M)$ is called a closed homomorphism if f(M) is a closed subsystem of M_s . The following theorems and lemma give a characterization of pseudo C-QP-injective S-systems:

Theorem(4.14): Let M_s be an S-system . Then M_s is pseudo C-QP-injective S-system if and only if $\ker(\alpha) = \ker(\beta)$, implies $T\alpha = T\beta$ for all closed homomorphisms$\alpha, \beta \in T = End(M_s)$.

Proof: ⇒) Let $\alpha, \beta \in T$ with $\ker(\alpha) = \ker(\beta)$. Define $\varphi : \alpha(M) \to M_s$ by $\varphi(\alpha(m)) = \beta(m)$ for every $m \in M_s$. Let $\alpha(m_1)$, $\alpha(m_2) \in \alpha(M)$ such that $\alpha(m_1) = \alpha(m_2)$. Then $(m_1 , m_2) \in \ker(\alpha) = \ker(\beta)$, so $\beta(m_1) = \beta(m_2)$Hence $\varphi(\alpha(m_1)) = \varphi(\alpha(m_2))$ and φ is well-defined , the reverse steps gives that φ is S-monomorphism. For every $m \in M_s$ and $s \in S$, we have

$\varphi(\alpha(ms)) = \beta(ms) = \beta(m)s = \varphi(\alpha(m))\, s$. This shows that φ is an S-homomorphism. Since M_s is pseudo C-QP- injective S-system and $\alpha(M)$ is closed M_s-cyclic subsystem of M_s , so there exists S-homomorphism $\psi:M_s \to M_s$ such that $\psi o\ i = \varphi$, where i is the inclusion map of $\alpha(m)$ into M_s . Thus , $\beta = \varphi o \alpha = \psi o i o \alpha = \psi o \alpha \in T\alpha$. Then, $T\beta \subseteq T\alpha$. Similarly , $T\alpha \subseteq T\beta$, therefore $T\alpha = T\beta$.

$\Leftarrow$) Let $\alpha \in T$ and $f : \alpha(M) \to M_s$ be S-monomorphism from closed M_s-cyclic subsystem $\alpha(M)$ of M_s into S-system M_s . Then ker f = ker i , where i is the inclusion map from $\alpha(M)$ into M_s . Since $f(\alpha(M)) \cong \alpha(M)$, and similarly $i(\alpha(M)) \cong \alpha(M)$, so this means f , i $\in$ T . Then by assumption , Tf = Ti , so we have f $\in$ Ti . Thus , f = hi , for some h $\in$ T . This shows that M_s is pseudo C-QP-injective S-system

Theorem(4.15): Let M_s be pseudo C-QP-injective S-system and $T = End(M_s)$ with closed homomorphism α , $\beta \in T$. Then:

(1) If $\alpha(M)$ embeds in $\beta(M)$, then $T\alpha$ is an image of $T\beta$.

(2) If $\alpha(M) \cong \beta(M)$, then $T\alpha \cong T\beta$.

Proof : (1) Let $f : \alpha(M) \to \beta(M)$ be S-monomorphism . Let i_1 (respectively i_2) be the inclusion maps of $\alpha(M)$ (respectively $\beta(M)$) into M_s . Since $i_2 o f$ is S-monomorphism and M_s is pseudo C-QP-injective S-system , so there exists S-homomorphism $\bar{f} : M_s \to M_s$ such that $\bar{f} \circ i_1 = i_2 \circ f$. Define $\sigma : T\beta \to T\alpha$ by $\sigma(\lambda\, \beta) = \lambda \bar{f}\alpha$, $\lambda \in T$. If $\lambda_1\beta = \lambda_2\beta$ for m $\in M_s$. $\bar{f}\alpha(m) = (\bar{f} \circ i_1)(\alpha(m)) = (i_2 \circ f)(\alpha(m)) = f(\alpha(m))$ and hence $\lambda\, \bar{f}\alpha(m) = \lambda f(\alpha(m))$, so σ is well-defined . It is clear that σ is T-homomorphism , in fact , let $\lambda\, \beta \in T\beta$ and $g \in T$, then $\sigma(g(\lambda\beta)) = \sigma((g\,\lambda)\beta) = g\lambda\bar{f}\alpha = g\,(\lambda\bar{f}\alpha) = g\sigma(\lambda\beta)$ .We claim that $\ker(\bar{f}\alpha) = \ker\alpha$. Let $(x_1,x_2) \in \ker(\bar{f}\alpha)$ which implies $\bar{f}\alpha(x_1)=\bar{f}\alpha(x_2)$. This implies $f(\alpha(x_1)) = f(\alpha(x_2))$, since f is S-monomorphism , so $\alpha(x_1) = \alpha(x_2)$. Thus , $(x_1,x_2) \in \ker\alpha$. Also , it is clear that $\ker\alpha \subseteq \ker(\bar{f}\alpha)$. Thus , $\ker(\bar{f}\alpha) = \ker\alpha$. Hence , by theorem(3.2.15), we have $T\alpha = T\bar{f}\alpha$, so there exists $\lambda \in T$ such that $\alpha = \lambda\bar{f}\alpha$, then $\alpha = \lambda\bar{f}\alpha = \sigma(\lambda\, \beta) \in \sigma(T\beta)$. This implies $T\alpha = \sigma(T\beta)$. Then σ is T-epimorphism

