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Pointwise weighted approximation of functions with inner singularities by Bernstein operators

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Abstract.

We consider the pointwise weighted approximation by Bernstein operators with inner singularities. The related weight functions are weights $\bar{w}(x) = |x-\xi|^{\alpha}(0<\xi<1,\ \alpha>0)$.

In this paper we give direct and inverse results of this type of Bernstein polynomials.

Keywords: Pointwise weighted approximation; Bernstein operators; inner singularities.

1 Introduction

The set of all continuous functions, defined on the interval *I*, is denoted by *C(I)*. For any $f \in C([0,1])$, the corresponding *Bernstein operators* are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x),$$

Where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ k=0,1,2,\ldots,n,\ x\in[0,1].$$

Approximation properties of Bernstein operators have been studied very well. Berens and Lorentz showed in [1] that

$$B_n(f,x) - f(x) = O((\frac{1}{\sqrt{n}}\delta_n(x))^{\alpha_1}) \Longleftrightarrow \omega^1(f,t) = O(t^{\alpha_1}),$$

where $0<\alpha_1<1,\ \delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}},\ \varphi(x) = \sqrt{x(1-x)}.$

It is well known that approximation of functions with singularities by polynomial is of special value in both theories and applications. As an important type of polynomial approximation, approximation of functions by Bernstein operators is an important topic in both approximation theory and computational theory, which plays an important role in neural networks, fitting date, curves, and surfaces. Some work has been done by [2]. Throughout the paper, C denotes a positive constant independent of n and x, which may be different in different cases.

Let $\bar{w}(x) = |x-\xi|^{\alpha},\ 0 < \xi < 1,\ \alpha > 0$ and $C_{\bar{w}} := \{f \in C([0,1] \setminus \{\xi\}) : \lim_{x \longrightarrow \xi} (\bar{w} f)(x) = 0\}$. The *norm* in $C_{\bar{w}}$ is defined by $\|f\|_{C_{\bar{w}}} := \|\bar{w} f\| = \sup_{0 \leqslant x \leqslant 1} |(\bar{w} f)(x)|$. Define

$$W^2_{\bar{w},\lambda} := \{f \in C_{\bar{w}} : f' \in A.C.((0,1)),\ \|\bar{w}\varphi^{2\lambda} f''\| < \infty\}.$$

For $f \in C_{\bar{w}}$, the *weighted modulus of smoothness* is defined by

$$\omega^2_{\varphi}(f,t)_{\bar{w}} := \sup_{0<h\leqslant t} \{\|\bar{w}\Delta^2_{h\varphi} f\|_{[16h^2,1-16h^2]} + \|\bar{w}\overrightarrow{\Delta}^2_h f\|_{[0,16h^2]} + \|\bar{w}\overleftarrow{\Delta}^2_h f\|_{[1-16h^2,1]}\},$$

where

$$\begin{aligned}
\Delta^2_{h\varphi} f(x) &= f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)),\\
\overrightarrow{\Delta}^2_h f(x) &= f(x+2h) - 2f(x+h) + f(x),\\
\overleftarrow{\Delta}^2_h f(x) &= f(x-2h) - 2f(x-h) + f(x),
\end{aligned}$$

and $\varphi(x) = \sqrt{x(1-x)},\ \delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$.

Let

$$\psi(x) = \begin{cases} 10x^3 - 15x^4 + 6x^5, & 0 < x < 1,\\ 0, & x \leqslant 0,\\ 1, & x \geqslant 1. \end{cases}$$

Obviously, ψ is non-decreasing on the real axis, $\psi \in C^2((-\infty,+\infty)),\ \psi^{(i)}(0) = 0,\ i = 0,1,2.\ \psi^{(i)}(1) = 0,\ i = 1,2$ and $\psi(1) = 1$. Further, let

$$x_1 = \frac{[n\xi - 2\sqrt{n}]}{n},\ x_2 = \frac{[n\xi - \sqrt{n}]}{n},\ x_3 = \frac{[n\xi + \sqrt{n}]}{n},\ x_4 = \frac{[n\xi + 2\sqrt{n}]}{n},$$

and

$$\bar{\psi}_1(x) = \psi(\frac{x-x_1}{x_2-x_1}),\ \bar{\psi}_2(x) = \psi(\frac{x-x_3}{x_4-x_3}).$$

Consider

$$P(x) := \frac{x-x_4}{x_1-x_4} f(x_1) + \frac{x_1-x}{x_1-x_4} f(x_4),$$

the linear function joining the points $(x_1, f(x_1))$ and $(x_4, f(x_4))$. And let

$$\bar{F}_n(f,x) := \bar{F}_n(x) = f(x)(1-\bar{\psi}_1(x)+\bar{\psi}_2(x)) + \bar{\psi}_1(x)(1-\bar{\psi}_2(x))P(x).$$

From the above definitions it follows that

$$\bar{F}_n(f,x) = \begin{cases} f(x), & x \in [0,x_1] \cup [x_4,1],\\ f(x)(1-\bar{\psi}_1(x)) + \bar{\psi}_1(x)P(x), & x \in [x_1,x_2],\\ P(x), & x \in [x_2,x_3],\\ P(x)(1-\bar{\psi}_2(x)) + \bar{\psi}_2(x) f(x), & x \in [x_3,x_4]. \end{cases}$$

Evidently, $\bar{F}_n$ is a positive linear operator which depends on the functions values $f(k/n),\ 0 \leqslant k/n \leqslant x_2$ or $x_3 \leqslant k/n \leqslant 1$, it reproduces linear functions, and $\bar{F}_n \in C^2([0,1])$ provided

