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Strong Algebrability of Certain Subset of $C_0(X)$

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Abstract

In 2013, strong algebrability of $c_0 \setminus \cup_p \ell^p$ is proved by Bartoszewics and Glab. In this paper we expand this result to an arbitrary finite dimensional Banach space. The main theorem of this paper concerns that $C_0(X) \setminus \cup_p L^p(X)$ contains an infinitely generated free algebra, where Xisa finite dimensional Banach space.

Keywords: Strong Algebrability; Pathological Properties; Free Algebra.

1. Introduction

Finding infinite dimensional algebraic structures and infinitely generated algebras in different subsets of various spaces is relatively new trend in mathematical analysis.

Recall that a subset S of a vector space V is called lineable if $S \cup \{0\}$ contains an infinite dimensional vector subspace of V. Also if Visa topological vector space then S is called spaceable if $S \cup \{0\}$ contains a closed infinite dimensional vector subspace of V. This notation was first appeared in an unpublished notes of Enflo and Gurariy. Aronand Gurariy was firstly published these notation in [2]. We should mention that Enflo'sand gurariy'sun published notes were completed in collaboration with Seoane-Sepulvida and finally published in[8].

The term algebrability was introduced later in[3]: letκisacardinal number, if V is a linear algebra, S is said to be κ –algebrable if $S \cup \{0\}$ contains a κ-generated algebra, i.e. the minimal system of generators of this algebra has cardinality κ([3]). Aron, Perez-Garcia and Seoane-Sepulvid approved in[4] that these to f non-convergent Fourier series is 1-algebrable. Aronand Seoane-Sepulvida established the algebrability of every where surjective functions on $\mathbb{C}$ in[5], which was strengthened in [1]. Other algebrability results can be found in[10] and [11]. Here we will discuss some strengthening of the notion of algebrability.

Definition1.1. Let κ is a cardinal number. We say that a subset $S$of an algebra A is strongly κ-algebrable if there exists a κ-generated free algebra B contained in $S \cup \{0\}$. In addition, if B is also dense in A, then A is called densely strongly κ-algebrable.

Were call that, for a cardinal number κ, to say that an algebra A is a κ-generated free algebra, means that there exists a subset $Z = \{z_\alpha : \alpha < \kappa\} \subset A$ such that any function f from Z into some algebra A' can be uniquely extended to a homomorphism from A into A'.The set Z is called a set offeregenerators of the algebra A. If Z is a set offreegenerators of some subalgebra $B \subset A$, we say that Z is a set offreegenerators in the subalgebra A. If A is commutative, then a subset $Z = \{z_\alpha : \alpha < \kappa\} \subset A$ is a set offreegenerators in A ifforeach polynomial P and any $z_{\alpha_1}, \dots, z_{\alpha_n} \in Z$ we have

$$P(z_{\alpha_1}, \dots, z_{\alpha_n}) = 0 \; if \; and \; only \; if \; P = 0.$$

The definition of strongly κ-algebrability was introduced in[9], though in several papers, sets which are shown t o be algebrable are infact strongly algebrable, and that is clear by the proofs(see[3] and [11]). Strong algebrability is in effect a stronger condition than algebrability: for example,$c_{00}$is$\aleph_0$-algebrablein$c_0$but itisnotstrong1-algebrable (see[6]).

2. Strong Algebrability in $C_0(X)$

In 2013, Bartoszewicz and Glabhave proved in[6] that $c_0 \setminus \cup_p \ell^p$is densely strongly $\mathfrak{c}$-algebrable. Themain theorem in this section extends this result to an arbitrary finite dimensional Banach space. Throughout this section, let X be a finite dimensional Banach space.

Theorem2.1. The set$C_0(X) \setminus \cup_p L^p(X)$ is densely strongly c-algebrable in$C_0(X)$.

Proof. Let$\{r_\alpha : \alpha < \mathfrak{c}\}$ be a linearly independent subset of positive realscontaining1. Let A be the linear algebra generated by the set

$S = \{f_\alpha : X \to \mathbb{R}, \alpha < \mathfrak{c}\}$, where for each $\alpha < \mathfrak{c}$

$$f_\alpha = \frac{1}{ln^{r_\alpha}(|x| + 2)}.$$

We will show that for each $\alpha < \mathfrak{c}$,$f_\alpha \in C_0(X)$,and also we will show that any non-trivial algebraic combination of elements of A is either a null function or is not in each $L^p(X)$, $(1 \le p < \infty)$.

