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Pointwise weighted approximation of functions with inner singularities by combinations of Bernstein operators

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Abstract.
We introduce another new type of combinations of Bernstein operators in this paper, which can be used to approximate the functions with inner singularities. The direct and inverse results of the weighted approximation of this new type combinations are obtained.

Keywords: Combinations of Bernstein polynomials; Functions with inner singularities; Weighted approximation; Direct and inverse results.

1 Introduction

The set of all continuous functions, defined on the interval *I*, is denoted by *C(I)*. For any $f \in C([0,1])$, the corresponding *Bernstein operators* are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x),$$

Where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ k = 0,1,2,\ldots,n,\ x \in [0,1].$$

Approximation properties of Bernstein operators have been studied very well (see [2], [3], [5]-[8], [12]-[14], for example). In order to approximate the functions with singularities, Della Vecchia et al. [3] and Yu-Zhao [12] introduced some kinds of modi_ed Bernstein operators. Throughout the paper, *C* denotes a positive constant independent of *n* and *x*, which may be different in different cases.

Let $\bar{w}(x) = |x-\xi|^{\alpha},\ 0<\xi<1,\ \alpha>0$ and $C_{\bar{w}} := \{f \in C([0,1] \setminus \xi) : \lim_{x \longrightarrow \xi} (\bar{w}f)(x) = 0\}$.

The norm in $C_{\bar{w}}$ is defined as $\|f\|_{C_{\bar{w}}} := \|\bar{w}f\| = \sup_{0 \leqslant x \leqslant 1} |(\bar{w}f)(x)|$. Define

$$W^r_{\bar{w},\lambda} := \{f \in C_{\bar{w}} : f^{(r-1)} \in A.C.((0,1)),\ \|\bar{w}\varphi^{r\lambda} f^{(r)}\| < \infty\}.$$

For $f \in C_{\bar{w}}$, define the *weighted modulus of smoothness* by

$$\omega^r_{\varphi^\lambda}(f,t)_{\bar{w}} := \sup_{0<h\leqslant t} \{\|\bar{w}\Delta^r_{h\varphi^\lambda} f\|_{[16h^2,1-16h^2]} + \|\bar{w}\overrightarrow{\Delta}^r_h f\|_{[0,16h^2]} + \|\bar{w}\overleftarrow{\Delta}^r_h f\|_{[1-16h^2,1]}\},$$

where

$$\begin{aligned}\Delta^r_{h\varphi^\lambda} f(x) &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (\frac{r}{2} - k)h\varphi^\lambda(x)),\\ \overrightarrow{\Delta}^r_h f(x) &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r-k)h),\\ \overleftarrow{\Delta}^r_h f(x) &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x - kh),\end{aligned}$$

and $\varphi(x) = \sqrt{x(1-x)}$. The *weighted K-function* is given by

$$K_{r,\varphi^\lambda}(f, t^r)_{\bar{w}} := \inf_g \{\|\bar{w}(f-g)\| + t^r \|\bar{w}\varphi^{r\lambda} g^{(r)}\| : g \in W^r_{\bar{w},\lambda}\}.$$

It was shown in [5] that $K_{r,\varphi^\lambda}(f, t^r)_{\bar{w}} \sim \omega^r_{\varphi^\lambda}(f,t)_{\bar{w}}$. On the other hand, since the *Bernstein polynomials* cannot be used for the investigation of higher orders of smoothness, Butzer [1] introduced the *combinations of Bernstein polynomials* which have higher orders of approximation. Ditzian and Totik [5] extended this method of combinations and defined the following combinations of Bernstein operators:

$$B_{n,r}(f,x) := \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f,x).$$

with the conditions
(a) $n = n_0 < n_1 < \cdots < n_{r-1} \leqslant Cn$,
(b) $\sum_{i=0}^{r-1} |C_i(n)| \leqslant C$, (c) $\sum_{i=0}^{r-1} C_i(n) = 1$,
(d) $\sum_{i=0}^{r-1} C_i(n) n_i^{-k} = 0$, for $k = 1, \cdots, r-1$.

2 The main results

For any positive integer r, we consider the determinant

$$A_r := \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2r+1 & 2r+2 & 2r+3 & \cdots & 4r+1 \\ (2r)(2r+1) & (2r+1)(2r+2) & (2r+2)(2r+3) & \cdots & (4r)(4r+1) \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 2\cdots(2r+1) & 3\cdots(2r+2) & 4\cdots(2r+3) & \cdots & (2r+2)\cdots(4r+1) \end{vmatrix}.$$

We obtain $A_r = \prod_{j=2}^{2r} j!$. Thus, there is a unique solution for the system of nonhomogeneous linear equations:

$$\left\{\begin{array}{ccccccccc} a_1 & + & a_2 & + & \cdots & + & a_{2r+1} & = & 1, \\ (2r+1)a_1 & + & (2r+2)a_2 & + & \cdots & + & (4r+1)a_{2r+1} & = & 0, \\ (2r+1)(2r)a_1 & + & (2r+1)(2r+2)a_2 & + & \cdots & + & (4r)(4r+1)a_{2r+1} & = & 0, \\ & & & & \vdots & & & & \\ (2r+1)!a_1 & + & 3\cdots(2r+2)a_2 & + & \cdots & + & (2r+2)\cdots(4r+1)a_{2r+1} & = & 0. \end{array}\right. \quad (2.1)$$

Let

$$\psi(x) = \begin{cases} a_1 x^{2r+1} + a_2 x^{2r+2} + \cdots + a_{2r+1} x^{4r+1}, & 0 < x < 1, \\ 0, & x \leqslant 0, \\ 1, & x = 1. \end{cases}$$

with the coefficients $a_1,\ a_2,\ \cdots,\ a_{2r+1}$ satisfying (2.1). From (2.1), we see that $\psi(x) \in C^{(2r)}(-\infty,+\infty)$, $0 \leqslant \psi(x) \leqslant 1$ for $0 \leqslant x \leqslant 1$. Moreover, it holds that $\psi(1)=1,\ \psi^{(i)}(0)=0,\ i=0,1,\cdots,2r$ and $\psi^{(i)}(1)=0,\ i=1,2,\cdots,2r$.

