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A note on the π-complemented algebras.

J. C. Cabello
Departamento de Análisis Matemático. Facultad de Ciencias.
e-mail: jcabello@ugr.es
Universidad de Granada. 18071 Granada. Spain.

Abstract.

π-complemented algebras are defined as those (not necessarily associative or unital) algebras such that each annihilator ideal is complemented by other annihilator ideal. Let $\mathcal{B}_A$ denote the set of all idempotents of the extended centroid of a semiprime algebra A. We prove that if there is a maximal ideal $\mathcal{P}$ of $\mathcal{B}_A$ such that $\mathcal{P}A \subseteq A$ then A and $\mathcal{P}A$ are two π-complemented algebras. As a consequence, we give a characterization of the π-complementation of the unitisation, and the multiplication ideal, of a semiprime algebra.

Key Words: Semiprime algebra; complemented algebra; extended centroid and central closure.

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1 Introduction

In this paper, we will deal with nonzero algebras over a field $\mathbb{K}$ of zero characteristic, which are not necessarily associative or unital. Throughout the paper we assume that A is a semiprime algebra. As usual, for each ideal I of an algebra A, the *annihilator* of I in A, denoted here by $\mathrm{A}nn_A(I)$ (or simply $\mathrm{A}nn(I)$ when no confusion can arise), is defined as the largest ideal J of A such that $IJ = JI = 0$. The π*-closure* of I is defined by

$$\overline{I} := \mathrm{A}nn(\mathrm{A}nn(I)).$$

Note that I is an essential ideal if, and only if, $\overline{I} = A$. The ideal I is said to be π*-closed* whenever $\overline{I} = I$. It is easy to see that this closure coincides with the one given in [11] in the associative setting. We will put $\mathcal{I}_A^\pi$ to denote the set of all π-closed ideals of A. Recall that A is said to be π*- complemented* if for any π-closed ideal I of A there exists a π-closed ideal J of A such that $A = I \oplus J$. It is clear that every π-complemented algebra is *semiprime* (that is, $I^2 \neq 0$ whenever I is a nonzero ideal of A). A structure theory for π-complemented algebras has been recently developed in [5]. In [7] was obtained a characterization of the π-complementation of a semiprime algebra in terms of the set of all idempotents of its extended centroid. Different approaches to the concepts of extended centroid, C_A, and central closure, $Q_A = C_A A$, for a semiprime algebra A appear in the literature (see [1], [14], and [15]). It is worth pointing out that in [13] it is proved that there exists a bijection between the closed ideals in A and the ideals which are direct summands in Q_A (see also [12]). For a recent treatment of these concepts we refer to reader to [8, Section 2].

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The starting point of this path relies on the possibility of associating an idempotent of the extended centroid of the algebra to each subset of the algebra itself. This result is well-known in an associative context [2] and it was established in [7] in a general context as follows

Proposition 1.1. [7, Proposition 1.6] and [2, Theorem 2.3.9.(iii)] Let A be a semiprime algebra and let S be a nonempty subset of Q_A or C_A. Then

(1) There exists a unique $e_{[S]}$ in $\mathcal{B}_A$ such that

$$\{\lambda \in C_A; \lambda S = 0\} = (1 - e_{[S]})C_A;$$

(2) $e_{[S]}p = p \;\; for every p \in S$ and for any $e \in \mathcal{B}_A$, $e_{[eS]} = ee_{[S]}$.

The second point of this path relies in the mentioned characterization of the π-complementation. The set $\mathcal{B}_A$ of all idempotents of the extended centroid of a semiprime algebra A has a partial order given by $e \le f$ iff $e = ef$. In fact, $\mathcal{B}_A$ is a Boolean algebra for the operations

$$e \wedge f = ef, \quad e \vee f = e + f - ef, \quad and \quad e^* = 1 - e.$$

Theorem 1.2. [7, Theorem 1.8] *Let A be a semiprime algebra. Then the map $e \mapsto eA \cap A$ is a lattice isomorphism from $\mathcal{B}_A$ onto $\mathcal{I}_A^\pi$, and its inverse is the map $I \mapsto e_{[I]}$. As a consequence, we have A is π-complemented if, and only if, $\mathcal{B}_A \subseteq \Gamma_A$. In this case,*

$$\mathcal{I}_A^\pi = \{eA \,:\, e \in \mathcal{B}_A\}.$$

We prove that if $\mathcal{J}$ is an *ideal* of $\mathcal{B}_A$ (that is, a subset such that $e, f \in \mathcal{J}$ implies that $e \vee f \in \mathcal{J}$, and, $e \wedge g \in \mathcal{J}$ whenever $e \in \mathcal{J}$ and $g \in \mathcal{B}_A$) then the subalgebra $A^{\mathcal{J}} = \sum_{e \in \mathcal{J}} eA$ of Q_A is a π-complemented algebra and $e_{[\mathcal{J}]} = e_{[A^{\mathcal{J}}]}$. This result is well known for $\mathcal{J} = \mathcal{B}_A$ (see [7]). As a first application we prove that A is π-complemented iff there is a maximal ideal $\mathcal{P}$ of $\mathcal{B}_A$ such that $A^{\mathcal{P}} \subseteq A$. On the other hand, any algebra (either without or with a unit) can be embedded in another algebra which does possess a unit element $\mathbf{1}$. The *unitisation* of an algebra A over $\mathbb{K}$, denoted by A^1, is the algebra consisting of the vector space $A \times \mathbb{K}$ with the product defined by

$$(a, \alpha)(b, \beta) := (ab + \alpha b + \beta a, \alpha\beta).$$

It is well known (see [8]) that

$$A^1 \; \pi\text{-}complemented \Rightarrow A \;\; \pi\text{-}complemented,$$

and, if A is unital then the converse is also true. Here we prove that in a general setting: A^1 is π-complemented if, and only if, there is a maximal ideal $\mathcal{P}$ of $\mathcal{B}_A$ such that $\mathcal{P}A^1 \subseteq A^1$. If moreover A has nonzero π-socle, then A^1 is a π-complemented algebra if, and only if, there is $e \in \mathcal{B}_A$ maximal such that $(A^1)^{[0,e]} \subseteq A^1$.