$f \in W^2_{\bar{w},\lambda}$. Now for every $f \in C_{\bar{w}}$ define the *Bernstein type operator*

$$\begin{aligned}\bar{B}_n(f,x) &:= B_n(\bar{F}_n(f),x)\\ &= \sum_{k/n\in[0,x_1]\cup[x_4,1]} p_{n,k}(x)f(\frac{k}{n}) + \sum_{x_2<k/n<x_3} p_{n,k}(x)P(\frac{k}{n})\\ &+ \sum_{x_1<k/n<x_2} p_{n,k}(x)\{f(\frac{k}{n})(1-\bar{\psi}_1(\frac{k}{n})) + \bar{\psi}_1(\frac{k}{n})P(\frac{k}{n})\}\\ &+ \sum_{x_3<k/n<x_4} p_{n,k}(x)\{P(\frac{k}{n})(1-\bar{\psi}_2(\frac{k}{n})) + \bar{\psi}_2(\frac{k}{n})f(\frac{k}{n})\}\end{aligned} \tag{1.1}$$

Obviously, $\bar{B}_n$ is a positive linear operator, $\bar{B}_n(f)$ is a polynomial of degree at most n, it preserves linear functions, and depends only on the function values $f(k/n),\ k/n \in [0,x_2]\cup[x_3,1]$. Now we state our main results as follows:

Theorem 1. *If* $\alpha > 0$, *for any* $f \in C_{\bar{w}}$, *we have* $\|\bar{w}\bar{B}''_n(f)\| \leqslant Cn^2\|\bar{w}f\|$.

Theorem 2. *For any* $\alpha > 0,\ 0 \leqslant \lambda \leqslant 1$, *we have*

$$|\bar{w}(x)\varphi^{2\lambda}(x)\bar{B}''_n(f,x)| \leqslant \begin{cases} Cn\{max\{n^{1-\lambda}, \varphi^{2(\lambda-1)}\}\}\|\bar{w}f\|, & f \in C_{\bar{w}},\\ C\|\bar{w}\varphi^{2\lambda}f''\|, & f \in W^2_{\bar{w},\lambda}.\end{cases}$$

Theorem 3. *For* $f \in C_{\bar{w}},\ 0 < \xi < 1,\ \alpha > 0,\ \alpha_0 \in (0,2)$, *we have*

$$\bar{w}(x)|f(x) - \bar{B}_n(f,x)| = O((n^{-\frac{1}{2}}\varphi^{-\lambda}(x)\delta_n(x))^{\alpha_0}) \Longleftrightarrow \omega^2_{\varphi^\lambda}(f,t)_{\bar{w}} = O(t^{\alpha_0}).$$

2 Lemmas

Lemma 1.([9]) *For any non-negative real* u *and* v, *we have*

$$\sum_{k=1}^{n-1}(\frac{k}{n})^{-u}(1-\frac{k}{n})^{-v}p_{n,k}(x) \leqslant Cx^{-u}(1-x)^{-v}. \tag{2.1}$$

Lemma 2.([2]) *For any* $\alpha > 0,\ f \in C_{\bar{w}}$, *we have*

$$\|\bar{w}B_n(f)\| \leqslant C\|\bar{w}f\|. \tag{2.2}$$

Lemma 3.([8]) *If* $\varphi(x) = \sqrt{x(1-x)},\ 0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \beta \leqslant 1$, *then*

$$\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \varphi^{-r\beta}(x + \sum_{k=1}^{r} u_k)du_1\cdots du_r \leqslant Ch^r\varphi^{r(\lambda-\beta)}(x). \tag{2.3}$$

Lemma 4.([2]) *If* $\gamma \in R$, *then*

$$\sum_{k=0}^{n} p_{n,k}(x)|k-nx|^\gamma \leqslant Cn^{\frac{\gamma}{2}}\varphi^\gamma(x). \tag{2.4}$$

Lemma 5. *Let* $A_n(x) := \bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}} p_{n,k}(x)$. *Then* $A_n(x) \leqslant Cn^{-\alpha/2}$ *for* $0 < \xi < 1$ *and* $\alpha > 0$.

Proof. If $|x-\xi| \leqslant \frac{3}{\sqrt{n}}$, then the statement is trivial. Hence assume $0 \leqslant x \leqslant \xi - \frac{3}{\sqrt{n}}$ (the case $\xi + \frac{3}{\sqrt{n}} \leqslant x \leqslant 1$ can be treated similarly). Then for a fixed x the maximum of $p_{n,k}(x)$ is attained for $k = k_n := [n\xi - \sqrt{n}]$. By using Stirling's formula, we get

$$\begin{aligned} p_{n,k_n}(x) &\leqslant C\frac{(\frac{n}{e})^n\sqrt{n}x^{k_n}(1-x)^{n-k_n}}{(\frac{k_n}{e})^{k_n}\sqrt{k_n}(\frac{n-k_n}{e})^{n-k_n}\sqrt{n-k_n}} \\ &\leqslant \frac{C}{\sqrt{n}}(\frac{nx}{k_n})^{k_n}(\frac{n(1-x)}{n-k_n})^{n-k_n} \\ &= \frac{C}{\sqrt{n}}(1-\frac{k_n-nx}{k_n})^{k_n}(1+\frac{k_n-nx}{n-k_n})^{n-k_n}. \end{aligned}$$

