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Vertex Magic Total labeling in Hamiltonian graphs

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Abstract

A vertex magic total labeling on a graph with $\boldsymbol{v}$ vertices and $\boldsymbol{e}$ edges is a one - to - one map taking the vertices and edges onto the integers $\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots \boldsymbol{v} + \boldsymbol{e}$ with the property that the sum of the label on the vertex and the labels of its incident edges is constant, independent of the choice of the vertex. It is proved that all cycles have vertex magic total labeling. The Hamiltonian graphs have necessarily a cycle in it. Hence we study the relation of vertex magic total labeling in Hamiltonian graphs.

AMS MSC Classification: 05C78

Key words: edge magic total labeling; vertex magic total labeling; Hamiltonian graphs; magic constant.

1. INTRODUCTION

A graph $\boldsymbol{G} = \{\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{f}\}$ has vertex set $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{G})$, an edge set $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{G})$, a relation f that associates each edge with two vertices and $v = |V|$ and $e = |E|$ .The members of $\boldsymbol{V} \cup \boldsymbol{E}$ are called elements of G. An introduction to Graph theory by D. B. West [13] is a general reference for the graph theoretic notions. A labeling for a graph is a map that takes graph elements to numbers (non negative integers). In this paper the domain is the set of all vertices and edges, giving rise to total labelings. J. A. Gallian [2] in the year 1998 completely compiled the survey of graph labelings. W. D. Wallis and others [12], introduced edge-magic total labelings that generalize the idea of a magic square. J.A.MacDougall and others [7], introduced vertex-magic total labelings of graphs and the magic labelings was made as a standardized terminology. It is an assignment of the integers from 1 to $v + e$ to the vertices and edges of G so that at each vertex the vertex labels and edge labels incident with the vertex add to a fixed constant k, called the valence or weight or magic constant k. In this paper, we identify some graphs that are Hamiltonian as well as that have vertex magic total labeling.

2. VERTEX MAGIC TOTAL LABELING

In general, by labeling a graph we mean an injective map defined from the set of vertices to the set of natural numbers, the same is extended to the set of edges.

2.1. Definition.

An edge magic total labeling on a graph G is a one to one map f from $V \cup E$ onto the integers $1,2, \dots, v + e$, with the property that, given any edge xy, $f(x) + f(xy) + f(y) = h$, for some constant h, h is called the magic sum of G. Any graph with an edge magic total labeling abbreviated as EMTL is called edge magic.

2.2. Definition.

A one to one and onto mapping f from $V \cup E$ to the finite subset $\{1,2,\dots v+e\}$ of natural numbers such that for every vertex x, $f(x) + \sum_i f(xy_i) = k$, where y_i's are vertices adjacent to x, is called vertex magic total labeling on a graph whose sum of labels at vertex x is the weight of the vertex, $wt(x) = k$ for all x. The constant k is called as *magic constant* for f. Any graph with a vertex magic total labeling abbreviated as VMTL is called vertex magic.

2.3. Example.

Consider the path P_3 .This is a graph with 3 vertices and 2 edges as given below.

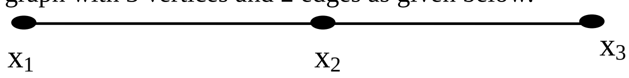

Fig.1: Path P_3

Let

$f(x_1) = 5, f(x_2) = 3, f(x_3) = 4,$

$f(x_1x_2) = 1, f(x_2x_3) = 2$

Then,

$f(x_1) + f(x_1x_2) = f(x_3) + f(x_3x_2)$

$= f(x_2) + f(x_2x_1) + f(x_2x_3) = 6$

Hence this graph has vertex magic total labeling with constant $k = 6$.

3. HAMILTONIAN GRAPHS

Hamiltonian graphs was first studied by Kirkman in 1856.He was followed by the Irish mathematician Sir William Rowan Hamilton(1805-1865).The Hamiltonian cycles was named after him in a game of a regular dodecahedron wooden version in which the vertices were named for 20 important cities. The game named as Icosian game on the dodecahedron is one player specified a 5-vertex path and the other must extend it to a spanning cycle[13].

