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Generalized* ⊕ Z* Supplemented Modules and Generalized* ⊕ Co- finitely Supplemented Modules

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Abstract: Let R be a commutative ring with identity, an R- module M is called G*⊕Z* supplemented modules, if every sub module containing Z*(M) has generalized* supplemented in M that is a direct summand of M . and an R-module M is called generalized* Co-finitely supplemented, if every Co-finitely has sub module of M has a generalized* supplemented in M and M is called ⊕Co-finitely generalized* supplemented, if every Co-finite sub module of M has G*S that is direct summand of M**.**

Keywords: Generalized*⊕Z* supplemented; ⊕ Co-finitely generalized* supplemented modules; Generalized*Co-finitely supplemented modules.

In this paper we will prove some properties of these types of modules.

§1 Introduction

Let R be an associative ring with identity and let M be a unital left R-module a submodule N of an R-module M is called small in denoted by $N \ll M$, if whenever N + L = M for some submodule L of M, then L = M .equivalently for any proper submodule L of M, N + L ≠ M[**1**], let N and L be a submodule of M .N is called supplement of L in M if M = N + L and N is minimal with respect to this property equivalently, M = N + L and $N \cap L \ll N$ **[1]**.M is called supplemented, if every submodule of M has a supplemented in M **[2]**

For any R-module M, Z*(M) = {m ∊ M, Rm ≪ E (M)} where E(M) is the injective hull of M . equivalently Z*(M) = M ∩ Rad (E(M)) = M ∩ Rad(E1) for any injective E1 ≥ M , where Rad(E(M)) and Rad(E1) denoted the Jacobson radical of E(M).E1 respectively .Z*(M) is called the co-singular submodule of M see**[3],** Notice that Rad (M) ≤ Z*(M).But the converse does not hold in general for example Z as Z-module, Z*(Z) = Z ≠ Rad (Z).

Let M be module , if N ,L ≤ M , and M = N + L , then L is called generalized supplement of N in case N ∩ L ≤ Rad(L), M is called generalized supplemented or (briefly GS)in case each submodule N has generalized supplement in M[**4**]**.** A module M is called generalized ⊕ supplemented, if every submodule has a generalized supplement that is a direct summand of M **[5]** As a generalization of ⊕ supplemented modules .In **[6]** anther were notation introduced called generalized⊕ radical supplemented module .A module M is called generalized ⊕ radical supplemented, if every submodule containing radical has a generalized supplement that is a direct summand of M.

The concept generalized* supplement modules were introduced in **[7]** .Let M be a module, if N, L ≤ M and M = N + L then L is called a generalized* supplement of L in case N ∩ L ≤ Z*(L) .A submodule K of M is called co- finite if M / K is finitely generated. In **[8]** M is called ⊕ co-finitely supplemented (briefly) ⊕ cof- supplemented, if every co-finitely submodule of M has a supplement that is a direct summand of M.

In this paper we define generalized*⊕ Z* supplemented as a generalization of ⊕ supplemented module .A module M is called G* ⊕ Z* supplemented if every submodule containing Z*(M) has generalized* supplement in M that is a direct summand of M (i.e. ∀ N ≤ M, Z*(M) ≤ N, N has generalized* supplement L in M and L is direct summand .Clearly every semi-simple module is a G*⊕ Z*S and every ⊕

As a generalization of ⊕ cof-supplemented we define ⊕ co-finitely generalized* supplement (briefly) ⊕cof-generalized* supplement, if every co-finitely submodule of M has generalized* supplement that is direct summand of M for short G*⊕CS. Clearly ⊕ supplement are G*⊕CS module and the converse is true if M is finitely generated.

§2 Generalized* ⊕ Z* supplemented

In this section we define G*⊕Z* supplemented module as a generalization of ⊕ supplemented in **[6]** and study some of properties of G*⊕Z*S supplemented module. Cleary every ⊕ supplemented module is a G*⊕Z*S module, but the converse does not hold in general, for example Z-module, Q is a G*⊕Z*S module which is not ⊕ supplemented module.

Definition (2.1) :- A module M is called G*⊕ Z* supplemented if every sub module containing Z*(M) has generalized* supplement in M that is a direct summand of M (i.e. ∀ N ≤ M, Z*(M) ≤ N, N has generalized* supplement L in M and L is direct summand (i.e. there exist L , K ≤ M such that M = L + N = L⊕ K and N ∩ L ≤ Z*(L).