Now from the inequalities

$$k_n - nx = [n\xi - \sqrt{n}] - nx > n(\xi - x) - \sqrt{n} - 1 \geqslant \frac{1}{2}n(\xi - x),$$

and

$$1 - u \leqslant e^{-u-\frac{1}{2}u^2},\ 1 + u \leqslant e^u,\ u \geqslant 0,$$

it follows that the second inequality is valid. To prove the first one we consider the function $\lambda(u) = e^{-u-\frac{1}{2}u^2} + u - 1$. Here $\lambda(0) = 0,\ \lambda'(u) = -(1+u)e^{-u-\frac{1}{2}u^2} + 1,\ \lambda'(0) = 0,\ \lambda''(u) = u(u+2)e^{-u-\frac{1}{2}u^2} \geqslant 0$, whence $\lambda(u) \geqslant 0$ for $u \geqslant 0$. Hence

$$\begin{aligned} p_{n,k_n}(x) &\leqslant \frac{C}{\sqrt{n}}exp\{k_n[-\frac{k_n-nx}{k_n} - \frac{1}{2}(\frac{k_n-nx}{k_n})^2] + k_n - nx\} \\ &= \frac{C}{\sqrt{n}}exp\{-\frac{(k_n-nx)^2}{2k_n}\} \leqslant e^{-Cn(\xi-x)^2}. \end{aligned}$$

Thus $A_n(x) \leqslant C(\xi - x)^\alpha e^{-Cn(\xi-x)^2}$. An easy calculation shows that here the maximum is attained when $\xi - x = \frac{C}{\sqrt{n}}$ and the lemma follows. □

Lemma 6. *For $0 < \xi < 1,\ \alpha,\ \beta > 0$, we have*

$$\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^\beta p_{n,k}(x) \leqslant Cn^{(\beta-\alpha)/2}\varphi^\beta(x). \tag{2.5}$$

Proof. By (2.4) and the lemma 5, we have

$$\bar{w}(x)^{\frac{1}{2n}}(\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}p_{n,k}(x))^{\frac{2n-1}{2n}}(\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^{2n\beta}p_{n,k}(x))^{\frac{1}{2n}} \leqslant Cn^{(\beta-\alpha)/2}\varphi^\beta(x).\square$$

Lemma 7. *For any $\alpha > 0,\ f \in W^2_{\bar{w},\lambda}$, we have*

$$\bar{w}(x)|f(x) - P(f,x)|_{[x_1,x_4]} \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^\lambda(x)})^2\|\bar{w}\varphi^{2\lambda}f''\|. \tag{2.6}$$

Proof. If $x \in [x_1, x_4]$, for any $f \in W^2_{\bar{w},\lambda}$, we have

$$f(x_1) = f(x) + f'(x)(x_1 - x) + \int_{x_1}^x (t - x_1)f''(t)dt,$$

$$f(x_4) = f(x) + f'(x)(x_4 - x) + \int_{x_4}^x (t - x_4)f''(t)dt,$$

$$\delta_n(x) \sim \frac{1}{\sqrt{n}},\ n = 1, 2, \cdots.$$

So

$$\begin{aligned}\bar{w}(x)|f(x)-P(f,x)| &\leqslant \bar{w}(x)|\frac{x-x_4}{x_1-x_4}|\int_{x_1}^{x}|(t-x_1)f''(t)|dt \\ &\quad +\bar{w}(x)|\frac{x_1-x}{x_1-x_4}|\int_{x_4}^{x}|(t-x_4)f''(t)|dt \\ &:= I_1+I_2.\end{aligned}$$

Whence t is between x_1 and x. We have $\frac{|t-x_1|}{\bar{w}(t)} \leqslant \frac{|x-x_1|}{\bar{w}(x)}$, then

$$\begin{aligned} I_1 &\leqslant C\|\bar{w}\varphi^{2\lambda}f''\||(x-x_1)(x-x_4)|\int_{x_1}^{x}\varphi^{-2\lambda}(t)dt \\ &\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}f''\|.\end{aligned}$$

Analogously, we have

$$I_2 \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}f''\|.$$

Now the lemma follows from combining these results together. □

3 Proof of Theorem

3.1 Proof of Theorem 1

If $f\in C_{\bar{w}}$, when $x\in[\frac{1}{n},1-\frac{1}{n}]$, by [2], we have

$$\begin{aligned}|\bar{w}(x)\bar{B}_n''(f,x)| &\leqslant n\varphi^{-2}(x)\bar{w}(x)|\bar{B}_n(f,x)| \\ &+ \bar{w}(x)\varphi^{-4}(x)\sum_{k=0}^{n}p_{n,k}(x)|k-nx||\bar{F}_n(\frac{k}{n})| \\ &+ \bar{w}(x)\varphi^{-4}(x)\sum_{k=0}^{n}(k-nx)^2|\bar{F}_n(\frac{k}{n})|p_{n,k}(x) \\ &:= A_1+A_2+A_3.\end{aligned}$$