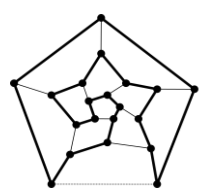

Fig.2 A regular dodecahedron

3.1. Definition

A Hamiltonian path is a spanning path containing all vertices of the graph G.

3.2. Definition

A Hamiltonian cycle is a cycle which passes once and exactly once through every vertex of G.

3.3. Definition

A Hamiltonian graph is a graph with a spanning cycle called as a Hamiltonian cycle.

3.4. Definition

A graph is k-connected if its connectivity (the minimum size of a vertex set S such that G-S is disconnected or has only one vertex) is atleast k.

3.5. Theorem

If G is a Hamiltonian graph, then (i) G is connected, (ii) G contains a Hamiltonian cycle, (iii) $o(G) \geq 3$, (iv) G is 2-connected (v) G has no cut vertices

Proof:

(i) A path between two vertices in a graph is a Hamiltonian path if it passes through every vertex of the graph. So a Hamiltonian graph is connected.

(ii) A closed Hamiltonian path is called a Hamiltonian cycle. By definition a Hamiltonian graph is a graph that has a Hamiltonian cycle.

(iii) For a simple graph, to get a minimum cycle atleast three vertices are essential. So, $o(G) \geq 3$

(iv) Any Hamiltonian graph is necessarily 2-connected as the deletion of a vertex from a graph results in a connected graph that has a Hamiltonian path.

(v) Consequently from(iv), no graph with a cut vertex is Hamiltonian.

4. Vertex Magic Total labeling Graphs and Hamiltonian Graphs

As cycles include spanning cycles, it is simple observation that they are Hamiltonian graphs.

4.1. Theorem

All cycles $C_n, n \geq 3$ have vertex magic total labeling.

Proof:

To prove that all cycles $C_n, n \geq 3$ have vertex magic total labeling , first let us see that all cycles $C_n, n \geq 3$ are edge magic total labeling(EMTL) f'.

Case(i): If v is odd, then C_v has an EMTL with $k = \frac{1}{2}(5v+3)$

Let v=$2n$+1. Consider the cyclic vertex labeling $(1, n+1, 2n+1, n, \ldots, n+2)$ where each label is derived from the preceding one by adding $n \; mod(2n+1)$. The successive pairs of vertices have sums

$n+2,\ 3n+2, 3n+1, \ldots,\qquad n+3$ which are all different. If $h = 5n+4$, the edge labels are

$4n+2, 2n+2, 2n+3, \ldots, 4n+1$ as required $h = 5n+4 = \frac{1}{2}(5v+3)$

Case(ii): If v is even, then C_v has an EMTL with h $= \frac{1}{2}(5v+4)$

Let v=$2n$.

(a)If n is odd,then

$$f(x_i) = \begin{cases} \dfrac{(i+1)}{2}; & i = 1,3,\dots,n \\ 3n; & i = 2 \\ \dfrac{(2n+i+2)}{2}; & i = 4,6,\dots,n-1 \\ \dfrac{(n+3)}{2}; & i = n+1 \\ \dfrac{(i+3)}{2}; & i = n+2, n+4, \dots 2n-1 \\ \dfrac{(2n+i)}{2}; & i = n+3, n+5, \dots, 2n-2 \\ n+2; & i = 2n \end{cases}$$

(b)If n is even, then

$$f(x_i) = \begin{cases} \dfrac{(i+1)}{2}; & i = 1,3,\dots,n+1 \\ 3n; & i = 2 \\ \dfrac{(2n+i)}{2}; & i = 4,6,\dots,n \\ \dfrac{(i+2)}{2}; & i = n+2, n+4, \dots 2n \\ \dfrac{(2n+i-1)}{2}; & i = n+3, n+5, \dots, 2n-1 \end{cases}$$

Based on these $f(x_i)$ and h value, $f(x_i x_{i+1})$ can be calculated. If we define a new mapping f by $f(x_i) = f'(x_i x_{i+1})$ and $f(x_i x_{i+1}) = f'(x_{i+1})$ where the subscripts are integer modulo n. Then at each vertex let k be magic constant, yielding vertex magic total labeling for cycles.