Recall that a sub module N of M is called fully invariant if for every h ∈End (M), h (N) ≤ N and M is called a duo module, if every sub module of M is fully invariant.**[1]**

Lemma (2.2):-Let M be a duo module, if M= M_1⊕M_2 then N= (N∩M_1) ⊕ (N∩M_2) for N is submodule of M.

Proof:- see**[9]**

Lemma (2.3):- Let M be any R-module, and let N be a submodule of M, then Z*(N) = Z*(M) ∩ N. **[10]**

Proposition (2.4):- Let M be a G*⊕Z*S module, if N is a fully invariant submodule of M, then N is a G*⊕Z*S module.

Proof:- Let K ≤ N ≤ M with Z*(N) ≤ K, then Z*(M) ≤ K + Z(M) since M is a G*⊕Z*S module, then there exist L , L´ ≤ M such that M = (K + Z*(M)) + L = L ⊕ L´ and (K + Z*(M)) ∩ L ≤ Z*(L).Now N = (K + Z*(M)) + L) ∩ N = K + (Z*(M) ∩ N) + (L ∩ N), hence N = (K + Z*(N)) + ((L ∩ N) = K + (L ∩ N) since Z*(N) ≤ K and K ∩ (L ∩ N) = (K ∩ L) ∩ N ≤ ((K + Z*(M)) ∩ L) ∩ N ≤ Z*(L) ∩ N = (Z*(M) ∩ (L ∩ N)) = Z* (L ∩ N) since M = L ⊕ L´, hence N = (L ∩ N) ⊕ (L´ ∩ N) therefore N is a G*⊕Z*S submodule of M.

Proposition (2.5):-If M is a G*⊕Z*S module, then M / N is a G*⊕Z*S for every fully invariant submodule of M.

Proof:- Let N be a fully invariant submodule of M and let K / N ≤ M / N with Z*(M / N) ≤ K / N , since Z*(M) + K / N ≤ Z* (M / N)by [11],then Z*(M) ≤ K, then by assumption there exist L ≤M such that M = L + K with L ∩ K ≤ Z*(L) and M = L ⊕ L´ for some L´ ≤ M ,thus M / N = K / N + (L + N) / N and K / N ∩ (L + N) / N = (K ∩ L)+ N / N ≤ Z*(L) + N / N ≤

Z*(L + N)/ N. Since N is a fully invariant submodule of M then N = (N ∩ L) + (N ∩ L´) and (N + L) / N ∩ (N + L´) / N = 0 then M / N = N + L / N ⨁ N + L´ / , hence M / N is a G*⨁Z*S module.

Corollary (2.6) :- The homomorphic image of a duo G*⨁Z*S is a G*⨁Z*S module

Proof:- Clear sine every homomorphic image is isomorphic to a quotient module.

The following theorem shows that when M is a duo module, the direct sum of G*⨁Z*S is again a G*⨁Z*S.

Theorem (2.7):- If $M = M_1 \oplus M_2$, if M is a duo module and M_1, M_2 are G*⨁Z*S, then M is a G*⨁Z*S.

Proof:- Let N ≤ M with Z*(M) ≤ N, then $Z^*(M_1) \leq N \cap M_1$ ∀ i= 1,2 hence there exist V_i , V'_i of M_i (∀ i=1,2) such that $M_i = (N \cap M_i) + V_i$, $(N \cap M_i) \cap V_i \leq Z^*(V_i)$and $M_i = V_i \oplus V'_i$ since N is fully invariant submodule of M hence $N = N \cap M_1 \oplus N \cap M_2$, let $V = V_1 \oplus V_2$, $V'= V'_1 \oplus V'_2$, hence there exist V, V´ ≤ M such that $M = M_1 \oplus M_2 = (N \cap M_1) \oplus (N \cap M_2) + (V_1 \oplus V_2)= N + V$ and $N \cap V = (N \cap M_1) + (N \cap M_2) \cap (V_1 \oplus V_2) \leq Z^*(V_1) \oplus Z^*(V_2) = Z^*(V)$ by **[11]** and $V \oplus V' =(V_1 \oplus V_2)\oplus(V'_1 \oplus V'_2) = M_1 \oplus M_2= M$, hence M is G*⨁Z*S

Corollary (2.8) :- Let $\{ M_i\}_{i= 1}$ be any infinite collection of R-modules and $M= \oplus_{i\in I} M_i$ is duo module, then M is G*⨁Z*S if M_i are G*⨁Z*S for each i∈I.