Let

$$H(f,x) := \sum_{i=1}^{r+1} f(x_i) l_i(x),$$

and

$$l_i(x) := \frac{\prod_{j=1, j\neq i}^{r+1}(x-x_j)}{\prod_{j=1,j\neq i}^{r+1}(x_i-x_j)},\ x_i = \frac{[n\xi-((r-1)/2+i)]}{n},\ i=1,2,\cdots r+1.$$

Further, let

$$x_1^{'} = \frac{[n\xi-2\sqrt{n}]}{n},\ x_2^{'} = \frac{[n\xi-\sqrt{n}]}{n},\ x_3^{'} = \frac{[n\xi+\sqrt{n}]}{n},\ x_4^{'} = \frac{[n\xi+2\sqrt{n}]}{n},$$

and

$$\bar{\psi}_1(x) = \psi(\frac{x-x_1^{'}}{x_2^{'}-x_1^{'}}),\ \bar{\psi}_2(x) = \psi(\frac{x-x_3^{'}}{x_4^{'}-x_3^{'}}).$$

Set

$$\bar{F}_n(f,x) := \bar{F}_n(x) = f(x)(1-\bar{\psi}_1(x)+\bar{\psi}_2(x)) + \bar{\psi}_1(x)(1-\bar{\psi}_2(x))H(x).$$

We have

$$\bar{F}_n(f,x) = \begin{cases} f(x), & x \in [0, x_{r-5/2}] \cup [x_{r+3/2}, 1], \\ f(x)(1-\bar{\psi}_1(x)) + \bar{\psi}_1(x)H(x), & x \in [x_{r-5/2}, x_{r-3/2}], \\ H(x), & x \in [x_{r-3/2}, x_{r+1/2}], \\ H(x)(1-\bar{\psi}_2(x)) + \bar{\psi}_2(x)f(x), & x \in [x_{r+1/2}, x_{r+3/2}]. \end{cases}$$

Obviously, $\bar{F}_n(f,x)$ is linear, reproduces polynomials of degree r, and $\bar{F}_n(f,x) \in C^{(2r)}([0,1])$, provided that $f \in C^{(2r)}([0,1])$. Now, we can define our new combinations of Bernstein operators as follows:

$$\bar{B}_{n,r}(f,x) := B_{n,r}(\bar{F}_n,x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(\bar{F}_n,x), \tag{2.2}$$

where $C_i(n)$ satisfy the conditions (a)-(d). Our main result is the following:

Theorem 1. *For $f \in C_{\bar{w}},\ 0 \leqslant \lambda \leqslant 1,\ 0<\xi<1,\ \alpha>0,\ 0<\alpha_0<r$, we have*

$$\bar{w}(x)|f(x)-\bar{B}_{n,r-1}(f,x)| = O((n^{-\frac{1}{2}}\varphi^{-\lambda}(x)\delta_n(x))^{\alpha_0}) \Longleftrightarrow \omega_{\varphi^\lambda}^r(f,t)_{\bar{w}} = O(t^{\alpha_0}).$$

3 Lemmas

Lemma 1.([3]) *If $\gamma \in R$, then*

$$\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^\gamma \leqslant Cn^{\frac{\gamma}{2}}\varphi^\gamma(x). \tag{3.1}$$

Lemma 2.([3]) *Let $A_n(x) := \bar{w}(x) \sum_{|k-n\xi|\leqslant\sqrt{n}} p_{n,k}(x)$. Then $A_n(x) \leqslant Cn^{-\alpha/2}$ for $0<\xi<1$ and $\alpha>0$.*

Lemma 3. For $0<\xi<1,\ \alpha,\beta>0$, we have

$$\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^{\beta}p_{n,k}(x)\leqslant Cn^{(\beta-\alpha/2)}\varphi^{\beta}(x). \tag{3.2}$$

Proof. By lemma 2, we have

$$\bar{w}(x)^{\frac{1}{2n}}(\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}p_{n,k}(x))^{\frac{2n-1}{2n}}(\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^{2n\beta}p_{n,k}(x))^{\frac{1}{2n}}\leqslant Cn^{(\beta-\alpha/2)}\varphi^{\beta}(x).$$