4.2. Example.

Consider Cycle C_3.

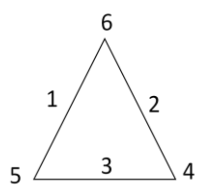

Fig.3 Cycle C_3

The Hamiltonian cycles here are mentioned as vertex - edge - vertex - … are 5-1-6-2-4-3-5,

6-2-4-3-5-1-6, … . The vertex magic total labeling here is obtained as follows

Let

$f(x_1) = 5, f(x_2) = 6, f(x_3) = 4,$

$f(x_1x_2) = 1, f(x_2x_3) = 2, f(x_1x_3) = 3$

Then,

$f(x_1) + f(x_1x_2) + f(x_1x_3)$

$= f(x_2) + f(x_2x_1) + f(x_2x_3)$

$= f(x_3) + f(x_3x_1) + f(x_3x_2) = 9$

This graph has vertex magic total labeling with constant $k = 9$.

Therefore the Cycle C_3 is Hamiltonian and has vertex magic total labeling.

4.3. Conjecture[3]

Every r-regular (r>1) graph except $2\mathrm{k}_3$ and k_2 has vertex magic total labeling.

A general result is that if there is less variation among the degrees of the vertices, then the graph possesses a vertex magic total labeling and if there is much variation, then it does not hold. Hence, a regular graph possesses a vertex magic total labeling. But for the graph k_2, since $f(x) \neq f(y)$ then $f(x) + f(xy) \neq f(y) + f(xy)$and so vertex magic total labeling is not possible . In the same manner for the graph $2\mathrm{k}_3$, while assigning the vertices and edges with the minimum and maximum values of $1,2,\dots,v+e$ in the first k_3 and remaining middle values in the second k_3, though magic constant k is reached separately in each k_3 a common constant will never be reached with the difference between these constants always remains to be 3. Hence $2\mathrm{k}_3$ and k_2 are not r-regular.

The 2- regular graphs are the cycles. Hence they have VMTL.

4.4. Remark

All r -regular graphs need not be Hamiltonian.

4.5. Example

Consider the P(6,2) Petersen graph is a 3- regular graph with 12 vertices and 18 edges,

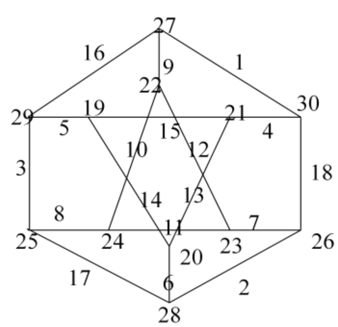

Fig.5. $P(6,2)$ Petersen graph

which has a vertex magic total labeling with magic constant 53 [8] and has a Hamiltonian path but does not have a Hamiltonian cycle as the outer edges alone or inner edges alone may form cycles . But the outer edges, the spokes connecting the outer edges to the inner edges together with the inner edges cannot cover all the vertices to form a single cycle ---------(i)

4.6. Example

Now consider the 3- regular graph, $P(3,1)$ - graph has a vertex magic total labeling and is Hamiltonian.

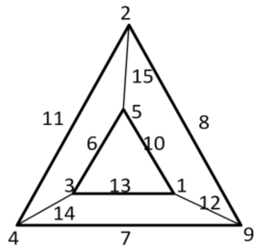

Fig.6. $P(3,1)$- graph

The Hamiltonian cycles here are 2-8-9-7-4-14-13-1-10-5-15-2,

5-6-3-13-1-12-9-7-4-11-2-15-5,...