Let $L(A)$ denote the algebra of all linear operators from A into A. For $a \in A$, L_a and R_a mean the operators of left and right, respectively, multiplication by a on A. $M(A)$ denotes the *multiplication algebra* of A, namely the subalgebra of $L(A)$ generated by the identity operator Id_A and the set $\{L_a, R_a : a \in A\}$. The *multiplication ideal* of an algebra A, denoted by $M^{\#}(A)$, is the subalgebra of $L(A)$

generated by the set $\{L_a, R_a : a \in A\}$. It is also well known (see [8, Theorem 4.9])) that in an associative context we have

$$A\,\pi-complemented \Leftrightarrow M^{\#}(A)\ \ \pi-complemented. \tag{1}$$

and

$$A^{1}\ \pi-complemented \Leftrightarrow M(A)\ \ \pi-complemented. \tag{2}$$

In this paper is proved that these equivalences remain true in a general context whenever $M(A^1)$ is semiprime (this condition is automatic for a semiprime associative algebra).

Throughout the paper frequently use is made of these properties of Proposition 1.1, often without explicit mention, and of the term "π-theorem" to refer to Theorem 1.2.

2 Ideals of $\mathcal{B}_A$ generating π-complemented algebras.

A nonempty subset $\mathcal{J}$ of $\mathcal{B}_A$ is said to be an *ideal* if for any $e, f \in \mathcal{J}$, $g \in \mathcal{B}_A$, we have $e \vee f \in \mathcal{J}$, and, $e \wedge g \in \mathcal{J}$.

In that follows, we will denote, for any idempotent $e \in \mathcal{B}_A$, by $[0,e]$ the ideal of $\mathcal{B}_A$ defined by

$$[0,e] = \{f \in \mathcal{B}_A :\ f \leq e\}.$$

Before to begin to study the properties of these ideals, let us obtain an elemental property

Proposition 2.1. *For any subset* $\mathcal{T}$ *of* Q_A *or* C_A, $e_{[\mathcal{T}]} = \vee\{e_{[s]};\ s \in \mathcal{T}\}$.

Proof. Suppose that $\mathcal{T}$ is a subset of Q_A or C_A. Since

$$\{\lambda \in C_A;\ \lambda\mathcal{T} = 0\} = \cap_{s\in\mathcal{T}}\{\lambda \in C_A;\ \lambda s = 0\}$$

we have $(1 - e_{[\mathcal{T}]})C_A = \cap_{s\in\mathcal{T}}(1 - e_{[s]})C_A$. Hence, keeping in mind π-theorem, we obtain that $1 - e_{[\mathcal{T}]} = \wedge_{s\in\mathcal{S}}(1 - e_{[s]})$ or equivalently $e_{[\mathcal{T}]} = \vee\{e_{[s]};\ s \in \mathcal{T}\}$. □

From the above Proposition, it is clear that for any ideal $\mathcal{J}$ of $\mathcal{B}_A$ and $e \in \mathcal{J}$, we have

$$[0,e] \subseteq \mathcal{J} \subseteq [0, e_{[\mathcal{J}]}].$$

The non existence of proper ideals of $\mathcal{B}_A$ is determined by the primeness of the algebra. Recall that an algebra A is said to be *prime* if $IJ \neq 0$ for all nonzero ideals I, J of A.

Proposition 2.2.

The following assertions are equivalent:

1. *A is prime.*
2. $\mathcal{B}_A = \{0,1\}$.
3. *$\mathcal{B}_A$ lacks proper ideals.*

Proof. In fact, combining [5, Propostion 3.3] and π-theorem, we deduce that A is prime if, and only if, $\mathcal{B}_A = \{0,1\}$. On the other hand, it is clear that $[0,e]$ is a proper ideal of $\mathcal{B}_A$ if, and only if, there is $e \in \mathcal{B}_A$

with $e \neq 0,1$.

Given an ideal $\mathcal{J}$ of $\mathcal{B}_A$, consider the subset $A^{\mathcal{J}}$ of Q_A defined by

$$A^{\mathcal{J}} := \sum_{e \in \mathcal{J}} eA.$$

Note that $\mathcal{J}$ is contained in Γ_A, if, and only if $A^{\mathcal{J}}$ is an ideal of A. In general, for any ideal $\mathcal{J}$ of $\mathcal{B}_A$, $A^{\mathcal{J}}$ is an ideal of $A^{[0,1]}$.