Lemma (2.9):- For any R-module M ≠ 0, Z*(M) = 0 if and only if Rad(E (M)) = 0.

Proof:- see [11]

Proposition (2.10) :- Let M be a non –zero R-module with Rad(E(M)) = 0 then M is G*⨁Z*S if and only if M is semi- simple.

Proof:- ⇒) Clear since Rad (E(M)) = 0 then by Lemma (2.9) Z*(M) = 0, hence ∀ 0 ≤ N ≤ M, N has generalized* supplement in M i.e. there exist K ≤ M such that M = N + K and N ∩ K = Z*(K) = 0 ⇒ N is a direct summand of M

⇐) Clearly since every semi-simple is a G*⨁Z*S.

§3 ⨁Co-finitely generalized* supplemented modules

In this section we introduce a ⨁co-finitely generalized* supplemented module as a generalization ⨁ co-finitely generalized module.**[8]**

Definition (3.1):- An R-module M is called generalized*co-finitely supplemented if every co-finite has submodule of M has a generalized* supplement in M for short we will refer to these module by G*CS. M is called ⨁ co- finitely generalized* supplemented or (briefly) ⨁-cof G*S, if every co- finite submodule of M has G*S that is direct summand of M for short (G*⨁CS)(i.e. $\forall$ $N \leq M$ with M / N is finitely generated, there exist $L \leq M$ such that L is a G*S of N in M ($M = N + L$), $N \cap L \leq Z^*(L)$ and there exist $K \leq M$ such that $M = L \oplus K$.

It easy to see that ⨁ supplement modules are G*⨁CS module and the converse is true if M is finitely generated, Notice that hollow modules are G*⨁CS modules.

The following proposition shows that under certain condition the quotient of G*⨁CS is a G*⨁CS

Proposition (3.2):- Assume that M is a G*⨁CS due module then M / N is a G*⨁CS module.

Proof:- Let $N \leq K \leq M$ with K / N a co- finite submodule of M / N, then $M / K \simeq (M / N) / (K / N)$ is finitely generated since M is a G*⨁CS module, there exist a submodule L and L´ of M such that $M = K + L = L \oplus L'$ and $K \cap L \leq Z^*(L)$.

Notice that $M / N = K/ N + (L + N) / N$ by modularity $K \cap (L + N) = (K \cap L) + N$ since $K \cap L \leq Z^*(L)$, we have $K / N\cap (L + N) / N = [(K \cap L) + N] / N \leq Z^* (L + N) / N$ this implies that (L + N) / N is G*S of K / N in M / N .Now $N = (N \cap L) \oplus (N \cap L')$ by lemma(2.2) implies that $(L + N) \cap (L' + N) = N + (L + N + L +N \cap L') \cap L'$ it following that $(L + N)\cap (L' + N) \leq N$ and $M / N = (L + N) / N \oplus (L' +N) / N$ then (L + N) / N is direct summand of M / N. Consequently M / N is a G*⨁CS module.

Proposition (3.3):-):- For any ring R, if $M = M_1 \oplus M_2$ with M_1 and M_2 are G*⨁CS module if M is a duo module, then M is a G*⨁CS.

Proof: - Let L be a co-finite submodule of M i.e. M / L is finitely generated. Now $M = M_1 + M_2 + L$ then $M_1 + M_2 + L$ has a G*S 0 in M i.e. $M = M_1 + M_2 + L + 0$ and $M_1 + M_2 + L \cap \{0\} \leq Z^*(0) = 0$ and so $M / L + M_1$ is finitely generated. Notice that $M / L + M_1 = (M_2+ (M_1 + L) / M_1 + L \simeq M_2 / M_2 \cap (M_1+ L)$ hence $M_2 / M_2 \cap (M_1+ L)$ is finitely generated , but M_2 is a G*⨁ CS module, then there exist $H \leq M_2$ such that $M_2 = M_2 \cap (L + M_1) + H$ and $M_2 = H \oplus H'$ for some $H' \leq M_2$ also $M = M_1 + M_2 = M_1 + M_2 \cap (L + M_1) + H$,hence $M = (M_1+ L) + H$ and $(M_1 + L) \cap H \leq Z^*(H)$, thus H is a G*S of $M_1 + L$ in M .Note that $M / L + H = M_1 + (L + H) / (L + H) \simeq M_1 / M_1 \cap (L + H)$, then $M_1 \cap (L + H)$ is co-finite submodule of M, since M_1 is a G*⨁CS module then there exist $K \leq M_1$ such that $M_1 = M_1 \cap (L + H) + K$ with $M_1 \cap (L + H) \cap K = (L + H) \cap K \leq Z^*(K)$ and there exist $K' \leq M_1$ such that $M_1= K \oplus K'$, hence L is a G*S of H + K in M i.e. $M = M_1 + M_2 = (L + H + K) +(M_2 \cap (L + M_1)$

$+ H = L + H + K$ and $L \cap (H + K) \leq Z^*(K) \leq Z^*(K + H)$ and $H + K = H \oplus K$ since H is a direct summand of M_1,hence $H \oplus K$ is a direct summand of M.

