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On The Diophantine Equation $\sum x_i^2 + a = y^2$ And $\sum x_i^3 + a = y^3$

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Abstract:

In this paper, the Diophantine equations $\sum x_i^2 + a = y^2$ and $\sum x_i^3 + a = y^3$ where $x_1 \neq x_2 \neq x_3 \neq \cdots$ and a is a positive integer have been discussed for possible positive integral solutions.

Keywords: Diophantine equation and integral solution.

Subject Classification: 11D45.

1 Introduction:

Several researchers have discussed the non-linear Diophantine equations. Most famous non-linear Diophantine equations are **Fermat's Last Problem** (1637) and **Beal's Conjecture** (1993). Fermat's Last Problem was solved by **Andrew Wiles** (1995). Beal's Conjecture is still an open unsolved problem. **Gandhi & Sarma** (2013) attempted to disprove the Beal's conjecture. **Gregorio, L.T.D.** (2013) presented a proof for the the Beal's conjecture and a new proof for the Fermat's last theorem.

Gola, L.W. (2014) presented a proof of Beal's conjecture. **Ghanouchi, J.** (2014) presented an elementary proof of Fermat-Wiles theorem and generalization to Beal's conjecture. **Thiagrajan, R.C.** (2014) provided computational results and a proof of Beal's conjecture.

In this paper, we have discussed the Diophantine equations $\sum x_i^2 + a = y^2$ and $\sum x_i^3 + a = y^3$ where $x_1 \neq x_2 \neq x_3 \neq \cdots$ and a is a positive integer. The physical interpretation of first Diophantine equation is that some squares of different dimensions and a strip of dimension $a \times 1$ is equal to a big square while the physical interpretation of second Diophantine equation is that some cubes of different dimensions and solid strip of dimensions $a \times 1 \times 1$ is equal to a big cube.

2 Analysis: (A) Diophantine equation $\sum x_i^2 + a = y^2$: This Diophantine equation will be discussed for $a = 10, 20, 30, 40, 50$ and 60.

(a) For $a = 10$ the given Diophantine equation becomes

$$\sum x_i^2 + 10 = y^2. \quad \ldots(1)$$

(i) If we take $x_1 = 3$, $x_2 = 9$ and $y = 10$ then equation (1) is satisfied. Thus $(x_1, x_2, y) = (3,9,10)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 10 = y^2$.

(ii) If we take $x_1 = 1$, $x_2 = 2, x_3 = 3, x_4 = 5$ and $y = 7$ then equation (1) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (1,2,3,5,7)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 10 = y^2$.

(iii) If we take $x_1 = 2$, $x_2 = 3, x_3 = 4, x_4 = 5$ and $y = 8$ then equation (1) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (2,3,4,5,8)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 10 = y^2$.

(b) For $a = 20$ the given Diophantine equation becomes

$$\sum x_i^2 + 20 = y^2. \quad \ldots(2)$$

(i) If we take $x_1 = 7$, $x_2 = 10$ and $y = 13$ then equation (2) is satisfied. Thus $(x_1, x_2, y) = (7,10,13)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(ii) If we take $x_1 = 4$, $x_2 = 8$ and $y = 10$ then equation (2) is satisfied. Thus $(x_1, x_2, y) = (4,8,10)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(iii) If we take $x_1 = 6$, $x_2 = 7, x_3 = 8$ and $y = 13$ then equation (2) is satisfied. Thus $(x_1, x_2, x_3, y) = (6,7,8,13)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(iv) If we take $x_1 = 6$, $x_2 = 13$ and $y = 15$ then equation (2) is satisfied. Thus $(x_1, x_2, y) = (6,13,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 20 = y^2$.

(c) For $a = 30$ the given Diophantine equation becomes

$$\sum x_i^2 + 30 = y^2. \quad \ldots(3)$$

(i) If we take $x_1 = 3$, $x_2 = 5, x_3 = 6$ and $y = 10$ then equation (3) is satisfied. Thus $(x_1, x_2, x_3, y) = (3,5,6,10)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 30 = y^2$.

(ii) If we take $x_1 = 3$, $x_2 = 7, x_3 = 9$ and $y = 13$ then equation (3) is satisfied. Thus $(x_1, x_2, x_3, y) = (3,7,9,13)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 30 = y^2$.

(iii) If we take $x_1 = 6$, $x_2 = 7, x_3 = 9$ and $y = 14$ then equation (3) is satisfied. Thus $(x_1, x_2, x_3, y) = (6,7,9,14)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 30 = y^2$.

(iv) If we take $x_1 = 5, x_2 = 7, x_3 = 11$ and $y = 15$ then equation (3) is satisfied. Thus $(x_1, x_2, x_3, y) = (5,7,11,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 30 = y^2$.