By (2.2), we have

$$A_1(x)=n\varphi^{-2}(x)\bar{w}(x)|\bar{B}_n(f,x)|\leqslant Cn^2\|\bar{w}f\|.$$

and

$$\begin{aligned}A_2 &= \bar{w}(x)\varphi^{-4}(x)[\sum_{k/n\in A}|k-nx||\bar{F}_n(\frac{k}{n})|p_{n,k}(x)+\sum_{x_2\leqslant k/n\leqslant x_3}|k-nx||P(\frac{k}{n})|p_{n,k}(x)] \\ &:= \sigma_1+\sigma_2.\end{aligned}$$

thereof $A:=[0,x_2]\cup[x_3,1]$. If $\frac{k}{n}\in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n})}\leqslant C(1+n^{-\frac{\alpha}{2}}|k-nx|^{\alpha})$, we have $|k-n\xi|\geqslant\frac{\sqrt{n}}{2}$, by (2.4), then

$$\begin{aligned}\sigma_1 &\leqslant C\|\bar{w}f\|\varphi^{-4}(x)\sum_{k=0}^{n}p_{n,k}(x)|k-nx|[1+n^{-\frac{\alpha}{2}}|k-nx|^{\alpha}] \\ &= C\|\bar{w}f\|\varphi^{-4}(x)\sum_{k=0}^{n}p_{n,k}(x)|k-nx|+Cn^{-\frac{\alpha}{2}}\|\bar{w}f\|\varphi^{-4}(x)\sum_{k=0}^{n}p_{n,k}(x)|k-nx|^{1+\alpha} \\ &\leqslant Cn^{\frac{1}{2}}\varphi^{-3}(x)\|\bar{w}f\|+Cn^{\frac{1}{2}}\varphi^{-3+\alpha}(x)\|\bar{w}f\| \\ &\leqslant Cn^2\|\bar{w}f\|.\end{aligned}$$

For σ_2, P is a linear function. We note $|P(\frac{k}{n})| \leqslant max(|P(x_1)|, |P(x_4)|) := P(a)$. If $x \in [x_1, x_4]$, we have $\bar{w}(x) \leqslant \bar{w}(a)$. So, if $x \in [x_1, x_4]$, by (2.4), then

$$\sigma_2 \leqslant C\bar{w}(a)P(a)\varphi^{-4}(x)\sum_{k=0}^{n} p_{n,k}(x)|k-nx| \leqslant Cn^2\|\bar{w}f\|.$$

If $x \notin [x_1, x_4]$, then $\bar{w}(a) > n^{-\frac{\alpha}{2}}$, by (2.5), we have

$$\begin{aligned}\sigma_2 &\leqslant C\varphi^{-4}(x)\bar{w}(x)\sum_{x_2\leqslant k/n\leqslant x_3}|P(a)(k-nx)|p_{n,k}(x)\\ &\leqslant Cn^{\frac{\alpha}{2}}\|\bar{w}f\|\varphi^{-4}(x)\bar{w}(x)\sum_{x_2\leqslant k/n\leqslant x_3}|k-nx|p_{n,k}(x)\\ &\leqslant Cn^2\|\bar{w}f\|.\end{aligned}$$

So, $A_2 \leqslant Cn^2\|\bar{w}f\|$. Similarly, $A_3 \leqslant Cn^2\|\bar{w}f\|$. It follows from combining the above inequalities that the inequality is proved.

When $x \in [0, \frac{1}{n}]$ (The same as $x \in [1-\frac{1}{n}, 1]$), by [6], then

$$\bar{B}_n''(f,x) = n(n-1)\sum_{k=0}^{n-2}\overrightarrow{\Delta}_{\frac{1}{n}}^2\bar{F}_n(\frac{k}{n})p_{n-2,k}(x).$$

We have

$$\begin{aligned}|\bar{w}(x)\bar{B}_n''(f,x)| &\leqslant Cn^2\bar{w}(x)\sum_{k=0}^{n-2}|\overrightarrow{\Delta}_{\frac{1}{n}}^2\bar{F}_n(\frac{k}{n})|p_{n-2,k}(x)\\ &= Cn^2\bar{w}(x)[\sum_{k/n\in A}p_{n-2,k}(x)|\overrightarrow{\Delta}_{\frac{1}{n}}^2\bar{F}_n(\frac{k}{n})| + \sum_{x_2\leqslant k/n\leqslant x_3}p_{n-2,k}(x)|\overrightarrow{\Delta}_{\frac{1}{n}}^2P(\frac{k}{n})|].\end{aligned}$$

We can deal with it in accordance with the former proofs, and prove it immediately, then the theorem is done. □

3.2 Proof of Theorem 2

(1) We prove the first inequality of Theorem 2.

Case 1. If $0 \leqslant \varphi(x) \leqslant \frac{1}{\sqrt{n}}$, by Theorem 1, we have

$$|\bar{w}(x)\varphi^{2\lambda}(x)\bar{B}_n''(f,x)| \leqslant Cn^{-\lambda}|\bar{w}(x)\bar{B}_n''(f,x)| \leqslant Cn^{2-\lambda}\|\bar{w}f\|.$$

Case 2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, by [3], we have

$$\begin{aligned}\bar{B}_n''(f,x) = B_n''(\bar{F}_n,x) &= (\varphi^2(x))^{-2}\sum_{i=0}^{2}Q_i(x,n)n^i\sum_{k=0}^{n}(x-\frac{k}{n})^i\bar{F}_n(\frac{k}{n})p_{n,k}(x),\\ Q_i(x,n) &= (nx(1-x))^{[(2r-i)/2]},\\ (\varphi^2(x))^{-2}Q_i(x,n)n^i &\leqslant C(n/\varphi^2(x))^{1+i/2}.\end{aligned}$$