Remark 2.3. The algebra $A^{\pi} := A^{[0,1]}$ appears in the literature with the name of *idempotent closure* of A [15, §.32.5]. In [7, Corollary 2.5], it is proved that A^{π} is the smallest π-complemented subalgebra of Q_A containing A. In particular, by [7, Theorem 2.3], $C_{A^{\pi}} = C_A$, and so, $\mathcal{B}_{A^{\pi}} = \mathcal{B}_A$. Moreover, it is clear that A is a π-complemented algebra if, and only if, $A = A^{\pi}$, and for any $e \in \mathcal{B}_A$, $A^{[0,e]} = eA^{\pi}$. In particular, $A^{[0,e]}$ is a π-closed ideal of A^{π} because of π-theorem.

The *centroid* Γ_A of A is defined as the subalgebra of C_A consisting of all elements $\lambda \in C_A$ such that $\lambda A \subseteq A$.

Proposition 2.4. *Let I be a π-closed ideal of A. Then $I = A^{[0,e_{[I]}]} \cap A$. As a consequence, if $[0,e_{[I]}] \subseteq \Gamma_A$ then $I = A^{[0,e_{[I]}]}$.*

Proof. By π-theorem, there is $e \in \mathcal{B}_A$ such that $I = eA \cap A$ and so $I = eA^{\pi} \cap A = A^{[0,e_{[I]}]} \cap A$. Note that if $[0,e_{[I]}] \subseteq \Gamma_A$ then $A^{[0,e_{[I]}]} \subseteq A$.

A subset $\mathcal{U}$ of $\mathcal{B}_A$ is said to be *dense* if $\{\lambda \in C; \lambda \mathrm{U} = 0\} = 0$, in fact, the above definition can be rephrased as follows, $\mathcal{U}$ is dense if, and only if, $e_{[\mathrm{U}]} = 1$.

A subalgebra B of A is said to be *dense* whenever the condition $T(B) = 0$, for $T \in M(A)$, implies $T = 0$. It is clear that A is a dense subalgebra of A^{π}.

Proposition 2.5. *Let $\mathcal{J}$ be an ideal of $\mathcal{B}_A$ Then $A^{\mathcal{J}}$ is a dense ideal of the algebra $A^{[0,e_{[\mathcal{J}]}]}$.*

Proof. Firstly we claim that if $\mathcal{J}$ is a dense ideal of $\mathcal{B}_A$ then $A^{\mathcal{J}}$ is a dense ideal of the algebra A^{π}. Indeed, if $T \in M(A^{\pi})$ such that $T(A^{\mathcal{J}}) = 0$ then $eT(A) = 0$, for all $e \in \mathcal{J}$, and so, $\mathcal{J}e_{[T(A)]} = 0$. Hence, $e_{[T(A)]} = 0$, thus $T(A) = 0$, and so, by the density of A, $T = 0$. On the other hand, by [9, Theorem 3.2 and Corollary 3.3] $e_{[\mathcal{J}]}A$ is a semiprime algebra and $\mathcal{B}_{e_{[\mathcal{J}]}A} = [0, e_{[\mathcal{J}]}]$. Since $\mathcal{J}$ is a dense ideal of $[0,e_{[\mathrm{J}]}]$ and $A^{\mathcal{J}} = (e_{[\mathcal{J}]}A)^{\mathcal{J}}$, by the claim, $A^{\mathcal{J}}$ is a dense ideal of $(e_{[\mathcal{J}]}A)^{\pi} = A^{[0,e_{[\mathcal{J}]}]}$.

It is well known, see [5, Proposition 4.3] that every π-closed ideal of A^{π} is a π-complemented algebra, in particular, $A^{[0,e]}$ is a π-complemented algebra. In fact,

Theorem 2.6. *Let $\mathcal{J}$ be an ideal of $\mathcal{B}_A$ then $A^{\mathcal{J}}$ is a π-complemented algebra, and $e_{[\mathcal{J}]} = e_{[A^{\mathcal{J}}]}$.*

Proof. Firstly we claim that if K is an ideal of $A^{\mathcal{J}}$ then $KA \subseteq K$. Indeed, if $k \in K$ and $a \in A(\subseteq A^{\pi})$, then it is clear that $ak \in A^{\mathcal{J}}$, and so, there are $e_1, e_2, \cdots, e_n \in \mathcal{J}$ and $a_1, a_2, \cdots, a_n \in A$ such that $ak = \sum_{i=1}^{n} e_i a_i$. Take $e = \vee_{i=1}^{n} e_i \in \mathcal{J}$. Therefore, since $ea \in A^{\mathcal{J}}$ and $ak = eak$, we have $ak = k(ea) \in K$. Now suppose that $K^2 = 0$. In particular, by the claim, $K \cap A$ is an ideal of A (whose square is zero) and so, by semiprimeness of A, $K \cap A = 0$. Given $q \in K$, by [8, Proposition 2.1.(i)], there exists an essential ideal D of A such that $D\mathcal{M}(A)(q) \subseteq A$. Then $D\mathcal{M}(A)(q) \subseteq K \cap A$,, and so $D\mathcal{M}(A)(q) = 0$. Hence $q = 0$ because of [8, Proposition 2.1.(ii)]. Since q is an arbitrary element of K, we conclude that $K = 0$. Thus, $A^{\mathcal{J}}$ is semiprime. On the other hand, by Proposition 2.5, $A^{\mathcal{J}}$ is a dense ideal of $e_{[\mathcal{J}]}A^{\pi}$, and hence, by [9, Theorems 2.6 and 3.2], $C_{A^{\mathcal{J}}} = C_{e_{[\mathcal{J}]}A^{\pi}} = e_{[\mathcal{J}]}C_{A^{\pi}} = e_{[\mathcal{J}]}C_A$, and therefore, $e_{[\mathcal{J}]} = e_{[A^{\mathcal{J}}]}$. Moreover, since $e_{[\mathcal{J}]}\mathcal{B}_A \subseteq \Gamma_{A^{\mathcal{J}}}$, we have $\mathcal{B}_{A^{\mathcal{J}}} \subseteq \Gamma_{A^{\mathcal{J}}}$. Thus, by π-theorem, $A^{\mathcal{J}}$ is a π-complemented algebra. □