(d) For $a = 40$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^2} + \mathbf{40} = \boldsymbol{y^2}. \qquad \ldots(4)$$

(i) If we take $x_1 = 3$and $y = 7$ then equation (4) is satisfied. Thus $(x_1, y) = (3,7)$ is the solution of the Diophantine equation $x_1^2 + 40 = y^2$.

(ii) If we take $x_1 = 9$and $y = 11$ then equation (8.4) is satisfied. Thus $(x_1, y) = (9,11)$ is the solution of the Diophantine equation $x_1^2 + 40 = y^2$.

(iii) If we take $x_1 = 4$, $x_2 = 5$ and $y = 9$ then equation (4) is satisfied. Thus $(x_1, x_2, y) = (4,5,9)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 40 = y^2$.

(iv) If we take $x_1 = 2$, $x_2 = 10$ and $y = 12$ then equation (4) is satisfied. Thus $(x_1, x_2, y) = (2,10,12)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 40 = y^2$.

(v) If we take $x_1 = 2, x_2 = 9, x_3 = 10$ and $y = 15$ then equation (4) is satisfied. Thus $(x_1, x_2, x_3, y) = (2,9,10,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 40 = y^2$.

(vi) If we take $x_1 = 4, x_2 = 5, x_3 = 12$ and $y = 15$ then equation (4) is satisfied. Thus $(x_1, x_2, x_3, y) = (4,5,12,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 40 = y^2$.

(vii) If we take $x_1 = 6, x_2 = 7, x_3 = 10$ and $y = 15$ then equation (4) is satisfied. Thus $(x_1, x_2, x_3, y) = (6,7,10,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 40 = y^2$.

(viii) If we take $x_1 = 4$, $x_2 = 13$ and $y = 15$ then equation (4) is satisfied. Thus $(x_1, x_2, y) = (4,13,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 40 = y^2$.

(ix) If we take $x_1 = 2, x_2 = 4, x_3 = 14$ and $y = 16$ then equation (4) is satisfied. Thus $(x_1, x_2, x_3, y) = (2,4,14,16)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 40 = y^2$.

(x) If we take $x_1 = 4$, $x_2 = 6, x_3 = 8, x_4 = 10$ and $y = 16$ then equation (8.4) is satisfied. Thus $(x_1, x_2, x_3, y) = (4,6,8,10,16)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 40 = y^2$.

(e) For $a = 50$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^2} + \mathbf{50} = \boldsymbol{y^2}. \qquad \ldots(5)$$

(i) If we take $x_1 = 5$, $x_2 = 11$ and $y = 14$ then equation (5) is satisfied. Thus $(x_1, x_2, y) = (5,11,14)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + 50 = y^2$.

(f) For $a = 60$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^2} + \mathbf{60} = \boldsymbol{y^2}. \qquad \ldots(6)$$

(i) If we take $x_1 = 2$, $x_2 = 5, x_3 = 6$ and $y = 15$ then equation (8.6) is satisfied. Thus $(x_1, x_2, x_3, y) = (2,5,6,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + 60 = y^2$.

(ii) If we take $x_1 = 4$, $x_2 = 6, x_3 = 7, x_4 = 8$ and $y = 15$ then equation (6) is satisfied. Thus $(x_1, x_2, x_3, y) = (4,6,7,8,15)$ is the solution of the Diophantine equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 60 = y^2$.

(B) Diophantine equation $\sum x_i^3 + a = y^3$:

(a) For $a = 0$ the given Diophantine equation becomes

$$\sum x_i^3 = y^3. \quad \ldots(7)$$

If we take $x_1 = 3$, $x_2 = 4, x_3 = 5$ and $y = 6$ then equation (7) is satisfied. Thus $(x_1, x_2, x_3, y) = (3,4,5,6)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + x_3^3 = y^3.$$

(b) For $a = 1$ the given Diophantine equation becomes

$$\sum x_i^3 + 1 = y^3. \quad \ldots(8)$$

If we take $x_1 = 6$, $x_2 = 8$ and $y = 9$ then equation (8) is satisfied. Thus $(x_1, x_2, y) = (6,8,9)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + 1 = y^3.$$

(c) For $a = 2$ the given Diophantine equation becomes

$$\sum x_i^3 + 2 = y^3. \quad \ldots(9)$$

If we take $x_1 = 5$, $x_2 = 6$ and $y = 7$ then equation (9) is satisfied. Thus $(x_1, x_2, y) = (5,6,7)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + 2 = y^3.$$

(d) For $a = 16$ the given Diophantine equation becomes

$$\sum x_i^3 + 16 = y^3. \quad \ldots(10)$$

If we take $x_1 = 3$, $x_2 = 6$ $x_3 = 7$, $x_4 = 9$ and $y = 11$ then equation (10) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (3,6,7,9,11)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3.$$