So

$$
\begin{aligned}
& |\bar{w}(x)\varphi^{2\lambda}(x)\bar{B}_n''(f,x)| \\
\leqslant\ & C\bar{w}(x)\varphi^{2\lambda}(x)\sum_{i=0}^{2}(\frac{n}{\varphi^2(x)})^{1+i/2}\sum_{k=0}^{n}|(x-\frac{k}{n})^i\bar{F}_n(\frac{k}{n})|p_{n,k}(x) \\
=\ & C\bar{w}(x)\varphi^{2\lambda}(x)\sum_{i=0}^{2}(\frac{n}{\varphi^2(x)})^{1+i/2}\sum_{k/n\in A}|(x-\frac{k}{n})^i\bar{F}_n(\frac{k}{n})|p_{n,k}(x) \\
& + C\bar{w}(x)\varphi^{2\lambda}(x)\sum_{i=0}^{2}(\frac{n}{\varphi^2(x)})^{1+i/2}\sum_{x_2\leqslant k/n\leqslant x_3}|(x-\frac{k}{n})^iP(\frac{k}{n})|p_{n,k}(x) \\
:=\ & \sigma_1+\sigma_2,
\end{aligned}
$$

where $A := [0, x_2] \cup [x_3, 1]$. Working as in the proof of Theorem 1, we can get $\sigma_1 \leqslant Cn^{2-\lambda}\|\bar{w}f\|$, $\sigma_2 \leqslant Cn^{2-\lambda}\|\bar{w}f\|$. By bringing these facts together, we can immediately get the first inequality of Theorem 2.

(2) If $f \in W^2_{\bar{w},\lambda}$, by $\bar{B}_n(f,x) = B_n(\bar{F}_n(f),x)$, then

$$
\begin{aligned}
|\bar{w}(x)\varphi^{2\lambda}(x)\bar{B}_n''(f,x)| &\leqslant n^2\bar{w}(x)\varphi^{2\lambda}(x)\sum_{k=0}^{n-2}|\overrightarrow{\Delta}^2_{\frac{1}{n}}\bar{F}_n(\frac{k}{n})|p_{n-2,k}(x) \\
&= n^2\bar{w}(x)\varphi^{2\lambda}(x)\sum_{k=1}^{n-3}|\overrightarrow{\Delta}^2_{\frac{1}{n}}\bar{F}_n(\frac{k}{n})|p_{n-2,k}(x) \\
&+ n^2\bar{w}(x)\varphi^{2\lambda}(x)|\overrightarrow{\Delta}^2_{\frac{1}{n}}\bar{F}_n(0)|p_{n-2,0}(x) \\
&+ n^2\bar{w}(x)\varphi^{2\lambda}(x)|\overrightarrow{\Delta}^2_{\frac{1}{n}}\bar{F}_n(\frac{n-2}{n})|p_{n-2,n-2}(x) \\
&:= I_1+I_2+I_3.
\end{aligned}
$$

By [3], if $0 < k < n-2$, we have

$$
|\overrightarrow{\Delta}^2_{\frac{1}{n}}\bar{F}_n(\frac{k}{n})| \leqslant Cn^{-1}\int_0^{\frac{2}{n}}|\bar{F}_n''(\frac{k}{n}+u)|du, \tag{3.1}
$$

If $k = 0$, we have

$$
|\overrightarrow{\Delta}^2_{\frac{1}{n}}\bar{F}_n(0)| \leqslant C\int_0^{\frac{2}{n}}u|\bar{F}_n''(u)|du, \tag{3.2}
$$

Similarly

$$
|\overrightarrow{\Delta}^2_{\frac{1}{n}}\bar{F}_n(\frac{n-2}{n})| \leqslant Cn^{-1}\int_{1-\frac{2}{n}}^{1}(1-u)|\bar{F}_n''(u)|du. \tag{3.3}
$$

By (3.1), then

$$
\begin{aligned}
I_1 &\leqslant Cn\bar{w}(x)\varphi^{2\lambda}(x)\sum_{k=1}^{n-3}\int_0^{\frac{2}{n}}|\bar{F}_n''(\frac{k}{n}+u)|du\,p_{n-2,k}(x) \\
&= Cn\bar{w}(x)\varphi^{2\lambda}(x)\sum_{k/n\in A}\int_0^{\frac{2}{n}}|\bar{F}_n''(\frac{k}{n}+u)|du\,p_{n-2,k}(x) \\
&\quad + Cn\bar{w}(x)\varphi^{2\lambda}(x)\sum_{x_2\leqslant k/n\leqslant x_3}\int_0^{\frac{2}{n}}|P''(\frac{k}{n}+u)|du\,p_{n-2,k}(x) \\
&:= T_1+T_2,
\end{aligned}
\tag{3.4}
$$

where $A := [0, x_2] \cup [x_3, 1]$, P is a linear function. If $\frac{k}{n} \in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n})} \leqslant C(1+n^{-\frac{\alpha}{2}}|k-nx|^{\alpha})$, we have $|k-n\xi| \geqslant \frac{\sqrt{n}}{2}$, by (2.1), (2.4) and the Theorem 2, then

$$\begin{aligned}
T_1 &\leqslant C\bar{w}(x)\varphi^{2\lambda}(x)\|\bar{w}\varphi^{2\lambda}\bar{F}_n''\| \sum_{k/n\in A} p_{n-2,k}(x)\bar{w}^{-1}(\frac{k}{n})\varphi^{-2\lambda}(\frac{k}{n}) \\
&\leqslant C\varphi^{2\lambda}(x)\|\bar{w}\varphi^{2\lambda}\bar{F}_n''\| \sum_{k=0}^{n-2} p_{n-2,k}(x)[1+n^{-\frac{\alpha}{2}}|k-nx|^{\alpha}]\varphi^{-2\lambda}(\frac{k}{n}) \\
&\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{F}_n''\| \\
&\leqslant C\|\bar{w}\varphi^{2\lambda}f''\|.
\end{aligned}$$