From Proposition 2.5, equality $e_{[\mathcal{J}]} = e_{[A^{\mathcal{J}}]}$ and [9, Corollary 1.6.(3)] we deduce that

Corollary 2.7. *Let $\mathcal{J}$ be an ideal of $\mathcal{B}_A$. Then $\mathcal{J}$ is dense if, and only if, $A^{\mathcal{J}}$ is a dense ideal of A^{π}.*

There are ideals I of semiprime (even π-complemented) algebras A such that I is a π-complemented algebra but $I \neq A^{\mathcal{J}}$ for all ideal $\mathcal{J}$ of $\mathcal{B}_A$

Example 2.8.

Consider the algebra $I = c_0$ of all null sequences and the algebra $A = l_{\infty}$ of all bounded sequences (endowed with the operations of algebra coordinate to coordinate). Keeping in mind [9, Corollary 5.2] and [7, Theorem 2.3], we deduce that the central closure and the extended centroid of the algebra c_{00} of all quasinull sequences, and so of l_{∞} and c_0, is the algebra s of all sequences. Note that

$$\mathcal{B}_A := \{\{x_n\}_n;\ \text{either } x_n = 0 \text{ or } x_n = 1\}.$$

It is clear that if $A^{\mathcal{J}} \subseteq c_0$, then $\mathcal{J} \subseteq c_{00}$, and so, $A^{\mathcal{J}} \subseteq c_{00}$.

Finally, taking into account π-theorem, we have c_{00}, c_0 and l_{∞} are π-complemented algebras. Note that c_0 is a π-dense ideal of l_{∞}.

In fact,

Corollary 2.9. *Let I be an ideal of A. Then $I = A^{[0,e_{[I]}]}$ if, and only if, I is a π-complemented algebra, $e_{[I]} \in \Gamma_A$, and I is a π-closed ideal of A.*

Proof. Assume that $I = A^{[0,e_{[I]}]}$. By assumption $[0,e_{[I]}]A \subseteq I$, and so, $[0,e_{[I]}] \subseteq \Gamma_A$. By Theorem 2.6, I is a π-complemented algebra and by π-theorem, $\bar{I} = e_{[I]}A \cap A = e_{[I]}A \subseteq I$. Conversely, suppose that I is a π-closed ideal which is a π-complemented algebra such that $e_{[I]} \in \Gamma_A$. By π-theorem, $I = e_{[I]}A \cap A$, and so, by assumption, $I = e_{[I]}A$. Then by [9, Corollary 3.3] and by π-theorem

$$[0,e_{[I]}]A \subseteq [0,e_{[I]}]e_{[I]}A = [0,e_{[I]}](e_{[I]}A \cap A) = [0,e_{[I]}]I \subseteq I.$$

Therefore $A^{[0,e_{[I]}]} = I$ because of Proposition 2.4.

3 Maximal ideals and π-complementation

In this section will highlight the relevance of maximal ideals.

Remark 3.1. *For any proper ideal $\mathcal{J}$ of $\mathcal{B}_A$ there is a maximal ideal , $\mathcal{P}$of $\mathcal{B}_A$ such that $\mathcal{J} \subseteq \mathcal{P}$.* In fact, by applying Zorn's lemma to the set

$$\mathrm{B}^{\mathcal{J}} := \{\mathcal{K} \text{ ideal of } \mathcal{B}_A : \ \mathcal{J} \subseteq \mathcal{K} \subsetneqq \mathcal{B}_A\},$$

we deduce that there is $\mathcal{P} \in \mathrm{B}^{\mathcal{J}}$ maximal.

The next result includes a well-known characterization of the maximal ideals (cf. [3, Appendix B]).

Lemma 3.2.

Let $\mathcal{P}$ be an ideal of $\mathcal{B}_A$ Then $\mathcal{P}$s maximal if, and only if, for every $e \in \mathcal{B}_A$ either $e \in \mathcal{P}$ or $1-e \in \mathcal{P}$ but not both. In this case, if $q \in Q_A$, then $q \in A^{\mathrm{P}}$ if, and only if $e_{[q]} \in \mathcal{P}$.

Proof. Suppose that $\mathcal{P}$ is a maximal ideal of $\mathcal{B}_A$ and let $e \in \mathcal{B}_A \backslash \mathcal{P}$. Let

$$Q := \{g \vee he : \ g \in \mathcal{P}, \ h \in \mathcal{B}_A\}.$$

It is clear that Q is an ideal of $\mathcal{B}_A$ containing strictly to the ideal $\mathcal{P}$. Therefore, since $\mathcal{P}$ is maximal, there are $g \in \mathcal{P}$ and $f \in [0,e]$ such that $1 = g \vee f$, and so, multiplying by $1-e$, we obtain that $1-e = g(1-e) \in \mathcal{P}$.