(e) For $a = 17$ the given Diophantine equation becomes

$$\sum x_i^3 + 17 = y^3. \quad \ldots(11)$$

If we take $x_1 = 3$, $x_2 = 5, x_3 = 7$ and $y = 8$ then equation (11) is satisfied. Thus $(x_1, x_2, x_3, y) = (3,5,7,8)$ is the solution of the Diophantine equation

$x_1^3 + x_2^3 + x_3^3 + 17 = y^3.$

(f) For $a = 18$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{18} = \boldsymbol{y^3}. \qquad \ldots(12)$$

If we take $x_1 = 2$, $x_2 = 4$, $x_3 = 8$, $x_4 = 9$ and $y = 11$ then equation (12) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (2,4,8,9,11)$ is the solution of the Diophantine equation

$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3.$

(g) For $a = 20$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{20} = \boldsymbol{y^3}. \qquad \ldots(13)$$

If we take $x_1 = 5$, $x_2 = 7, x_3 = 8$ and $y = 10$ then equation (8.12) is satisfied. Thus $(x_1, x_2, x_3, y) = (5,7,8,10)$ is the solution of the Diophantine equation

$x_1^3 + x_2^3 + x_3^3 + 17 = y^3.$

(h) For $a = 20$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{20} = \boldsymbol{y^3}. \qquad \ldots(14)$$

If we take $x_1 = 2$, $x_2 = 3$, $x_3 = 6$, $x_4 = 9$ and $y = 10$ then equation (14) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (2,3,6,9,10)$ is the solution of the Diophantine equation

$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3.$

(i) For $a = 30$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{30} = \boldsymbol{y^3}. \qquad \ldots(15)$$

If we take $x_1 = 5$, $x_2 = 6, x_3 = 7$ and $y = 9$ then equation (15) is satisfied. Thus $(x_1, x_2, x_3, y) = (5,6,7,9)$ is the solution of the Diophantine equation

$x_1^3 + x_2^3 + x_3^3 + 17 = y^3.$

(j) For $a = 50$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{50} = \boldsymbol{y^3}. \qquad \ldots(16)$$

If we take $x_1 = 1$, $x_2 = 4$, $x_3 = 6$, $x_4 = 10$ and $y = 11$ then equation (16) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (1,4,6,10,11)$ is the solution of the Diophantine equation

$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3.$

(k) For $a = 55$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{55} = \boldsymbol{y^3}. \qquad \ldots(17)$$

If we take $x_1 = 3$, $x_2 = 4$, $x_3 = 5$, $x_4 = 9$ and $y = 10$ then equation (17) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (3,4,5,9,10)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 16 = y^3.$$

(l) For $a = 55$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{55} = \boldsymbol{y^3}. \qquad \ldots(18)$$

If we take $x_1 = 1$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$, $x_5 = 6$, $x_6 = 8$ and $y = 10$ then equation (18) is satisfied. Thus $(x_1, x_2, x_3, x_4, x_5, x_6, y) = (1,3,4,5,6,8,10)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + 55 = y^3.$$

(m) For $a = 19$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{19} = \boldsymbol{y^3}. \qquad \ldots(19)$$

If we take $x_1 = 2$, $x_2 = 3$, $x_3 = 7$, $x_4 = 11$ and $y = 12$ then equation (19) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (2,3,7,11,12)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 19 = y^3.$$

(n) For $a = 19$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{19} = \boldsymbol{y^3}. \qquad \ldots(20)$$

If we take $x_1 = 5$, $x_2 = 7$, $x_3 = 8$, $x_4 = 9$ and $y = 12$ then equation (20) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (5,7,8,9,12)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 19 = y^3.$$

(o) For $a = 21$ the given Diophantine equation becomes

$$\sum \boldsymbol{x_i^3} + \mathbf{21} = \boldsymbol{y^3}. \qquad \ldots(21)$$

If we take $x_1 = 2$, $x_2 = 3$, $x_3 = 6$, $x_4 = 10$ and $y = 12$ then equation (22) is satisfied. Thus $(x_1, x_2, x_3, x_4, y) = (2,3,6,10,12)$ is the solution of the Diophantine equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + 21 = y^3.$$

8.3 Concluding Remarks: Here the Diophantine equation $\sum \boldsymbol{x_i^2} + \boldsymbol{a} = \boldsymbol{y^2}$ has been discussed for $a = 10, 20, 30, 40, 50$ and 60. Also the Diophantine equation $\sum \boldsymbol{x_i^3} + \boldsymbol{a} = \boldsymbol{y^3}$ has been discussed for $a = 0, 1, 2, 16, 17, 18, 19, 20, 21, 30, 50$ and 55. Possible solutions of these Diophantine equations have been obtained. These Diophantine equations can be discussed for other values of a.

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