Working as the Theorem 1, we can get

$$T_2 \leqslant C\|\bar{w}\varphi^{2\lambda}f''\|.$$

So, we can get

$$I_1 \leqslant C\|\bar{w}\varphi^{2\lambda}f''\|.$$

By (3.2) and the Theorem 2, we have

$$\begin{aligned}
I_2 &\leqslant Cn^2\bar{w}(x)\varphi^{2\lambda}(x)(1-x)^{n-2}\int_0^{\frac{2}{n}} u|\bar{F}_n''(u)|du \\
&\leqslant Cn^2\bar{w}(x)\varphi^{2\lambda}(x)(1-x)^{n-2}\|\bar{w}\varphi^{2\lambda}\bar{F}_n''\| \int_0^{\frac{2}{n}} u\bar{w}^{-1}(u)\varphi^{-2\lambda}(u)du \\
&\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{F}_n''\| \\
&\leqslant C\|\bar{w}\varphi^{2\lambda}f''\|. \qquad (3.5)
\end{aligned}$$

Similarly,

$$I_3 \leqslant C\|\bar{w}\varphi^{2\lambda}f''\|. \qquad (3.6)$$

By bringing (3.4), (3.5) and (3.6) together, we can get the second inequality of Theorem 2. □

Corollary 1. *If $\alpha > 0$ and $\lambda = 0$, we have*

$$|\bar{w}(x)\bar{B}_n''(f,x)| \leqslant \begin{cases} Cn^2\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C\|\bar{w}f''\|, & f \in W_{\bar{w}}^2. \end{cases}$$

Corollary 2. *If $\alpha > 0$ and $\lambda = 1$, we have*

$$|\bar{w}(x)\varphi^2(x)\bar{B}_n''(f,x)| \leqslant \begin{cases} Cn\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C\|\bar{w}\varphi^2 f''\|, & f \in W_{\bar{w}}^2. \end{cases}$$

3.3 Proof of Theorem 3

3.3.1 The direct theorem

We know

$$\bar{F}_n(t) = \bar{F}_n(x) + \bar{F}_n'(t)(t-x) + \int_x^t (t-u)\bar{F}_n''(u)du,$$
$$B_n(t-x, x) = 0.$$

According to the definition of $W^2_{\bar{w},\lambda}$, for any $g \in W^2_{\bar{w},\lambda}$, we have $\bar{B}_n(g,x) = B_n(\bar{G}_n(g),x)$.

(1) We first estimate $\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n,x)|$ under the condition of $x \in [\frac{1}{n}, 1-\frac{1}{n}]$, then $\varphi^2(x) < \frac{1}{n}$, $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, and

$$\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n,x)| = \bar{w}(x)|B_n(R_2(\bar{G}_n,t,x),x)|,$$

thereof $R_2(\bar{G}_n,t,x) = \int_x^t (t-u)\bar{G}''_n(u)du$.
It follows from $\frac{|t-u|}{\bar{w}(u)} \leqslant \frac{|t-x|}{\bar{w}(x)}$, u between t and x, we have

$$\begin{aligned}
\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n,x)| &\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{G}''_n\|\bar{w}(x)B_n(\int_x^t \frac{|t-u|}{\bar{w}(u)\varphi^{2\lambda}(u)}du,x) \\
&\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{G}''_n\|\bar{w}(x)(B_n(\int_x^t \frac{|t-u|}{\varphi^{4\lambda}(u)}|du,x))^{\frac{1}{2}}(B_n(\int_x^t \frac{|t-u|}{\bar{w}^2(u)}du,x))^{\frac{1}{2}},
\end{aligned}$$

also

$$\int_x^t \frac{|t-u|}{\varphi^{4\lambda}(u)}du \leqslant C\frac{(t-x)^2}{\varphi^{4\lambda}(x)}, \ \int_x^t \frac{|t-u|}{\bar{w}^2(u)}du \leqslant \frac{(t-x)^2}{\bar{w}^2(x)}. \tag{3.7}$$

By (2.4) and (3.7), we have

$$\begin{aligned}
\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n,x)| &\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{G}''_n\|\varphi^{-2\lambda}(x)B_n((t-x)^2,x) \\
&\leqslant Cn^{-1}\frac{\varphi^2(x)}{\varphi^{2\lambda}(x)}\|\bar{w}\varphi^{2\lambda}\bar{G}''_n\| \\
&\leqslant Cn^{-1}\frac{\delta_n^2(x)}{\varphi^{2\lambda}(x)}\|\bar{w}\varphi^{2\lambda}\bar{G}''_n\| \\
&= C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}\bar{G}''_n\|.
\end{aligned}$$

(2) We estimate $\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n,x)|$ under the condition of $x \in [0, \frac{1}{n})$ (The same as $x \in (1-\frac{1}{n}, 1]$), $\varphi(x) \sim \delta_n(x)$, now

$$\begin{aligned}
\bar{w}(x)|\bar{G}_n(x) - B_n(\bar{G}_n,x)| &\leqslant C\bar{w}(x)\sum_{k=1}^{n-1}p_{n,k}(x)\int_x^{\frac{k}{n}}|(\frac{k}{n}-u)\bar{G}''_n(u)|du \\
&+ C\bar{w}(x)p_{n,0}(x)\int_0^x u|\bar{G}''_n(u)|du \\
&+ C\bar{w}(x)p_{n,n}(x)\int_x^1 |(1-u)\bar{G}''_n(u)|du \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

If u between $\frac{k}{n}$ and x, we have

$$\frac{|\frac{k}{n}-u|}{\bar{w}^2(u)} \leqslant \frac{|\frac{k}{n}-x|}{\bar{w}^2(x)}, \ \frac{|\frac{k}{n}-u|}{\varphi^{4\lambda}(u)} \leqslant \frac{|\frac{k}{n}-x|}{\varphi^{4\lambda}(x)}. \tag{3.8}$$