Conversely, suppose that $\mathcal{J}$ is an ideal of $\mathcal{B}_A$ ıch that $\mathcal{P} \subseteq \mathcal{J} \subseteq \mathcal{B}_A$ .here is $e \in \mathcal{J} \backslash \mathcal{P}$, $1-e \in \mathcal{P} \subseteq \mathcal{J}$, ;o $1 = e \vee (1-e) \in \mathcal{J}$, that is $\mathcal{J} = \mathcal{B}_A$.

Finally, let $q \in Q_A$ and let $\mathcal{P}$ be a maximal ideal of $\mathcal{B}_A$. If $e_{[q]} \in \mathcal{P}$, then $q = e_{[q]}q \in A^{\mathcal{P}}$. Conversely, suppose $q = \sum_{i=1}^{n} e_i q_i \in A^{\mathcal{P}}$. For each i we have $1 - e_i \notin \mathcal{P}$ and so, $e = \prod_{i=1}^{n}(1-e_i) \notin \mathcal{P}$ (note that if $e, f \notin \mathcal{P}$ then $(1-e) \vee (1-f) \in \mathcal{P}$, and hence $ef = 1 - ((1-e) \vee (1-f)) \notin \mathcal{P}$). But $eq = 0$, whence $0 = e_{[eq]} = ee_{[q]}$. Since $e \notin \mathcal{P}$ it follows that $e_{[q]} = (1-e)e_{[q]} \in \mathcal{P}$.

□

As a consequence of Lemma 3.2,

Corollary 3.3. If $\mathcal{P}$ is a maximal ideal of $\mathcal{B}_A$ then either $\mathcal{P}$ is dense or $\mathcal{P}$ is closed.

Proof. Let $\mathcal{P}$ be a maximal ideal of $\mathcal{B}_A$. If $e_{[\mathcal{P}]} \notin \mathcal{P}$ then $1 - e_{[\mathrm{P}]} \in \mathrm{P}$, and so, $1 - e_{[\mathcal{P}]} \in \mathcal{P}$, and hence $e_{[\mathcal{P}]} = 1$.

Note that if $\mathcal{P}$ is a maximal ideal of $\mathcal{B}_{c_0}$, keeping in mind Lemma 3.2, the sequence $x_0 = \{1/n\} \in c_0 \backslash c_0^{\mathcal{P}}$ because of $e_{[x_0]} = 1$. In fact, as a consequence of Lemma 3.2,

Corollary 3.4. *Suppose that there is $a \in A$ with $e_{[a]} = 1$. If $\mathcal{J}$ is a proper ideal of $\mathcal{B}_A$ then $A^{\mathcal{J}}$ is a* *proper ideal of* A.

Proof. Let $\mathcal{J}$ be a proper ideal of $\mathcal{B}_A$. By Remark 3.1, there is $\mathcal{P}$ a maximal ideal of $\mathcal{B}_A$ such that $\mathcal{J} \subseteq \mathcal{P}$. Take $I = A^{\mathcal{P}}$. Let $e \in \mathcal{B}_A \setminus \mathcal{P}$. Since $e = ee_{[a]} = e_{[ea]}$, by Lemma 3.2, $ea \in A^{\pi} \setminus A^{\mathcal{P}}$, thus I is proper, and so, $A^{\mathcal{J}}$ is a proper ideal of A^{π}.

In a certain sense, the next result is a rereading of the final assertion of π-theorem.

Theorem 3.5. Then A is a π-complemented algebra if, and only if, there is a maximal ideal $\mathcal{P}$ of $\mathcal{B}_A$ such that $\mathcal{P} \subseteq \Gamma_A$.

Proof. Suppose that A is a π-complemented algebra. By Remark 3.1 and π-theorem there is a maximal ideal $\mathcal{P}$ of $\mathcal{B}_A$ such that $\mathcal{P} \subseteq \mathcal{B}_A \subseteq \Gamma_A$. Conversely, let $\mathcal{P}$ be a maximal ideal $\mathcal{B}_A$ such that $\mathcal{P} \subseteq \Gamma_A$. Take $f \in \mathcal{B}_A$. If $f \notin \mathcal{P}$ then, by Lemma 3.2, $1 - f \in \mathcal{P} \subseteq \Gamma_A$. In any case, $f \in \Gamma_A$. Thus, $\mathcal{B}_A \subseteq \Gamma_A$, and so, by π-theorem, A is a π-complemented algebra.

□

Note that there are (dense) ideals $\mathcal{J}$ of $\mathcal{B}_A$ of a semiprime algebra A such that $\mathcal{J} \subseteq \Gamma_A$, but A is not π-complemented.

Example 3.6. Take $A = c$, the algebra of all convergent sequences, and $\mathcal{J} = \{ef_n : n \in \mathbb{N}, e \in \mathcal{B}_A\}$ $f_n(m) = 0$ in other case. By Proposition 2.1, $\mathcal{J}$ is a dense ideal of $\mathcal{B}_A$, and, since $\mathcal{J} = \mathcal{B}_A \cap c_{00}$, then $\mathcal{J} \subseteq \Gamma_c$. On the other hand, c is not π-complemented because of the product of the sequence $\mathbf{1} = \{a_n\} \in c$ where $a_n = 1$ for all $n \in \mathbb{N}$ and the sequence $e = \{b_n\} \in C_c$ with $a_{2n} = 1$ and $a_{2n-1} = 0$ for all $n \in \mathbb{N}$ is such that $e\mathbf{1} \notin c$. Note that by Theorem 2.6, $c^{\mathcal{J}} = c_{00}$ is a π-complemented algebra which is a dense ideal of c.