Lemma 4. *For any* $\alpha>0,\ 0\leqslant\lambda\leqslant 1,\ f\in C_{\bar{w}}$, *we have*

$$\|\bar{w}\bar{B}_{n,r-1}^{(r)}(f)\|\leqslant Cn^{r}\|\bar{w}f\|. \tag{3.3}$$

Proof. We first prove $x\in[0,\frac{1}{n})$ (The same as $x\in(1-\frac{1}{n},1]$), now

$$\begin{aligned}
|\bar{w}(x)\bar{B}_{n,r-1}^{(r)}(f,x)| &\leqslant \bar{w}(x)\sum_{i=0}^{r-2}\frac{n_i!}{(n_i-r)!}\sum_{k=0}^{n_i-r}|C_i(n)\overrightarrow{\Delta}_{\frac{1}{n_i}}^{r}\bar{F}_n(\frac{k}{n_i})|p_{n_i-r,k}(x)\\
&\leqslant C\bar{w}(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=0}^{n_i-r}|C_i(n)\overrightarrow{\Delta}_{\frac{1}{n_i}}^{r}\bar{F}_n(\frac{k}{n_i})|p_{n_i-r,k}(x)\\
&\leqslant C\bar{w}(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=0}^{n_i-r}\sum_{j=0}^{r}C_r^j|C_i(n)\bar{F}_n(\frac{k+r-j}{n_i})|p_{n_i-r,k}(x)\\
&\leqslant C\bar{w}(x)\sum_{i=0}^{r-2}n_i^r\sum_{j=0}^{r}C_r^j|C_i(n)\bar{F}_n(\frac{r-j}{n_i})|p_{n_i-r,0}(x)\\
&\quad +C\bar{w}(x)\sum_{i=0}^{r-2}n_i^r\sum_{j=0}^{r}C_r^j|C_i(n)\bar{F}_n(\frac{n_i-j}{n_i})|p_{n_i-r,n_i-r}(x)\\
&\quad +C\bar{w}(x)\sum_{i=0}^{r-2}n_i^r\sum_{k=1}^{n_i-r-1}\sum_{j=0}^{r}C_r^j|C_i(n)\bar{F}_n(\frac{k+r-j}{n_i})|p_{n_i-r,k}(x)\\
&:= H_1+H_2+H_3.
\end{aligned}$$

We have

$$\begin{aligned}
H_1 &\leqslant C\bar{w}(x)\sum_{i=0}^{r-2}n_i^r(\sum_{j=0}^{r-1}|C_i(n)\bar{F}_n(\frac{r-j}{n_i})|+|\bar{F}_n(0)|)p_{n_i-r,0}(x)\\
&\leqslant Cn^r\|\bar{w}f\|\sum_{i=0}^{r-2}\sum_{j=0}^{r-1}(\frac{n_i|x-\xi|}{r-j-n_i\xi})^{\alpha}(1-x)^{n_i-r}\\
&\leqslant Cn^r\|\bar{w}f\|\sum_{i=0}^{r-2}(n_i|x-\xi|)^{\alpha}(1-x)^{n_i-r}\\
&\leqslant Cn^r\|\bar{w}f\|.
\end{aligned}$$

Similarly, we can get $H_2\leqslant Cn^r\|\bar{w}f\|$, and $H_3\leqslant Cn^r\|\bar{w}f\|$.

When $x \in [\frac{1}{n}, 1-\frac{1}{n}]$, according to [5], we have

$$\begin{aligned} & |\bar{w}(x)\bar{B}^{(r)}_{n,r-1}(f,x)| \\ = & |\bar{w}(x)B^{(r)}_{n,r-1}(\bar{F}_n,x)| \\ = & \bar{w}(x)(\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{k/n_i\in A}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ & +\bar{w}(x)(\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{x_2'\leqslant k/n_i\leqslant x_3'}|(x-\frac{k}{n_i})^jH(\frac{k}{n_i})|p_{n_i,k}(x) \\ := & \sigma_1+\sigma_2. \end{aligned}$$

Where $A := [0,x_2']\cup[x_3',1]$, H is a linear function. If $\frac{k}{n_i}\in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n_i})}\leqslant C(1+n_i^{-\frac{\alpha}{2}}|k-n_ix|^{\alpha})$, we have $|k-n_i\xi|\geqslant\frac{\sqrt{n_i}}{2}$, also $Q_j(x,n_i)=(n_ix(1-x))^{[(r-j)/2]}$, and $(\varphi^2(x))^{-2r}Q_j(x,n_i)n_i^j\leqslant C(n_i/\varphi^2(x))^{r+j/2}$.
By (3.1), then

$$\begin{aligned} \sigma_1 \leqslant & C\varphi^{2r}(x)\bar{w}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ \leqslant & C\varphi^{2r}(x)\|\bar{w}f\|\sum_{i=0}^{r-2}\sum_{j=0}^{r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k=0}^{n_i}[1+n_i^{-\frac{\alpha}{2}}|k-n_ix|^{\alpha}]|x-\frac{k}{n_i}|^jp_{n_i,k}(x) \\ := & I_1+I_2. \end{aligned}$$

By a simple calculation, we have $I_1\leqslant Cn^r\|\bar{w}f\|$. By (3.1), then

$$I_2\leqslant C\|\bar{w}f\|\varphi^{2r}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}|C_i(n)|n_i^{-(\frac{\alpha}{2}+j)}(\frac{n_i}{\varphi^2(x)})^{j/2}\sum_{k=0}^{n_i}|k-n_ix|^{\alpha+j}p_{n_i,k}(x)\leqslant Cn^r\|\bar{w}f\|.$$

We note that $|H(\frac{k}{n_i})|\leqslant max(|H(x_1')|,|H(x_4')|):=H(a)$.