The vertex magic total labeling here is obtained as follows, Let

$f(x_1) = 2, f(x_2) = 4, f(x_3) = 9,$

$f(x_4) = 1, f(x_5) = 3, f(x_6) = 5$

$f(x_1x_2) = 11, f(x_2x_3) = 7, f(x_3x_4) = 12,$

$f(x_4x_6) = 10, f(x_1x_3) = 8, f(x_2x_5) = 14,$

$f(x_5x_6) = 6, f(x_4x_5) = 13, f(x_1x_6) = 15$

Then,

$f(x_1) + f(x_1x_2) + f(x_1x_6) + f(x_1x_3)$

$= f(x_2) + f(x_2x_1) + f(x_2x_3) + f(x_2x_5)$

$= f(x_3) + f(x_3x_2) + f(x_3x_4) + f(x_3x_1)$

$= f(x_4) + f(x_4x_5) + f(x_4x_3) + f(x_4x_6)$

$= f(x_5) + f(x_5x_4) + f(x_5x_2) + f(x_5x_6)$

$= f(x_6) + f(x_6x_1) + f(x_6x_4) + f(x_6x_5)$

$= 36$

This graph has vertex magic total labeling with constant $k = 36$

Therefore the 3- regular graph, $P(3,1)$- graph is Hamiltonian and has vertex magic total labeling.------(ii)

From (i) and (ii) we observe that the 3- regular graphs have vertex magic total labeling, but need not be Hamiltonian.

4.7. Definition

Given the vertex magic total labeling f for graph G, let us define the map f on $V \cup E$ by $f'(x) = v + e + 1 - f(x)$ for any vertex x and $f'(xy) = v + e + 1 - f(xy)$for any edge xy. Then $f'$is also a one-to-one map from the set $V \cup E$ to {1,2,…,v+e} and is called as the dual of f.

4.8. Theorem [7]

The path P_n has vertex magic total labeling for any $n \geq 3$.

Proof:

As $P_n, n \geq 3$ has edge magic total labeling f', by finding the dual f for the function f',we can define vertex magic total labeling for all $P_n, n \geq 3$.

4.9. Definition

A matrix $A = (a_{ij})$ with order rxc and its entries are $\{1,2,\ldots,rc\}$ each element appearing once, whose row sums and column sums are equal is called a magic rectangle. The sum of all entries in A is $\frac{1}{2}rc(rc+1)$. Then $\sum_{i=1}^{r} a_{ij} =$ *12c(rc*+1) for all j; *j=1caij= 12r(rc*+1) for all i.

4.10. Theorem [4]

Complete graphs K_n, have vertex magic total labeling[4].

Proof.

The method of magic rectangles is used to prove the result.

Case(i): When n is odd

Consider $a_{ij} = a_{i-1,j+1}$ with the subscript reduced modulo v and let S be the sequence $s_0, s_1, \ldots, s_{\frac{n-1}{2}}$.

Then B(S) = ($b_{i,j}$) where $b_{i,j} = b_{i-1,j-1}$ with the subscripts taken modulo v such that

$b_{1,1} = s_0, b_{1,2} = s_1, \ldots, b_{1,n} = s_1$.

The labeling $f_s(n)$ of K_n, given by

$f(x_i) = a_{i,i} + b_{i,i}; f\left(x_i x_j\right) = a_{i,j} + b_{i,j}$ and each vertex has the same weight.

$$wt(x) = s_0 + 2(s_1 + s_2 + \cdots + s_{\frac{n-1}{2}}) + 1 + 2 + \cdots + n$$

Case(ii): When n is even

If $n \equiv 2 \ mod\ 4$, we consider $n = 2v$ and so $K_n = 2K_v$ (i.e) union of two copies of K_v . The label $v^2 + v + m_{xy}$ is given to the edge connecting the vertex x of one K_v to the vertex y of the other K_v, yielding the magic constant $h + \frac{1}{2}(3v^3 + 2v^2 + v)$

4.11. Theorem

Complete graphs K_n are Hamiltonian graphs.