Obviously, $\mathcal{J}$ is not maximal because of $e = (1,0,1,0,\cdots)$ and $1-e$ do not belong to $\mathcal{J}$.

A natural way to made maximal ideals of $\mathcal{B}_A$ is to take maximal elements e in $\mathcal{B}_A$ and consider the ideals of the form $[0,e]$.

Proposition 3.7. *Let* $e \in \mathcal{B}_A$. *Then the following assertions are equivalent:*

1. e is maximal.
2. $[0,e]$ is a maximal ideal of $\mathcal{B}_A$.
3. $A^{[0,e]}$ is a maximal π-closed ideal of A^{π}.

Proof. 1) $\Rightarrow$ 2) Assume that e is maximal and suppose that $f \in \mathcal{B}_A$ is such that $f \notin [0,e]$. By π-theorem (since $eA \cap A$ is a maximal π-closed ideal of A, and so, $A = \overline{(eA \cap A) + (fA \cap A)}$ we have $1 = e \vee f$. In particular, $1 - e = 0 \vee (1-e)f = (1-e)f$, and so, $1 - f \in [0,e]$. Thus, by Lemma 3.2, $[0,e]$ is maximal.

2) $\Rightarrow$ 3) Assume that $[0,e]$ is maximal. By Corollary 2.9, $A^{[0,e]} = eA^{\pi}$ is a π-closed ideal of A^{π}. If I is a π-closed ideal of A^{π}, such that $A^{[0,e]} \subseteq I$, by π-theorem $eA^{\pi} = A^{[0,e]} \subseteq I = e_{[I]}A^{\pi}$, and so, $e \le e_{[I]}$. Hence either $[0,e] = [0,e_{[I]}]$, and so, $e = e_{[I]}$ or, $[0,e_{[I]}] = \mathcal{B}_A$, and so, $e_{[I]} = 1$.

3) $\Rightarrow$ 1) Since $A^{[0,e]} = eA^{\pi}$, e is maximal because of π-theorem.

However, there are maximal ideals of the set of all idempotents of the extended centroid which are not derived from a maximal idempotent.

Example 3.8. Indeed, the algebra $\mathbb{M}$ of all equivalence classes (under equality almost everywhere) of Lebesgue measurable functions on $[0,1]$ with pointwise operations is a π-complemented algebra which lacks maximal π-closed ideals (see [5, Example 5.12]), and so, by π-theorem $\mathcal{B}_{\mathbb{M}}$ lacks maximal elements.

Obviously, the situation is better when the algebra contains maximal π-closed ideals of A^{π}. We set $\pi - Soc(A)$ to denote the π- *socle* of A, i.e. the sum of its minimal π-closed ideals.

Corollary 3.9. *The following assertions are equivalent:*

1. *A is a π-complemented algebra which has nonzero π-Socle.*
2. *There is a maximal element $e \in \mathcal{B}_A$ such that $[0,e] \subseteq \Gamma_A$*
3. *A contains a maximal π-closed ideal of A^{π}.*

In this case, $A^{[0,e]}$ is a maximal π-closed ideal of A.

Proof. $(1) \Rightarrow (2)$ By assumption there is a maximal π-closed ideal I of A. By π-theorem, $e := e_{[I]}$ is maximal, and $[0,e] (\subseteq \mathcal{B}_A) \subseteq \Gamma_A$.

$(2) \Rightarrow (3)$ Note that, taking into account that $\mathcal{B}_A = \mathcal{B}_{A^{\pi}}$ and π-theorem, $A^{[0,e]} = eA^{\pi}$ is a maximal π-closed ideal of A^{π} and, of course, by assumption $[0,e] \subseteq \Gamma_A$, $A^{[0,e]} \subseteq A$

$(3) \Rightarrow (1)$ Let I be a maximal π-closed ideal of A^{π} contained in A. Since $\mathcal{B}_{A^{\pi}} = \mathcal{B}_A$, by π-theorem, there is $e \in \mathcal{B}_A$ maximal such that $I = eA^{\pi}$. In particular, $[0,e]A \subseteq eA^{\pi} = I \subseteq A$. Therefore, since by π-theorem, $[0,e]$ is a maximal ideal of $\mathcal{B}_A$, by Theorem 3.5, A is a π-complemented algebra and, by π-theorem, $I = eA \cap A$ is a maximal π-closed ideal of A.

4 π-complementation and unitisation

Recall that the *unitisation* of an algebra A over $\mathbb{K}$, denoted by A^1, is the algebra consisting of the vector space $A \times \mathbb{K}$ with the product defined by

$$(a,\alpha)(b,\beta) := (ab + \alpha b + \beta a, \alpha\beta).$$

It is routine matter to verify that $\mathbf{1} := (0,1)$ is the unit element of A^1, and that the map $a \mapsto (a,0)$ allows to regard A as a subalgebra of A^1 in such a way that $A^1 = A \oplus \mathbb{K}\mathbf{1}$. It is well known (cf. the implication $(i) \Rightarrow (iii)$ in [8, Theorem 3.22] and π-theorem) that

$$A^1 \quad \pi - complemented \Rightarrow A \quad \pi - complemented,$$

and, if A is unital then the converse is also true (see [8, Theorem 3.10]). The example 3.6 shows, since c_0^1 is isomorphic to c, that in a general setting the converse is not true.