Corollary (3.4):- Any finite direct sum of G*⊕CS module is a G*⊕CS module.

Proof:- follows from proposition (3.3)

Before we give next result we need the following definitions:-

Definition (3.5): A module M is said to be have the summand intersection property (SIP) if the intersection of any pair of direct summands of M is a direct summand of M (i.e. if N and K are direct summand of M then $N \cap K$ is also a direct summand of M).

A module M is said to have the summand sum property (SSP) if the sum of any pair of direct summand of M is a summand of M(i.e. if N and K are direct summand of M then N + K is also a direct summand of M.

Recall that a module M distributive if for submodule K, L , N of M $N+(K \cap L) = (N + K) \cap (N + L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$.

Hence we have the following:-

Theorem (3.6):- 1- Let M be a G*⊕CS-module and N a submodule of M, if for every direct summand K of M, (N+K)/ N is direct summand of N / M then M / N is a G*⊕CS module.

2- Let M be a G*⊕CS-distributive module then M / N is a G*⊕CS module for every submodule N of M.

3- Let M be a G*⊕CS module with SSP then every direct summand of M is a G*⊕CS module.

Proof: 1- Any co -finite submodule of M / N has the form L / N where L there exist a direct summand K of M such that $M = L + K = K \oplus K'$ and $L \cap K \leq Z^*(K)$ for some submodule $K' \leq M$ .Now $M / N = L / N + (K + N) / N$, by hypothesis $(K + N) / N$ is direct summand of M / N, Note that $(L / N) \cap (K + N) / N = [(L \cap (K + N)] / N= [N + (K \cap L)]/ N$ since $L \cap K \leq Z^*(K)$.we have $[(K \cap L + N] / N \leq Z^*(K + N) / N)$.This implies that (K+N)/ N is G*S submodule of L / N in M / N .hence M / N is a G*⊕CS module.

Proof: 2- Since M is a G*⊕CS then any co-finite submodule of M has a G*S that is a direct summand of M. Let L be a direct summand of M i.e. $M = L \oplus L'$ for some submodule L′ of M .Now $M/ N = [(L + N) / N] + [(L' + N) / N]$ and $N = N+ (L \cap L') = (N + L) \cap (N +$

L´) since M is distributive , thus M / N [(L + N) / N] ⊕ [(L´ + N) / N] by(1) hence M / N is G*⊕CS module.

Proof:3- Let N be a direct summand of M i.e. M = N ⊕N´ for some N´ ≤ M, to prove that M / N´ is a G*⊕CS module .Let L be a direct summand of M, since M has the SSP , then L + N´ is a direct summand of M. i.e. M = (L + N´) ⊕ K for some K≤M, then M / N´ = L + N´ / N´ ⊕ K + N´ / N´, hence by (1) M / N´ is a G*⊕CS module.

Weimin Xue in **[12]** introduce the notation of generalized projective covers to characterize semi perfect modules and rings.

An epimorphism f: P⟶ M is called a generalized cover in case kerf ≤ Rad (P), when P is a projective module then f is called a generalized projective cover.

As a generalization of this concept we introduce the following definition:

Definition (3.7):- If P and M are modules, we call an epimorphism f: P⟶ M a (generalized*) cover in case (kerf ≤ Z*(P)), If P is a projective module, then f is called (generalized*) projective cover .Clearly every projective cover is generalized* projective cover.

We have the following basis properties of generalized* cover.

Lemma (3.8):1- If f: P⟶ M and g: M⟶ N are generalized* cover for M and N, with f(Z*(P))=Z*(M) ,then g○f: P ⟶ N is a generalized*cover for N.