(2) Let $f : \alpha(M) \to \beta(M)$ is S-isomorphism . Let i_1 (respectively i_2) be the inclusion maps of $\alpha(M)$ (respectively $\beta(M)$) into M_s . Since $i_2 o f$ is S-monomorphism and M_s is pseudo C-QP-injective , so $i_2 o f$ can be extended to $\bar{f} : M_s \to M_s$ such that $\bar{f} o i_1 = i_2 o f$. Define $\sigma : T\beta \to T\alpha$ by $\sigma(\lambda\beta) = \lambda\bar{f}\alpha$, for every $\lambda \in T$. As in part(1) , σ is well-defined and T-epimorphism . Now , let $\sigma(\lambda_1\beta) = \sigma(\lambda_2\beta)$, then $\lambda_1\bar{f}\alpha = \lambda_2\bar{f}\alpha$. Since $\bar{f}\alpha(M) = \bar{f} o i_1(\alpha(M)) = i_2 o f(\alpha(M)) = f\alpha(M) = \beta(M)$, then $\lambda\bar{f}\alpha(M) = \lambda\beta(M)$, hence $\lambda_1\beta(M) = \lambda_1\bar{f}\alpha(M) = \lambda_2\bar{f}\alpha(M) = \lambda_2\beta(M)$,then $\lambda_1\beta = \lambda_2\beta$. Hence σ is T-monomorphism

Lemma(4.16) : Let M_s be a pseudo C-QP-injective system and $T = End(M_s)$. If $\alpha(M)$ is a simple S-system , where α is closed homomorphism such that $\alpha \in T$, then $T\alpha$ is a simple T-system .

Proof : Let $\Theta \neq f\alpha \in T\alpha$. Then $f : \alpha(M) \to f\alpha(M)$ is an S-isomorphism by hypothesis , so let $\sigma : f\alpha(M) \to \alpha(M)$ be the inverse . If $\bar{\sigma} \in T$ extends σ, then , for m $\in M_s$, we have $\alpha(m) = \sigma(f\alpha(m)) = \bar{\sigma}(f\alpha(m)) \in Tf\alpha$ and hence $T\alpha = Tf\alpha$.

Note that if N_s is M_s-projective ,then every S-epimorphism from S-system M_s into N_s is split. Also , retract of M_s-projective S-system is M_s-projective [11] .

The following proposition explain a relation between pseudo C-QP-injective system with quasi projective :

Proposition(4.17): Let M_s be a pseudo C-QP-injective S-system , and $\alpha \in T = End(M_s)$ is closed homomorphism . The following statements are equivalent :

(1) $\alpha(M)$ is a retract of M_s ,

(2) $\alpha(M)$ is a pseudo C-M-P-injective . In additional , if M_s is quasi projective S-system , then (1) and (2) are equivalent to :

(3) $\alpha(M)$ is M_s-projective .

Proof : ($1\rightarrow 2$) Follows from lemma(4.3) .

($2\rightarrow 1$) As $\alpha(M)$ is closed M_s-cyclic subsystem of M_s , so by corollary (4.6) , $\alpha(M)$ is a retract of M_s .

($2\rightarrow 3$) By (2) and corollary (4.6) , we have $\alpha(M)$ is a retract of M_s . Since M_s is quasi projective S-system , so $\alpha(M)$ is M_s-projective .