Next results highlight the special relationship between the π-complementation of an algebra and the π-complementation of its unitisation. Firstly note that if A lacks unit then $\mathcal{B}_A = \mathcal{B}_{A^1}$ because of [8, Theorem 3.15]. Secondly,

Lemma 4.1. *Let $e \in \mathcal{B}_A$ such that $e \in \Gamma_{A^1}$. If A lacks unit unit, then either $e\mathbf{1} \in A$ or $(1-e)\mathbf{1} \in A$.*

Proof. Since $\Gamma_{A^1} \subseteq \Gamma_A$ because of [8, Corollary 3.17], we have

$$eA \subseteq A \ \ and \ \ eA^1 \subseteq A^1. \tag{3}$$

Suppose that $e\mathbf{1} \notin A$, by (3) $e\mathbf{1} = a + \mu 1$, with $a \in A$ and $0 \neq \mu \in \mathbb{K}$. In particular $e\mathbf{1} = ea + \mu e\mathbf{1}$, and so, by (3), $(1-\mu)e\mathbf{1} \in A$. Hence, since $e\mathbf{1} \notin A$, $\mu = 1$. Therefore $e\mathbf{1} = a + \mathbf{1}$, thus $(1-e)\mathbf{1} \in A$.

□

Let A be a semiprime algebra and consider the set

$$\mathcal{J}_{A^1}^A := \{e \in \mathcal{B}_A : \ e\mathbf{1} \in A\}.$$

Note that $\mathcal{J}_{A^1}^A = \{e \in \mathcal{B}_A : \ eA^1 \subseteq A\}$, and so, $\mathcal{J}_{A^1}^A \subseteq \Gamma_{A^1}$.Indeed, if $e \in \mathcal{J}_{A^1}^A$, $a \in A$ and $\lambda \in \mathbb{K}$, we have $e(a + \lambda\mathbf{1}) = e\mathbf{1}a + \lambda e\mathbf{1} \in A$.

Lemma 4.2. *If A is a π-complemented algebra then $\mathcal{J}_{A^1}^A$ is an ideal of $\mathcal{B}_A$.*

Proof. Assume that A is a π-complemented algebra. Keeping in mind π-theorem, it is clear that

$$\mathcal{B}_A \mathcal{J}_{A^1}^A A^1 \subseteq \mathcal{B}_A A \subseteq A,$$

as a consequence, $\mathcal{J}_{A^1}^A$ is an ideal of $\mathcal{B}_A$.

As a consequence of Theorem 3.5, we complete [8, Theorem 3.22].

Corollary 4.3. *Let A be a semiprime algebra without unit. Then the following assertions are equivalent:*

1. *A^1 is a π complemented algebra.*
2. *$\mathcal{J}_{A^1}^A$ is a maximal ideal of $\mathcal{B}_A$.*
3. *There is a maximal ideal $\mathcal{P}$ of $\mathcal{B}_A$ such that $\mathcal{P} \subseteq \Gamma_{A^1}$.*

In this case, $(A^1)^{\mathcal{J}_{A^1}^A}$ and A are two π-complemented algebras and $(A^1)^{\mathcal{J}_{A^1}^A} \subset A$.

Proof. First of all we note that A^1 is semiprime, because of [8, Corollary 3.3], and $\mathcal{B}_A = \mathcal{B}_{A^1}$ because of [8, Theorem 3.15].

$1) \Rightarrow 2)$. By assumption and keeping in mind π-theorem, $\mathcal{B}_A \subseteq \Gamma_{A^1}$ and so, since by [8, Corolary 3.17]) $\mathcal{B}_A \subseteq \Gamma_A$, we have, again by π-theorem, A is a π complemented algebra. Combining Lemmas 4.2, 4.8, and 3.2, we deduce that $\mathcal{J}_{A^1}^A$ maximal ideal of $\mathcal{B}_A$.

$2) \Rightarrow 3)$ Take $\mathcal{P} = \mathcal{J}_{A^1}^A$.

$3) \Rightarrow 1)$ It follows from Theorem 3.5.

□

As a consequence of Corollary 3.9,

Corollary 4.4. *The following assertions are equivalent:*

1. *A has nonzero π-Socle and A^1 is a π complemented algebra.*
2. *There is a maximal element $e \in \mathcal{B}_A$ such that $[0,e] \subseteq \Gamma_{A^1}$.*

3. *A^1 contains a maximal π-closed ideal of $(A^1)^{\pi}$.*

In this case, $(A^1)^{[0,e]}$ is a maximal π-closed ideal of A^1.

We can now consider a particular case: the multiplication ideal and the multiplication algebra of an algebra A. It is clear that $M^{\#}(A)$ is an ideal of $M(A)$ such that $M(A) = M^{\#}(A) + \mathbb{K}\,\mathrm{Id}_A$. It is well known that (see [8, Theroem 4.3 and Corollary 4.4]) $\mathcal{B}_{M(A)} = \mathcal{B}_{M^{\#}(A)} = \mathcal{B}_A$ whenever A is m.s.p.

An algebra A is said to be *multiplicatively semiprime* or *m.s.p.* whenever both A and $M(A)$ are semiprime algebras.

Corollary 4.5. *Let A be an m.s.p. algebra. Then the following assertions are equivalent:*

1. *$M(A)$ is a π-complemented algebra.*
2. *The set $\{e \in \mathcal{B}_A : e\mathrm{Id}_A \in M^{\#}(A)\}$ is a maximal ideal of $\mathcal{B}_A$.*
3. *There is a maximal ideal $\mathcal{P}$ of $\mathcal{B}_A$ such that $\mathcal{P} \subseteq \Gamma_{M(A)}$.*

Let us see that the equivalences (1) and (2) can be extended to a non associative context.