If $x\in[x_1',x_4']$, we have $\bar{w}(x)\leqslant\bar{w}(a)$. So, if $x\in[x_1',x_4']$, then

$$\sigma_2\leqslant Cn^r\bar{w}(a)H(a)\leqslant Cn^r\|\bar{w}f\|.$$

If $x\notin[x_1',x_4']$, then $\bar{w}(a)>n_i^{-\frac{\alpha}{2}}$, by lemma 3, we have

$$\sigma_2\leqslant C\bar{w}(a)H(a)\varphi^{-2r}(x)\bar{w}(x)\sum_{i=0}^{r-2}C_i(n)n_i^{r+\frac{\alpha}{2}}\sum_{x_2'\leqslant k/n_i\leqslant x_3'}p_{n_i,k}(x)\leqslant Cn^r\|\bar{w}f\|.$$

It follows from combining the above inequalities that the lemma is proved. □

Lemma 5. ([15]) *If* $\varphi(x)=\sqrt{x(1-x)},\ 0\leqslant\lambda\leqslant1,\ 0\leqslant\beta\leqslant1,\ \alpha>0,$ *then*

$$\int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\cdots\int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\varphi^{-r\beta}(x+\sum_{k=1}^{r}u_k)du_1\cdots du_r\leqslant Ch^r\varphi^{r(\lambda-\beta)}(x). \tag{3.4}$$

Lemma 6. *For any* $r\in N,\ f\in W^r_{\bar{w},\lambda},\ 0\leqslant\lambda\leqslant1,\ \alpha>0,$ *we have*

$$\|\bar{w}\varphi^{r\lambda}\bar{F}_n^{(r)}\|\leqslant C\|\bar{w}\varphi^{r\lambda}f^{(r)}\|. \tag{3.5}$$

Proof. We first prove $x \in [x_{r-5/2}, x_{r-3/2}]$ (The same as the others), we have

$$\begin{aligned} |\bar{w}(x)\varphi^{r\lambda}(x)\bar{F}_n^{(r)}(x)| &\leqslant |\bar{w}(x)\varphi^{r\lambda}(x)f^{(r)}(x)| + |\bar{w}(x)\varphi^{r\lambda}(x)(f(x) - \bar{F}_n(x))^{(r)}| \\ &:= I_1 + I_2. \end{aligned}$$

Obviously

$$I_1 \leqslant C\|\bar{w}\varphi^{r\lambda}f^{(r)}\|.$$

For I_2, we have

$$I_2 = \bar{w}(x)\varphi^{r\lambda}(x)|(f(x) - \bar{F}_n(x))^{(r)}| = \bar{w}(x)\varphi^{r\lambda}(x)\sum_{i=0}^{r} n^{\frac{i}{2}}|(f(x) - \bar{F}_n(x))^{(r-i)}|.$$

By [5], we have

$$|(f(x) - \bar{F}_n(x))^{(r-i)}|_{[x_{r-5/2}, x_{r-3/2}]} \leqslant C(n^{(r-i)/2}\|f - H\|_{[x_{r-5/2}, x_{r-3/2}]} + n^{-i/2}\|f^{(r)}\|_{[x_{r-5/2}, x_{r-3/2}]}).$$

So

$$\begin{aligned} I_2 &\leqslant Cn^{\frac{r}{2}}\bar{w}(x)\varphi^{r\lambda}(x)\|f - H\|_{[x_{r-5/2}, x_{r-3/2}]} + C\bar{w}(x)\varphi^{r\lambda}(x)\|f^{(r)}\|_{[x_{r-5/2}, x_{r-3/2}]} \\ &:= T_1 + T_2. \end{aligned}$$

By Taylor expansion, we have

$$f(x_i) = \sum_{u=0}^{r-1}\frac{(x_i - x)^u}{u!}f^{(u)}(x) + \frac{1}{(r-1)!}\int_x^{x_i}(x_i - s)^{r-1}f^{(r)}(s)ds, \tag{3.6}$$

It follows from (3.6) and the identities

$$\sum_{i=1}^{r} x_i^v l_i(x) = Cx^v, \ v = 0, 1, \cdots, r.$$

we have

$$\begin{aligned} H(f, x) &= \sum_{i=1}^{r}\sum_{u=0}^{r}\frac{(x_i - x)^u}{u!}f^{(u)}(x)l_i(x) + \frac{1}{(r-1)!}\sum_{i=1}^{r} l_i(x)\int_x^{x_i}(x_i - s)^{r-1}f^{(r)}(s)ds \\ &= f(x) + \sum_{u=1}^{r} f^{(u)}(x)(\sum_{v=0}^{u} C_u^v(-x)^{u-v}\sum_{i=1}^{r} x_i^v l_i(x)) \\ &\quad + \frac{1}{(r-1)!}\sum_{i=1}^{r} l_i(x)\int_x^{x_i}(x_i - s)^{r-1}f^{(r)}(s)ds, \end{aligned}$$

which implies that

$$\bar{w}(x)\varphi^{r\lambda}(x)|f(x) - H(f, x)| = \frac{1}{(r-1)!}\bar{w}(x)\varphi^{r\lambda}(x)\sum_{i=1}^{r} l_i(x)\int_x^{x_i}(x_i - s)^{r-1}f^{(r)}(s)ds,$$

since $|l_i(x)| \leqslant C$ for $x \in [x_{r-5/2}, x_{r-3/2}]$, $i = 1, 2, \cdots, r$. It follows from $\frac{|x_i - s|^{r-1}}{\bar{w}(s)} \leqslant \frac{|x_i - x|^{r-1}}{\bar{w}(x)}$, s between x_i and x, then

$$\begin{aligned} \bar{w}(x)\varphi^{r\lambda}(x)|f(x) - H(f, x)| &= C\bar{w}(x)\varphi^{r\lambda}(x)\sum_{i=1}^{r}\int_x^{x_i}(x_i - s)^{r-1}|f^{(r)}(s)|ds \\ &\leqslant C\varphi^{r\lambda}(x)\|\bar{w}\varphi^{r\lambda}f^{(r)}\|\sum_{i=1}^{r}(x_i - x)^{r-1}\int_x^{x_i}\varphi^{-r\lambda}(s)ds \\ &\leqslant \frac{C}{n^{r/2}}\|\bar{w}\varphi^{r\lambda}f^{(r)}\|. \end{aligned}$$