First le t$\varepsilon > 0$ is arbitrary. We have

$$\{x \in X : |f_\alpha(x)| \ge \varepsilon\} = \left\{x \in X : |x| + 2 \le \exp\left(\frac{1}{\alpha^{\frac{1}{r_\alpha}}}\right)\right\}.$$

Since X is finite dimensional, so each closed d is kin X is compact. So the above set is compact and therefore $f_\alpha \in C_0(X)$.

Now we want to prove that any non-trivial algebraic combination of elements of A is either a null function or is not in each $L^p(X)$, $(1 \le p < \infty)$. To this end, let $\{k_i : 1 \le i \le n\}$ be a set of non-negative integers that are not simultaneously zero and $\beta_1, \dots, \beta_n \in \mathbb{R}$. We claim that $\sum_{i=1}^{n} \beta_i f_{\alpha_i}$ is not in each $L^p(X)$, $(1 \le p < \infty)$. Without loss of generality let $\beta_1 \neq 0$ and $k_1 r_{\alpha_1} \le k_2 r_{\alpha_2} \le \cdots \le k_n r_{\alpha_n}$. Now we have

$$\left|\frac{\beta_1}{ln^{k_1 r_{\alpha 1}}(|x|+2)} + \frac{\beta_2}{ln^{k_2 r_{\alpha 2}}(|x|+2)} + \cdots + \frac{\beta_n}{ln^{k_n r_{\alpha n}}(|x|+2)}\right| \le \frac{|\beta_1|}{ln^{k_1 r_{\alpha 1}}(|x|+2)} - \frac{|\beta_2|}{ln^{k_2 r_{\alpha 2}}(|x|+2)} - \cdots - \frac{|\beta_n|}{ln^{k_n r_{\alpha n}}(|x|+2)}.$$

Since $k_1 r_{\alpha_1}$ is smaller than each $k_2 r_{\alpha_2}, \dots, k_n r_{\alpha_n}$, there is $r > 0$ such that for al l $x \in X$ with $|x| \ge r$ we have

$$\frac{|\beta_2|}{ln^{k_2 r_{\alpha 2}}(|x|+2)} + \cdots + \frac{|\beta_n|}{ln^{k_n r_{\alpha n}}(|x|+2)} < \frac{|\beta_1|}{ln^{k_1 r_{\alpha 1}}(|x|+2)}.$$

Hence for each $x \in X$ with $|x| \ge r$ we have

$$\left|\frac{\beta_1}{ln^{k_1 r_{\alpha 1}}(|x|+2)} + \frac{\beta_2}{ln^{k_2 r_{\alpha 2}}(|x|+2)} + \cdots + \frac{\beta_n}{ln^{k_n r_{\alpha n}}(|x|+2)}\right| \ge \frac{|\beta_1|}{2\, ln^{k_1 r_{\alpha 1}}(|x|+2)}.$$

But $\frac{|\beta_1|}{2\, ln^q(|x|+2)} \notin L^p(X)$, for all $p \ge 1$ and all $q > 0$. This shows that any algebraic combination of elements of A is either a null function or is not in each $L^p(X)$, $(1 \le p < \infty)$.

Now, we claim that A is dense in $C_0(X)$. Consider the sub-algebra A_1 of $C(X)$ generated by two functions $f_1(x) = \frac{1}{\ln(|x|+2)}$ and $f_2(x) = 1$. Let A_2 e a sub-algebra of A generated by the function f_1. Note that

$$A_1 = \{f + a f_2 : f \in A_2, a \in \mathbb{R}\}.$$

Let $\tilde{X}$ be the one point compactification of X. Notice that $C(X)$ is isometrically isomorphic to $C(\tilde{X})$. Since A_1 separates the points of $\tilde{X}$ and never vanishes on it, so byS tone-Weierstrass theorem, A_1 is uniformly dense in $C(X)$. Take a function $f \in C_0(X)$. There exists $g \in A_1$ with $\|f - g\| < \varepsilon/2$. Note that $\left|\lim_{|x|\to\infty} g(x)\right| \le \varepsilon/2$. Put $\tilde{g} = g - \lim_{|x|\to\infty} g(x)$. Then $\|f - \tilde{g}\| < \varepsilon$. Now $\tilde{g} \in A_2 \subset A$. Thus A is dense in $C_0(X)$. This completes the proof.

□

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