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When M-small modules are simple

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Abstract.
In this paper, we define and study *SMSI*-modules. A module *M* is called an *SMSI*-module if every *M*-small module is simple in σ[M].

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1 Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M a unitary right R-module and $S = End(M)$ the ring of all R-endomorphisms of M. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules. We will use the notation $N \leq_e M$ to indicate that N is essential in M (i.e., $N \cap L \neq 0\ \forall 0 \neq L \leq M$); $N \ll M$ means that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M.

In this note, we define and study $SMSI$-modules. A module M is called an *$SMSI$-module* if every M-small module is simple in $\sigma[M]$. For example, every simple module is an $SMSI$-module (Example 2.3). In Section 2, we show that if M is an $SMSI$-module and $N \in \sigma[M]$, then every nonzero submodule of N is coatomic. It is shown that if M is an $SMSI$-module, then $Rad(M)$ is artinian and noetherian.

We denote the radical of M by $Rad(M)$. The second radical of M is defined to be the submodule $Rad^2(M)$ of M given by $Rad^2(M) = Rad(Rad(M))$.

Letting $Rad^1(M) = Rad(M)$ and proceeding in this fashion, we manufacture the radical series or (upper) Loewy series of M as the descending chain of submodules

$$M \geq Rad^1(M) \geq Rad^2(M) \geq ... \geq Rad^\alpha(M) \geq Rad^{\alpha+1}(M) \geq ...;$$

where, for each ordinal $\alpha > 0$,

$$Rad^{\alpha+1}(M) = Rad(Rad^\alpha(M));$$

and if α is a limit ordinal then

$$Rad^\alpha(M) = \bigcap_{0<\beta<\alpha} Rad^\beta(M).$$

Since M is a set, at some stage the radical series of M must become stationary, i.e., there is an ordinal ρ such that $Rad^\alpha(M) = Rad^\rho(M)$ for all ordinals $\alpha \geq \rho$.

2 When M-small modules are simple

Definition 2.1 A module M is called an *$SMSI$-module* if every M-small module is simple in $\sigma[M]$.

Example 2.2 Let p be a prime integer and M denote the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$ with $k \geq 3$. Let $N = p\mathbb{Z}/p^k\mathbb{Z}$. Since $\mathbb{Z}/p^k\mathbb{Z}$ is hollow, N is M-small. But N is not simple, so M is not an $SMSI$-module.

Example 2.3 Let M be a simple module. It is clear that every module in $\sigma[M]$ is semisimple. Now, if L is a M-small module, then there is a module $H \in \sigma[M]$ such that $L \ll H$. Since H is semisimple, L is a direct summand of H. Hence $L = 0$. Thus M is an $SMSI$-module.

Proposition 2.4 *Let M be a module. Then M is an $SMSI$-module if and only if every module in $\sigma[M]$ is an $SMSI$-module.*

Proof. ($\Rightarrow$) Let M be an $SMSI$-module and $N \in \sigma[M]$. Assume that $A \in \sigma[N]$ is N-small. Note that $A \in \sigma[M]$ and A is M-small. Since M is an $SMSI$-module, A is simple in $\sigma[M]$ and hence simple in $\sigma[N]$.

($\Leftarrow$) Clear. $\square$

Proposition 2.5 *Let M be an $SMSI$-module. Then:*

(1) *$Rad(N) \subseteq Soc(N)$ for every module $N \in \sigma[M]$.*

(2) *Every module $N \in \sigma[M]$ has a maximal submodule.*

Proof. (1) Clear.

(2) Let $N \in \sigma[M]$. By (1), $Rad(N) \subseteq Soc(N)$. If $Soc(N) = N$, then N has a maximal submodule. Assume that $Soc(N) \neq N$, then $Rad(N) \neq N$. This implies that N has a maximal submodule, again. □

A module M is called *coatomic* if every proper submodule is contained in a maximal submodule.

Theorem 2.6 *Let M be an $SMSI$-module and $N \in \sigma[M]$. Then every nonzero submodule of N is coatomic.*

Proof. Let L be a proper submodule of N. By Proposition 2.5, N/L has a maximal submodule T/L. So T is a maximal submodule of N which contains L. Hence N is coatomic, and the theorem is proved since every submodule of N belongs to $\sigma[M]$. □

The following example shows that a module for which every submodule is coatomic needs not be an $SMSI$-module.

Example 2.7 In Example 2.2, we show that the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$ with $k \geq 3$ is not an $SMSI$-module. It is clear that every submodule of M is coatomic.

Corollary 2.8 *Let M be an $SMSI$-module. Then for every module $N \in \sigma[M]$, $Rad(N) \ll N$.*

Corollary 2.9 *Let M be an $SMSI$-module and $N \in \sigma[M]$. Then $Rad^{\alpha+1}(N) = 0$ for all $\alpha \geq 1$.*

Proof. By Corollary 2.8, $Rad^{\alpha}(N) \ll N$ for all $\alpha \geq 1$. By hypothesis, $Rad^{\alpha}(N)$ is simple. Thus the zero submodule of $Rad^{\alpha}(N)$ is maximal. Hence $Rad^{\alpha+1}(N) = 0$ for all $\alpha \geq 1$. □

Proposition 2.10 *If M is an $SMSI$-module, then $Rad(M)$ is artinian and noetherian.*

Proof. By Corollary 2.8, every submodule K of $Rad(M)$ is small in M. By hypothesis, K is simple. Thus $Rad(M)$ is artinian and noetherian. □

Theorem 2.11 *Let $f: M \to N$ be an epimorphism and $M/Rad^2(M)$ is semisimple. Then $f(Rad^2(M)) = Rad^2(N)$ and $Rad^2(N) = Rad(N)$.*

Proof. It is clear that $f(Rad^2(M)) \subseteq Rad^2(N) \subseteq Rad(N)$. Consider the natural epimorphism $\overline{f}: M/Rad^2(M) \to N/f(Rad^2(M))$. Since $M/Rad^2(M)$ is semisimple, $N/f(Rad^2(M))$ is semisimple. Thus $Rad(N/f(Rad^2(M))) = 0$. But $Rad(N)$ is the smallest submodule K of N such that $Rad(N/K) = 0$, hence $f(Rad^2(M)) = Rad^2(N) = Rad(N)$. □

Corollary 2.12 *Let $f: M \to N$ be an epimorphism and let $\alpha \geq 1$ be any ordinal. If $M/Rad^\alpha(M)$ is semisimple, then $f(Rad^\alpha(M)) = Rad^\alpha(N)$ and $Rad^\beta(N) = Rad(N)$ for all $\beta \leq \alpha$.*

Proposition 2.13 *Assume that $Rad^\alpha(M)$ is essential submodule of M for some ordinal $\alpha \geq 1$. Then:*

(1) *Let $K \subseteq L \subseteq M$ be direct summands of M. Then $Rad^\alpha(K) = Rad^\alpha(L)$ if and only if $K = L$.*

(2) *If $Rad^\alpha(M)$ has ACC(DCC) on direct summands, then M has ACC(DCC) on direct summands.*

Proof. Let $M = K \oplus K'$. Then $L = K \oplus (L \cap K')$ and $Rad^\alpha(L) = Rad^\alpha(K) \oplus Rad^\alpha(L \cap K')$. If $Rad^\alpha(K) = Rad^\alpha(L)$, then $0 = Rad^\alpha(L \cap K') = Rad^\alpha(M) \cap (L \cap K')$. Since $Rad^\alpha(M)$ is essential in M, $L \cap K' = 0$ and so $K = L$.

(2) This is a consequence of (1). □

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