By (2.4) and (3.8), then

$$\begin{aligned}
I_1 &\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|\bar{w}(x)\sum_{k=1}^{n-1}p_{n,k}\int_x^{\frac{k}{n}}\frac{|\frac{k}{n}-u|}{\bar{w}(u)\varphi^{2\lambda}(u)}du \\
&\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|\bar{w}(x)\sum_{k=1}^{n-1}p_{n,k}(\int_x^{\frac{k}{n}}\frac{|\frac{k}{n}-u|}{\bar{w}^2(u)}du)^{\frac{1}{2}}(\int_x^{\frac{k}{n}}\frac{|\frac{k}{n}-u|}{\varphi^{4\lambda}(u)}du)^{\frac{1}{2}} \\
&\leqslant Cn^{-2}\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|\varphi^{-2\lambda}(x)\sum_{k=0}^{n-1}p_{n,k}(x)(k-nx)^2 \\
&\leqslant Cn^{-1}\frac{\varphi^2(x)}{\varphi^{2\lambda}(x)}\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\| \\
&\leqslant Cn^{-1}\frac{\delta_n^2(x)}{\varphi^{2\lambda}(x)}\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\| \\
&= C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|. \qquad (3.9)
\end{aligned}$$

For I_2, when u between $\frac{k}{n}$ and x, we let $k=0$, then $\frac{u}{\bar{w}(u)}\leqslant\frac{x}{\bar{w}(x)}$, and

$$\begin{aligned}
I_2 &\leqslant C\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|\bar{w}(x)p_{n,0}(x)\int_0^x u\bar{w}^{-1}(u)\varphi^{-2\lambda}(u)du \\
&\leqslant C(nx)(1-x)^{n-1}\cdot n^{-1}\frac{\varphi^2(x)}{\varphi^{2\lambda}(x)}\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\| \\
&\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|. \qquad (3.10)
\end{aligned}$$

Similarly, we have

$$I_3 \leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|. \qquad (3.11)$$

By bringing (3.9), (3.10) and (3.11), we get the result. Above all, we have

$$\bar{w}(x)|\bar{G}_n(x)-B_n(\bar{G}_n,x)|\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\|.$$

By (2.6) and the second inequality of Theorem 2, when $g\in W^2_{\bar{w},\lambda}$, then

$$\begin{aligned}
\bar{w}(x)|g(x)-\bar{B}_n(g,x)| &\leqslant \bar{w}(x)|g(x)-\bar{G}_n(g,x)|+\bar{w}(x)|\bar{G}_n(g,x)-\bar{B}_n(g,x)| \\
&\leqslant \bar{w}(x)|g(x)-P(g,x)|_{[x_1,x_4]}+C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}\bar{G}_n''\| \\
&\leqslant C(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}g''\|. \qquad (3.12)
\end{aligned}$$

For $f\in C_{\bar{w}}$, we choose proper $g\in W^2_{\bar{w},\lambda}$, by (2.2) and (3.12), then

$$\begin{aligned}
\bar{w}(x)|f(x)-\bar{B}_n(f,x)| &\leqslant \bar{w}(x)|f(x)-g(x)|+\bar{w}(x)|\bar{B}_n(f-g,x)|+\bar{w}(x)|g(x)-\bar{B}_n(g,x)| \\
&\leqslant C(\|\bar{w}(f-g)\|+(\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})^2\|\bar{w}\varphi^{2\lambda}g''\|) \\
&\leqslant C\omega^2_{\varphi^{\lambda}}(f,\frac{\delta_n(x)}{\sqrt{n}\varphi^{\lambda}(x)})_{\bar{w}}.\Box
\end{aligned}$$

3.3.2 The inverse theorem

We define the weighted main-part modulus for $D = R_+$ by

$$\Omega^2_{\varphi^\lambda}(C, f, t)_{\bar{w}} = \sup_{0<h\leqslant t} \|\bar{w}\Delta^2_{h\varphi^\lambda} f\|_{[Ch^*,\infty]},$$
$$\Omega^2_{\varphi^\lambda}(1, f, t)_{\bar{w}} = \Omega^2_{\varphi^\lambda}(f, t)_{\bar{w}}.$$

where $C > 2^{1/\beta(0)-1}$, $\beta(0) > 0$, and h^* is given by

$$h^* = \begin{cases} (Ar)^{1/1-\beta(0)} h^{1/1-\beta(0)}, & 0 \leqslant \beta(0) < 1, \\ 0, & \beta(0) \geqslant 1. \end{cases}$$

The main-part K-functional is given by

$$K^2_{\varphi^\lambda}(f, t^2)_{\bar{w}} = \sup_{0<h\leqslant t} \inf_g \{\|\bar{w}(f-g)\|_{[Ch^*,\infty]} + t^2 \|\bar{w}\varphi^{2\lambda} g''\|_{[Ch^*,\infty]},\ g' \in A.C.((Ch^*, \infty))\}.$$

By [3], we have

$$C^{-1}\Omega^2_{\varphi^\lambda}(f, t)_{\bar{w}} \leqslant \omega^2_{\varphi^\lambda}(f, t)_{\bar{w}} \leqslant C \int_0^t \frac{\Omega^2_{\varphi^\lambda}(f, \tau)_{\bar{w}}}{\tau} d\tau, \tag{3.13}$$
$$C^{-1}K^2_{\varphi^\lambda}(f, t^2)_{\bar{w}} \leqslant \Omega^2_{\varphi^\lambda}(f, t)_{\bar{w}} \leqslant CK^2_{\varphi^\lambda}(f, t^2)_{\bar{w}}. \tag{3.14}$$