So

$$I_2 \leqslant C\|\bar{w}\varphi^{r\lambda} f^{(r)}\|.$$

Then, the lemma is proved. □

Lemma 7. *For any* $g \in W^r_{\bar{w},\lambda},\ 0 \leqslant \lambda \leqslant 1,\ \alpha > 0,$ *we have*

$$\bar{w}(x)|g(x) - H(g,x)| \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|\bar{w}\varphi^{r\lambda}g^{(r)}\|. \tag{3.7}$$

Proof. According to the lemma 6, we only give a simple proof.

$$\bar{w}(x)|g(x) - H(g,x)| = \frac{1}{(r-1)!}\bar{w}(x)\sum_{i=1}^{r} l_i(x)\int_x^{x_i}(x_i - s)^{r-1}g^{(r)}(s)ds,$$

since $|l_i(x)| \leqslant C$ for $x \in [x_{r-5/2}, x_{r-3/2}],\ i = 1, 2, \cdots, r$. It follows from $\frac{|x_i - s|^{r-1}}{\bar{w}(s)} \leqslant \frac{|x_i - x|^{r-1}}{\bar{w}(x)}$, s between x_i and x, then

$$\begin{aligned}
\bar{w}(x)|g(x) - H(g,x)| &\leqslant C\bar{w}(x)\sum_{i=1}^{r}\int_x^{x_i}(x_i - s)^{r-1}|g^{(r)}(s)|ds \\
&\leqslant C\|\bar{w}\varphi^{r\lambda}g^{(r)}\|\sum_{i=1}^{r}(x_i - x)^{r-1}\int_x^{x_i}\varphi^{-r\lambda}(s)ds \\
&\leqslant C\frac{\varphi^r(x)}{\varphi^{r\lambda}(x)}\|\bar{w}\varphi^{r\lambda}g^{(r)}\|\sum_{i=1}^{r}(x_i - x)^{r-1}\int_x^{x_i}\varphi^{-r}(s)ds \\
&\leqslant C\frac{\delta_n^r(x)}{\varphi^{r\lambda}(x)}\|\bar{w}\varphi^{r\lambda}g^{(r)}\|\sum_{i=1}^{r}(x_i - x)^{r-1}\int_x^{x_i}\varphi^{-r}(s)ds \\
&\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|\bar{w}\varphi^{r\lambda}g^{(r)}\|.\square
\end{aligned}$$

Lemma 8. *If* $r \in N,\ 0 \leqslant \lambda \leqslant 1,\ f \in W^r_{\bar{w},\lambda},\ \alpha > 0,$ *we have*

$$|\bar{w}(x)\varphi^{r\lambda}(x)\bar{B}^{(r)}_{n,r-1}(f,x)| \leqslant C\|\bar{w}\varphi^{r\lambda}f^{(r)}\|. \tag{3.8}$$

Proof. It follows from $\frac{|t-u|}{\varpi(u)} \leqslant \frac{|t-x|}{\varpi(x)}$, u between t and x, let $t=0$, we have

$$\begin{aligned}
n_i^r|\overrightarrow{\Delta}^r_{\frac{1}{n_i}}\bar{F}_n(\frac{k}{n_i})| &= n_i^r\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}\cdots\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}\bar{F}_n^{(r)}(x+\frac{rh}{2}+u_1+\cdots+u_r)du_1\cdots du_r\\
&\leqslant Cn_i^r\|\varpi\varphi^{r\lambda}\bar{F}_n^{(r)}\|\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}\cdots\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}\varpi^{-1}(x+\frac{rh}{2}+u_1+\cdots+u_r).\\
&\quad\varphi^{-r\lambda}(x+\frac{rh}{2}+u_1+\cdots+u_r)du_1\cdots du_r\\
&= Cn_i^r\|\varpi\varphi^{r\lambda}\bar{F}_n^{(r)}\|\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}\cdots\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}\frac{(x+\frac{rh}{2}+u_1+\cdots+u_r)}{\varpi(x+\frac{rh}{2}+u_1+\cdots+u_r)}\cdot\\
&\quad\frac{\varphi^{-r\lambda}(x+\frac{rh}{2}+u_1+\cdots+u_r)}{(x+\frac{rh}{2}+u_1+\cdots+u_r)}du_1\cdots du_r\\
&\leqslant Cn_i^r\|\varpi\varphi^{r\lambda}\bar{F}_n^{(r)}\|\frac{x}{\varpi(x)}\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}\cdots\int_{-\frac{1}{2n_i}}^{\frac{1}{2n_i}}(x+\frac{rh}{2}+u_1+\cdots+u_r)^{-(\frac{r\lambda}{2}+1)}\cdot\\
&\quad[1-(x+\frac{rh}{2}+u_1+\cdots+u_r)]^{-\frac{r\lambda}{2}}du_1\cdots du_r\\
&\leqslant C\varpi^{-1}(x)\varphi^{-r\lambda}(x)\|\varpi\varphi^{r\lambda}\bar{F}_n^{(r)}\|\\
&\leqslant C\varpi^{-1}(x)\varphi^{-r\lambda}(x)\|\varpi\varphi^{r\lambda}f^{(r)}\|.
\end{aligned}$$