Proof: - If both f and g are covers, then g○f is cover by **[2]**, Now let both f and g be generalized* cover .It is enough to prove that ker (g○f) ≤ Z*(P). Let x ϵ ker (g○f), then g○f(x) = 0, hence f(x) ∊ kerg ≤ Z*(M) , since kerf ≤ Z*(P) ,then there exist x´ ∊ Z*(P) such that f(x) = f(x´), for some x´ ϵ Z*(P), hence x-x´ ∊ kerf ≤ Z*(P),therefore x ϵ Z*(P).

2- If each f_i : P_i ⟶ M_i, i = 1,…,n, is a generalized* cover, then $\oplus_{i=1} f_i$: $\oplus_{i=1} P_i$⟶ $\oplus_{i=1} M_i$ is a generalized* cover.

Proof:- Since kerf_i ≤ Z*(P_i), ∀ i= 1,2,…n we have ker($\oplus_{i=1} f_i$) = $\oplus_{i=1}$ ker f_i, thus ker ($\oplus_{i=1} f_i$) ≤ $\oplus_{i=1}$ Z*(P_i), i.e.$\oplus_{i=1} f_i$ is a generalized* cover.

Lemma (3.9):- Let N be a submodule of the module M and f: M⟶ M / N be canonical epimorphism also, let P any module, g: P⟶ M /N and h: P ⟶ M with h(Z*(P)) = Z*(M) such that g is h composed with f .Then the map g is a generalized* cover epimorphism if and only if Im (h) is a generalized*supplemented of N and kerh ≤ Z*(P).

Proof:- ⇒) Let x ∊ N ∩ Imh, then x ∊ N = kerf and x∊ Imh i.e. there exist y ∊ P such that x = h(y).Now g(y)= f(h(y)) = f (x) = 0(since x∊ kerf = N), thus y ∊ kerg and h(y) ∊ h(kerg).Now let x ∊ h(kerg), then x= h(y), y ∊ kerg(i.e. g(y) = 0, hence fh(y) = g(y) = 0, f(x)= g (y)= 0,thus N ∩ Imh =h (kerg) ≤ Z* (Imh) = Z*(h(P)) then Imh = h (P) is a generalized* supplement of N, since g is an epimorphism then kerh ≤ kerg thus kerh ≤ Z*(P).

⇐) the converse is clearly by lemma (3.8 (1))

Recall that an R-module M is called semi perfect module if every factor module has a projective cover. As a generalization of semi perfect modules, we will introduce the following **[1].**

An R-module M is called a generalized* co-finitely semi perfect, if every finitely generated factor has a generalized* projective cover. Clearly every semi perfect module is a generalized* co-finitely semi perfect.

Theorem (3.10):- Let M be a module in which every generalized* projective cover f satisfies f (Z*(P)) = Z*(M), the following are equivalent:

1. M is a generalized* co-finitely semi perfect module.

2. M is a generalized* co-finitely module by supplements which have generalized* projective cover.

Proof:- 1 ⇒ 2) Let M = N + L with M / N is finitely generated projective cover for M / N, P is a projective R-module .Now M / N = N + L / N ≃ L / L + N since P is projective ,then the map f lefts g : P ⟶ L, and since f is a generalized* cover ,then by Lemma(3.9),we get Img is a generalized* cover of (L ∩ N) i.e. Img + (L ∩ N) = L and Img ∩ (L ∩ N) ≤ Z* (Img), kerg ≤ ker (π○i○g) = kerf ≤ Z*(P).

2⇒1) Let N be a co-finite submodule of M, then M / N is finitely generated by (2) there exist L ≤ M such that M = L + N and L ∩ N ≤ Z*(L). Let f: P ⟶ L be a generalized* projective cover of L the natural epimorphism g: L⟶ L / L ∩ N ≃ N + K / N = M / N is a generalized* cover (for kerg = L ∩ N ≤ Z* (L)), hence h= g○f: P ⟶ M / N is generalized* projective cover for M / N by Lemma (3.8).

Corollary (3.11):-Let M be a projective G*⊕CS module, then M is a generalized* co-finitely semi perfect module.
Proof:- Let N be a co-finite submodule of M, i.e. M / N is finitely generated since M is a G*⊕CS module, then there exist K , K´ ≤ M such that M = N + K = K ⊕ K´ and N ∩ K ≤ Z* (K), K is projective, let i: K ⟶M be the inclusion homomorphism and let π :M ⟶ M / N be the natural epimorphism ,hence π○i:K ⟶ M / N is an epimorphism, ker(π○i) = N ∩ K ≤ Z*(K) thus M is a generalized* co-finitely semi perfect module.

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