($3\rightarrow 2$) Assume that $\alpha(M)$ is M_s-projective . Let A be closed M_s-cyclic subsystem of M_s and σ be S-monomorphism from A into $\alpha(M)$. Since $\alpha(M)$ is closed M_s-cyclic , so there exists S-epimorphism $\beta : M_s \rightarrow \alpha(M)$. Since $\alpha(M)$ is M_s-projective , so β split . This means there exists S-homomorphism k from $\alpha(M)$ into M_s , such that $\beta ok = I_{\alpha(M)}$. Then , define $f = ko\sigma$. Since f is S-monomorphism (whence $\beta ok = I_{\alpha(M)}$) and M_s is pseudo C-QP-injective , so there exists S-homomorphism $h : M_s \rightarrow M_s$ such that $hoi = f$. Since M_s is quasi projective , so $\beta oh = g$, where g is an S-homomorphism from M_s into $\alpha(M)$. Thus , we have $goi = \beta ohoi = \beta of = \beta oko\sigma = I_{\alpha(M)}o\sigma$. This means $\alpha(M)$ is pseudo C-M-P-injective S-system

Corollary(4.18): Let M_s be a pseudo C-QP-injective S-system and quasi projective . Then the following statements holds for closed M_s-cyclic subsystem N of M_s :

(1) N is a retract of M_s .

(2) N is pseudo C-M-P-injective . In additional , if M_s is quasi projective S-system , then (1) and (2) are equivalent to :

(3) N is M_s-projective .

Now, we study the relation among pseudo C-QP-injective with other class of injectivity :

The following proposition give a condition , under which pseudo C-QPinjective to be injective :

Theorem(4.19): The following statements are equivalent for S-system M_s :

(1) M_s is injective system ,

(2) M_s is pseudo C-N-P-injective system for every S-system N .

Proof: ($1\Longrightarrow 2$) It is obvious .

(2⟹1) Assume that M_s is pseudo C-N-P-injective system and E(M) is injective envelope of M_s . Then , since E(M) is injective , so it is pseudo C-N-P-injective system . Now , by corollary(4.9) , $M_s\oplus$ E(M) is pseudo C-N-P-injective , then put $N_s = M_s\oplus E(M)$. Thus , $M_s\oplus E(M)$ is pseudo C-M⊕E-P-injective . By proposition(3.7)(1) , M_s is pseudo C-M⊕E-P-injective system . Consider the injection maps $j_1: M_s\rightarrow E(M)$, $j_2: E(M) \rightarrow M_s\oplus E(M)$, $j_3: M_s\rightarrow M_s\oplus E(M)$ and $I_M : M_s \rightarrow M_s$ is the identity map of M_s . Let $\pi_M: M_s\oplus E(M)\rightarrow M_s$ be the projection map such that $\pi_M o j_3 = I_M$. Now , $M_s\oplus$ E(M) is pseudo C-quasi principally injective , so this implies there exists S-homomorphism g : $M_s\oplus$ E(M) → $M_s\oplus$ E(M) such that $g o j_2 o j_1 = j_3 o I_M$, then $\pi_M o g o j_2 o j_1 = \pi_M o j_3 o I_M$. Thus $I_M = \pi_M o g o j_2 o j_1$, so that $f = \pi_M o g o j_2$ and then $I_M = f o j_1$. Therefore M_s is a retract of E(M) and then it is injective .

Definition(4.20): An S-system M_s is satisfy CM-property if , every closed subsystem of M_s is an M_s-cyclic subsystem of M_s .

The following lemma give a condition , under which pseudo C-QP-injective to be pseudo C-quasi injective :

Lemma(4.21): The following statements are equivalent for S-system M_s :

(1) M_s is extending system (this means CS-system) ,

(2) Every S-system is pseudo C-M-injective ,

(3) Every S-system is pseudo C-M-P-injective and M_s satisfies CM-property .

Proof : (1⟹2) It is obvious .

(2⟹3) Let N be a closed subsystem of S-system M_s . By(2) , N is pseudo C-M-injective . Thus , by proposition(2.6) N is a retract of M_s and since every retract is M_s-cyclic by remarks and examples(2.3)(2)[2] , so M_s satisfies CM-property . Another part is obvious.

(3⟹1) Let N be any closed subsystem of S-system M_s . Since M_s satisfies CM-property , so N is M_s-cyclic . By(3) , N is pseudo C-M-P-injective system . By corollary(4.6) , N is a retract of M_s . Thus M_s is extending system .

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