By [5], we have

$$\bar{B}_{n,r-1}^{(r)}(f,x)=\sum_{i=0}^{r-2}\frac{n_i!}{(n_i-r)!}\sum_{k=0}^{n_i-r}C_i(n)\overrightarrow{\Delta}^r_{\frac{1}{n_i}}\bar{F}_n(\frac{k}{n_i})p_{n_i-r,k}(x).$$

Obviously

$$|\varpi(x)\varphi^{r\lambda}(x)\bar{B}_{n,r-1}^{(r)}(f,x)|\leqslant C\|\varpi\varphi^{r\lambda}f^{(r)}\|.\Box$$

Lemma 9. *If* $r\in N,\ 0\leqslant\lambda\leqslant 1,\ f\in C_{\varpi},\ \alpha>0,$ *we have*

$$|\varpi(x)\varphi^{r\lambda}(x)\bar{B}_{n,r-1}^{(r)}(f,x)|\leqslant Cn^{r/2}\{max\{n^{r(1-\lambda)/2},\varphi^{r(\lambda-1)}\}\}\|\varpi f\|. \tag{3.9}$$

Proof. *Case 1.* If $0\leqslant\varphi(x)\leqslant\frac{1}{\sqrt{n}}$, by (3.3), we have

$$|\varpi(x)\varphi^{r\lambda}(x)\bar{B}_{n,r-1}^{(r)}(f,x)|\leqslant Cn^{-r\lambda/2}|\varpi(x)\bar{B}_{n,r-1}^{(r)}(f,x)|\leqslant Cn^{r(1-\lambda/2)}\|\varpi f\|.$$

Case 2. If $\varphi(x)>\frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}
&|\bar{B}_{n,r-1}^{(r)}(f,x)|=|B_{n,r-1}^{(r)}(\bar{F}_n,x)|\\
\leqslant\ &(\varphi^2(x))^{-r}\sum_{i=0}^{r-2}\sum_{j=0}^{r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x),
\end{aligned}$$

where
$Q_j(x,n_i)=(n_ix(1-x))^{[(2r-j)/2]}$, and $(\varphi^2(x))^{-2r}Q_j(x,n_i)n_i^j\leqslant C(n_i/\varphi^2(x))^{r+j/2}$.

So

$$\begin{aligned}
&|\bar{w}(x)\varphi^{r\lambda}(x)\bar{B}^{(r)}_{n,r-1}(f,x)| \\
\leqslant\ & C\bar{w}(x)\varphi^{r(\lambda+2)}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\
=\ & C\bar{w}(x)\varphi^{r(\lambda+2)}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k/n_i\in A}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\
&+C\bar{w}(x)\varphi^{r(\lambda+2)}(x)\sum_{i=0}^{r-2}\sum_{j=0}^{r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{x_2'\leqslant k/n_i\leqslant x_3'}|(x-\frac{k}{n_i})^jH(\frac{k}{n_i})|p_{n_i,k}(x) \\
:=\ & \sigma_1+\sigma_2.
\end{aligned}$$

Where $A := [0, x_2'] \cup [x_3', 1]$. According to lemma 3, we can easily get $\sigma_1 \leqslant Cn^{\frac{r}{2}}\varphi^{r(\lambda-1)}(x)\|\bar{w}f\|$, and $\sigma_2 \leqslant Cn^{\frac{r}{2}}\varphi^{r(\lambda-1)}(x)\|\bar{w}f\|$. By bringing these facts together, the lemma is proved. □

4 Proof of Theorem

The direct theorem

We know

$$\bar{F}_n(t) = \bar{F}_n(x) + \bar{F}_n'(t)(t-x) + \cdots + \frac{1}{(r-1)!}\int_x^t (t-u)^{r-1}\bar{F}_n^{(r)}(u)du, \tag{4.1}$$

$$B_{n,r-1}((\cdot - x)^k, x) = 0,\ \ k = 1, 2, \cdots, r-1. \tag{4.2}$$

According to the definition of $W^r_{\bar{w},\lambda}$, for any $g \in W^r_{\bar{w},\lambda}$, we have $\bar{B}_{n,r-1}(g,x) = B_{n,r-1}(\bar{G}_n(g), x)$. and $\bar{w}(x)|\bar{G}_n(x) - B_{n,r-1}(\bar{G}_n, x)| = \bar{w}(x)|B_{n,r-1}(R_r(\bar{G}_n, t, x), x)|$, thereof $R_r(\bar{G}_n, t, x) = \int_x^t (t-u)^{r-1}\bar{G}_n^{(r)}(u)du$.
It follows from $\frac{|t-u|^{r-1}}{\bar{w}(u)} \leqslant \frac{|t-x|^{r-1}}{\bar{w}(x)}$, u between t and x, we have