Remark 4.6. Note that , by [8, Corollary 3.3 and Lemma 3.8], it is easy to deduce that, if A has a unit, then A is an m.s.p. algebra, if and only if, A^1 is an m.s.p. algebra.

However,

Lemma 4.7. *Let A a semiprime algebra without unit. Then A^1 is an m.s.p. algebra if, and only if, A is m.s.p. and A is a dense ideal of A^1.*

Proof. Suppose that A^1 is an m.s.p. algebra. By [6, Corollary 4.7], A is m.s.p. Since by [8, Theorem 3.15], $C_{A^1} = C_A$ and $e_{[A]} = 1$, keeping in mind [9, Corollary 4.5 and Corollary 4.18], we have $A^1 = \overline{A}$ because of π-theorem. Therefore by [4, Theorem 2.6], we deduce that A is dense.

Suppose that A is m.s.p. and A is a dense ideal of A^1. By [8, Corollary 3.3], A^1 is a semiprime algebra, and by [8, Lemma 3.8] A^1 is m.s.p.

□

Lemma 4.8. *Let A a semiprime algebra which is a dense ideal of A^1. Then*

$$\Gamma_{M^{\#}(A)} = \Gamma_A.$$

Proof. By [8, Proposition 4.5], $\Gamma_{M(A)} \subseteq \Gamma_A \subseteq \Gamma_{M^{\#}}(A)$.

Let $\lambda \in \Gamma_{M^{\#}(A)}$ and $a \in A$. If $A = A^2$, then, it follows that there are $b, c \in A$ such that

$$\lambda a = \lambda bc = \lambda(L_b(c)) = (\lambda L_b)(c) \in M^{\#}(A)(A) \subseteq A^2 = A$$

. In other case, there is $c \in A \setminus A^2$. Since $\lambda M^{\#}(A) \subseteq M^{\#}(A) \subseteq M(A) \subseteq M(A^1)$ because of the density of A, it follows that

$$\lambda a = \lambda(L_a(\mathbf{1})) = (\lambda L_a)(\mathbf{1}) \in A^1,$$

that is, there is $\alpha \in \mathsf{K}, b \in A$ such that $\lambda a = b + \alpha \mathbf{1}$. In particular, $ac = bc + \alpha c$, and so, by assumption on c, $\alpha = 0$. Therefore, in any case, $\lambda a \in A$ for all $a \in A$, and hence $\lambda \in \Gamma_A$.

□

Of course, there are simple algebras not dense in its unitisation.

Example 4.9. Let A be the subalgebra of matrix algebra $\mathcal{M}_2(\mathbb{R})$ whose trace is zero, equipped with the Lie product $[.,.]$. It is easy to show that $(A,[.,.])$ is a simple algebra (in particular m.s.p.) lacks unit of dimension 3 such that $M^{\#}(A) = L(A)$, and so, Id_A can write $\sum L_{a_1} \ldots L_{a_n}$ with $a_1, a_2, \cdots, a_n \in A$. Consider $F = \sum L_{a_1}^{A^1} \ldots L_{a_n}^{A^1} \in M(A^1)$. Note that $\mathrm{Id}_{A^1}(1) = 1 \neq F(1) \in A$, but $\mathrm{Id}_{A^1}(a) - F(a) = 0$ for every $a \in A$. Thus A is not dense in A^1.

Note that, taking into account [10, Corollary 4.3]) and [8, Corollary 3.3], if A is an associative algebra then, A is semiprime if, and only if, A^1 is m.s.p. Thus, the next result can be considered as an improvement of [8, Proposition 4.7].

Proposition 4.10. *If* A^1 *is an m.s.p. algebra then*

$$\Gamma_{M^{\#}(A)} = \Gamma_A.$$

Proof. Suppose that A is unital. In this case, $M^{\#}(A) = M(A)$, and therefore by [8, Proposition 4.5], $\Gamma_A = \Gamma_{M^{\#}(A)}$. If A lacks unit then, combining Lemmas 4.7 and 4.8, we have also $\Gamma_A = \Gamma_{M^{\#}(A)}$.

□

As a consequence we improved the equivalences (1) and (2).

Corollary 4.11. Let A be an algebra such that A^1 is m.s.p.

1. *A is π-complemented if, and only if, $M^{\#}(A)$ is a π-complemented algebra.*
2. *A^1 is π-complemented if, and only if, $M(A)$ is a π-complemented algebra.*

Proof. (1) By Proposition 4.10, $\Gamma_{M^{\#}(A)} = \Gamma_A$. Since $\mathcal{B}_A = \mathcal{B}_{M^{\#}(A)}$ because of [8, Corollary 4.4], by π-theorem, we deduce that A is π-complemented if, and only if, $M^{\#}(A)$ is a π-complemented algebra.

(2) If A has unit then, by [8, Theorem 3.10] and by (1), A^1 is π-complemented iff $M^{\#}(A) = M(A)$ is π-complemented. Suppose that A lacks unit. Keeping in mind Remark 4.6, by (1), A^1 is π-complemented iff $M^{\#}(A^1) = M(A^1)$ is a π-complemented algebra. On the other hand, since A is dense in A^1 because of Lemma 4.7, we have, by [8, Lemma 3.8], there is an algebra isomorphism from $M(A^1)$ onto $M(A)$, and so, $M(A^1)$ is complemented iff $M(A)$ is a π-complemented algebra.

□

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