Proof. Let $\delta > 0$, by (3.14), we choose proper g so that

$$\|\bar{w}(f-g)\| \leqslant C\Omega^2_{\varphi^\lambda}(f, \delta)_{\bar{w}},\ \|\bar{w}\varphi^{2\lambda} g''\| \leqslant C\delta^{-2}\Omega^2_{\varphi^\lambda}(f, \delta)_{\bar{w}}.$$

then

$$\begin{aligned}
|\bar{w}(x)\Delta^2_{h\varphi^\lambda} f(x)| &\leqslant |\bar{w}(x)\Delta^2_{h\varphi^\lambda}(f(x) - \bar{B}_n(f, x))| + |\bar{w}(x)\Delta^2_{h\varphi^\lambda}\bar{B}_n(f-g, x)| \\
&\quad + |\bar{w}(x)\Delta^2_{h\varphi^\lambda}\bar{B}_n(g, x)| \\
&\leqslant \sum_{j=0}^{2} C_2^j \Big(n^{-\frac{1}{2}} \frac{\delta_n(x + (1-j)h\varphi^\lambda(x))}{\varphi^\lambda(x + (1-j)h\varphi^\lambda(x))}\Big)^{\alpha_0} \\
&\quad + \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \bar{w}(x)\bar{B}''_n(f-g, x + \sum_{k=1}^{2} u_k) du_1 du_2 \\
&\quad + \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \bar{w}(x)\bar{B}''_n(g, x + \sum_{k=1}^{2} u_k) du_1 du_2 \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{3.15}$$

Obviously

$$J_1 \leqslant C(n^{-\frac{1}{2}}\varphi^{-\lambda}(x)\delta_n(x))^{\alpha_0}. \tag{3.16}$$

By Theorem 1, we have

$$\begin{aligned}
J_2 &\leqslant Cn^2\|\bar{w}(f-g)\| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} du_1 du_2 \\
&\leqslant Cn^2h^2\varphi^{2\lambda}(x)\|\bar{w}(f-g)\| \\
&\leqslant Cn^2h^2\varphi^{2\lambda}(x)\Omega^2_{\varphi^\lambda}(f, \delta)_{\bar{w}}.
\end{aligned} \tag{3.17}$$

By the second inequality of Corollary 2 and (2.3), we have

$$\begin{aligned} J_2 &\leqslant Cn\|\bar{w}(f-g)\| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \varphi^{-2}(x+\sum_{k=1}^{2} u_k) du_1 du_2 \\ &\leqslant Cnh^2\varphi^{2(\lambda-1)}(x)\|\bar{w}(f-g)\| \\ &\leqslant Cnh^2\varphi^{2(\lambda-1)}(x)\Omega^2_{\varphi^\lambda}(f,\delta)_{\bar{w}}. \end{aligned} \tag{3.18}$$

By the second inequality of Theorem 2 and (2.3), we have

$$\begin{aligned} J_3 &\leqslant C\|\bar{w}\varphi^{2\lambda}g''\|\bar{w}(x) \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \bar{w}^{-1}(x+\sum_{k=1}^{2} u_k)\varphi^{-2\lambda}(x+\sum_{k=1}^{2} u_k) du_1 du_2 \\ &\leqslant Ch^2\|\bar{w}\varphi^{2\lambda}g''\| \\ &\leqslant Ch^2\delta^{-2}\Omega^2_{\varphi^\lambda}(f,\delta)_{\bar{w}}. \end{aligned} \tag{3.19}$$

Now, by (3.16), (3.17), (3.18) and (3.19), we get

$$|\bar{w}(x)\Delta^2_{h\varphi^\lambda} f(x)| \leqslant C\{(n^{-\frac{1}{2}}\delta_n(x))^{\alpha_0} + h^2(n^{-\frac{1}{2}}\delta_n(x))^{-2}\Omega^2_{\varphi^\lambda}(f,\delta)_{\bar{w}} + h^2\delta^{-2}\Omega^2_{\varphi^\lambda}(f,\delta)_{\bar{w}}\}.$$

When $n \geqslant 2$, we have

$$n^{-\frac{1}{2}}\delta_n(x) < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x) \leqslant \sqrt{2}n^{-\frac{1}{2}}\delta_n(x),$$

Choosing proper $x, n \in N$, so that

$$n^{-\frac{1}{2}}\delta_n(x) \leqslant \delta < (n-1)^{-\frac{1}{2}}\delta_{n-1}(x),$$

Therefore

$$|\bar{w}(x)\Delta^2_{h\varphi^\lambda} f(x)| \leqslant C\{\delta^{\alpha_0} + h^2\delta^{-2}\Omega^2_{\varphi^\lambda}(f,\delta)_{\bar{w}}\}.$$

By Borens-Lorentz lemma, we get

$$\Omega^2_{\varphi^\lambda}(f,t)_{\bar{w}} \leqslant Ct^{\alpha_0}. \tag{3.20}$$

So, by (3.20), we get

$$\omega^2_{\varphi^\lambda}(f,t)_{\bar{w}} \leqslant C\int_0^t \frac{\Omega^2_{\varphi^\lambda}(f,\tau)_{\bar{w}}}{\tau} d\tau = C\int_0^t \tau^{\alpha_0-1} d\tau = Ct^{\alpha_0}.\square$$

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