$$\begin{aligned}
\bar{w}(x)|\bar{G}_n(x) - B_{n,r-1}(\bar{G}_n, x)| &\leqslant C\|\bar{w}\varphi^{r\lambda}\bar{G}_n^{(r)}\|\bar{w}(x)B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{\bar{w}(u)\varphi^{r\lambda}(u)}du, x) \\
&\leqslant C\|\bar{w}\varphi^{r\lambda}\bar{G}_n^{(r)}\|\bar{w}(x)(B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{\varphi^{2r\lambda}(u)}|du, x))^{\frac{1}{2}} \cdot \\
&\quad (B_{n,r-1}(\int_x^t \frac{|t-u|^{r-1}}{\bar{w}^2(u)}du, x))^{\frac{1}{2}}.
\end{aligned} \tag{4.3}$$

also

$$\int_x^t \frac{|t-u|^{r-1}}{\varphi^{2r\lambda}(u)}du \leqslant C\frac{|t-x|^r}{\varphi^{2r\lambda}(x)},\ \int_x^t \frac{|t-u|^{r-1}}{\bar{w}^2(u)}du \leqslant \frac{|t-x|^r}{\bar{w}^2(x)}. \tag{4.4}$$

By (3.1), (4.3) and (4.4), we have

$$\begin{aligned}
\bar{w}(x)|\bar{G}_n(x) - B_{n,r-1}(\bar{G}_n, x)| &\leqslant C\|\bar{w}\varphi^{r\lambda}\bar{G}_n^{(r)}\|\varphi^{-r\lambda}(x)B_{n,r-1}(|t-x|^r, x) \\
&\leqslant Cn^{-\frac{r}{2}}\frac{\varphi^r(x)}{\varphi^{r\lambda}(x)}\|\bar{w}\varphi^{r\lambda}\bar{G}_n^{(r)}\| \\
&\leqslant Cn^{-\frac{r}{2}}\frac{\delta_n^r(x)}{\varphi^{r\lambda}(x)}\|\bar{w}\varphi^{r\lambda}\bar{G}_n^{(r)}\| \\
&= C(\frac{\delta_n(x)}{\sqrt{n}\varphi^\lambda(x)})^r\|\bar{w}\varphi^{r\lambda}\bar{G}_n^{(r)}\|.
\end{aligned} \tag{4.5}$$

By (3.7) and (4.5), when $g \in W^r_{\bar{w},\lambda}$, then

$$\begin{aligned} \bar{w}(x)|g(x)-\bar{B}_{n,r-1}(g,x)| &\leqslant \bar{w}(x)|g(x)-\bar{G}_n(g,x)|+\bar{w}(x)|\bar{G}_n(g,x)-\bar{B}_{n,r-1}(g,x)| \\ &\leqslant \bar{w}(x)|g(x)-H(g,x)|_{[x_1,x_4]}+C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|\bar{w}\varphi^{r\lambda}\bar{G}_n^{(r)}\| \\ &\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|\bar{w}\varphi^{r\lambda}g^{(r)}\|. \qquad (4.6) \end{aligned}$$

For $f \in C_{\bar{w}}$, we choose proper $g \in W^r_{\bar{w},\lambda}$, by (4.6), then

$$\begin{aligned} \bar{w}(x)|f(x)-\bar{B}_{n,r-1}(f,x)| &\leqslant \bar{w}(x)|f(x)-g(x)|+\bar{w}(x)|\bar{B}_{n,r-1}(f-g,x)|+\bar{w}(x)|g(x)-\bar{B}_{n,r-1}(g,x) \\ &\leqslant C(\|\bar{w}(f-g)\|+(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^r\|\bar{w}\varphi^{r\lambda}g^{(r)}\|) \\ &\leqslant C\omega^r_{\varphi^{\lambda}}(f,\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})_{\bar{w}}.\square \end{aligned}$$

The inverse theorem

We define the weighted main-part modulus for $D = R_+$ by

$$\Omega^r_{\varphi^{\lambda}}(C,f,t)_{\bar{w}} = \sup_{0<h\leqslant t}\|\bar{w}\Delta^r_{h\varphi^{\lambda}}f\|_{[Ch^*,\infty]},$$
$$\Omega^r_{\varphi^{\lambda}}(1,f,t)_{\bar{w}} = \Omega^r_{\varphi^{\lambda}}(f,t)_{\bar{w}}.$$

where $C > 2^{1/\beta(0)-1}$, $\beta(0) > 0$, and h^* is given by

$$h^* = \begin{cases} (Ar)^{1/1-\beta(0)}h^{1/1-\beta(0)}, & 0 \leqslant \beta(0) < 1, \\ 0, & \beta(0) \geqslant 1. \end{cases}$$

The main-part K-functional is given by

$$H^r_{\varphi^{\lambda}}(f,t^r)_{\bar{w}} = \sup_{0<h\leqslant t}\inf_g\{\|\bar{w}(f-g)\|_{[Ch^*,\infty]}+t^r\|\bar{w}\varphi^{r\lambda}g^{(r)}\|_{[Ch^*,\infty]},\ g^{(r-1)} \in A.C.((Ch^*,\infty))\}.$$

By [5], we have

$$C^{-1}\Omega^r_{\varphi^{\lambda}}(f,t)_{\bar{w}} \leqslant \omega^r_{\varphi^{\lambda}}(f,t)_{\bar{w}} \leqslant C\int_0^t \frac{\Omega^r_{\varphi^{\lambda}}(f,\tau)_{\bar{w}}}{\tau}d\tau, \qquad (4.7)$$
$$C^{-1}H^r_{\varphi^{\lambda}}(f,t^r)_{\bar{w}} \leqslant \Omega^r_{\varphi^{\lambda}}(f,t)_{\bar{w}} \leqslant CH^r_{\varphi^{\lambda}}(f,t^r)_{\bar{w}}. \qquad (4.8)$$

Proof. Let $\delta > 0$, by (4.8), we choose proper g so that

$$\|\bar{w}(f-g)\| \leqslant C\Omega^r_{\varphi^{\lambda}}(f,t)_{\bar{w}},\ \|\bar{w}\varphi^{r\lambda}g^{(r)}\| \leqslant C\delta^{-r}\Omega^r_{\varphi^{\lambda}}(f,t)_{\bar{w}}. \qquad (4.9)$$

then

$$\begin{aligned} |\bar{w}(x)\Delta^r_{h\varphi^{\lambda}}f(x)| &\leqslant |\bar{w}(x)\Delta^r_{h\varphi^{\lambda}}(f(x)-\bar{B}_{n,r-1}(f,x))|+|\bar{w}(x)\Delta^r_{h\varphi^{\lambda}}\bar{B}_{n,r-1}(f-g,x)| \\ &+ |\bar{w}(x)\Delta^r_{h\varphi^{\lambda}}\bar{B}_{n,r-1}(g,x)| \\ &\leqslant \sum_{j=0}^{r}C_r^j(n^{-\frac{1}{2}}\delta_n(x+(\frac{r}{2}-j)h\varphi^{\lambda}(x)))^{\alpha_0} \\ &+ \int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\cdots\int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\bar{w}(x)\bar{B}^{(r)}_{n,r-1}(f-g,x+\sum_{k=1}^{r}u_k)du_1\cdots du_r \\ &+ \int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\cdots\int_{-\frac{h\varphi^{\lambda}(x)}{2}}^{\frac{h\varphi^{\lambda}(x)}{2}}\bar{w}(x)\bar{B}^{(r)}_{n,r-1}(g,x+\sum_{k=1}^{r}u_k)du_1\cdots du_r \\ &:= J_1+J_2+J_3. \end{aligned}$$

Obviously

$$J_1 \leqslant C(n^{-\frac{1}{2}}\delta_n(x))^{\alpha_0}. \tag{4.10}$$

By (3.3) and (4.9), we have

$$\begin{aligned} J_2 &\leqslant Cn^r\|\bar{w}(f-g)\| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} du_1 \cdots du_r \\ &\leqslant Cn^r h^r \varphi^{r\lambda}(x)\|\bar{w}(f-g)\| \\ &\leqslant Cn^r h^r \varphi^{r\lambda}(x)\Omega^r_{\varphi^\lambda}(f,\delta)_{\bar{w}}. \end{aligned} \tag{4.11}$$

By (3.9)we let $\lambda = 1$, and (3.4) as well as (4.9), we have

$$\begin{aligned} J_2 &\leqslant Cn^{\frac{r}{2}}\|\bar{w}(f-g)\| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \varphi^{-r}(x+\sum_{k=1}^r u_k) du_1 \cdots du_r \\ &\leqslant Cn^{\frac{r}{2}} h^r \varphi^{r(\lambda-1)}(x)\|\bar{w}(f-g)\| \\ &\leqslant Cn^{\frac{r}{2}} h^r \varphi^{r(\lambda-1)}(x)\Omega^r_{\varphi^\lambda}(f,\delta)_{\bar{w}}. \end{aligned} \tag{4.12}$$

By (3.8) and (4.9), we have

$$\begin{aligned} J_3 &\leqslant C\|\bar{w}\varphi^{r\lambda}g^{(r)}\|\bar{w}(x) \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \bar{w}^{-1}(x+\sum_{k=1}^r u_k)\varphi^{-r\lambda}(x+\sum_{k=1}^r u_k) du_1 \cdots du_r \\ &\leqslant Ch^r\|\bar{w}\varphi^{r\lambda}g^{(r)}\| \\ &\leqslant Ch^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_{\bar{w}}. \end{aligned} \tag{4.13}$$

Now, by (4.10), (4.11), (4.12) and (4.13), we get

$$|\bar{w}(x)\Delta^r_{h\varphi^\lambda}f(x)| \leqslant C\{(n^{-\frac{1}{2}}\delta_n(x))^{\alpha_0} + h^r(n^{-\frac{1}{2}}\delta_n(x))^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_{\bar{w}} + h^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_{\bar{w}}\}.$$

When $n \geqslant 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x) < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x) \leqslant \sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper $x, n \in N$, so that

$$n^{-\frac{1}{2}}\delta_n(x) \leqslant \delta < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x),$$

Therefore

$$|\bar{w}(x)\Delta^r_{h\varphi^\lambda}f(x)| \leqslant C\{\delta^{\alpha_0} + h^r\delta^{-r}\Omega^r_{\varphi^\lambda}(f,\delta)_{\bar{w}}\}.$$

By Borens-Lorentz lemma, we get

$$\Omega^r_{\varphi^\lambda}(f,t)_{\bar{w}} \leqslant Ct^{\alpha_0}. \tag{4.14}$$

So, by (4.14), we get

$$\omega^r_{\varphi^\lambda}(f,t)_{\bar{w}} \leqslant C\int_0^t \frac{\Omega^r_{\varphi^\lambda}(f,\tau)_{\bar{w}}}{\tau} d\tau = C\int_0^t \tau^{\alpha_0-1} d\tau = Ct^{\alpha_0}.\Box$$

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