Proof:

Every pair of the vertices of a complete graph K_n adjacent. Then, the vertices $v_1, \ldots, v_n$ form a Hamiltonian cycle in K_n. So K_n is a Hamiltonian graph for all n.

All complete graphs K_n have vertex magic total labeling and are also Hamiltonian graphs.

4.12. Example

Let us consider Complete graph K_5 .

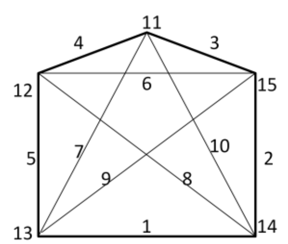

Fig.4. Complete graph K_5

The Hamiltonian cycles here are 11-3-15-2-14-1-13-5-12-4-11, 11-7-13-9-15-6-12-8-14-10-11, ...

The vertex magic total labeling here is obtained as follows

Let $f(x_1) = 11,\ f(x_2) = 12,\ f(x_3) = 13,\ \ f(x_4) = 14,\ f(x_5) = 15$

$f(x_1x_2) = 4, f(x_2x_3) = 5, f(x_3x_4) = 1, f(x_4x_5) = 2, f(x_5x_6) = 3$

$f(x_1x_3) = 7, f(x_1x_4) = 10, f(x_2x_5) = 6, f(x_2x_4) = 8, f(x_3x_5) = 9$

Then,

$$\begin{aligned}
&\ f(x_1) + f(x_1x_2) + f(x_1x_3) + f(x_1x_4) + f(x_1x_5)\\
&= f(x_2) + f(x_2x_1) + f(x_2x_3) + f(x_2x_4) + f(x_2x_5)\\
&= f(x_3) + f(x_3x_1) + f(x_3x_2) + f(x_3x_4) + f(x_3x_5)\\
&= f(x_4) + f(x_4x_1) + f(x_4x_2) + f(x_4x_3) + f(x_4x_5)\\
&= f(x_5) + f(x_5x_1) + f(x_5x_2) + f(x_5x_3) + f(x_5x_4)\\
&= 35
\end{aligned}$$

This graph has vertex magic total labeling with constant k = 35

Therefore the Complete graph K_5 is Hamiltonian and has vertex magic total labeling.

4.13. Theorem

The Wheel W_n has vertex magic total labeling for $3 \le n \le 11$.

Proof:

The Wheel W_n has $v = n + 1, e = 2n$ and $v + e = 3n + 1$. Let c be the hub vertex (vertex at the centre) and we have n other rim vertices $x_1, \dots, x_n$. Then

$$k \ge wt(c) \ge 1 + 2 + \cdots + (n + 1) = \frac{(n+1)(n+2)}{2} \qquad \text{.........(a)}$$

Now the sum of the vertices of all the rim vertices is considered. The n largest labels are assigned to the rim edges and spoke edges to find the upper bound.

$$wt(x_1) + \cdots + wt(x_n) \le \sum_{2}^{3n+1}(i) + \sum_{2n+2}^{3n+1}(i)$$

$$= 2\frac{(3n+1)(3n+2)}{2} - \frac{(2n+1)(2n+2)}{2} - 1 = n(7n + 6) \qquad \text{.........(b)}$$

Therefore for the rim vertices, $k \le 7n + 6$. Also (b) is less than(a) for all $n > 11$. For the Wheel W_n, $3 \le n \le 11$, the minimum magic constant is smaller than the maximum magic constant but for $n > 11$, the minimum magic constant is larger than the maximum magic constant (i.e) (b) is less than (a) when n > 11. So, vertex magic total labeling is possible only for Wheel W_n when $3 \le n \le 11$.

4.14. Theorem

The Wheel W_n is Hamiltonian.

Proof:

As all wheels have a cycle in it, W_n is Hamiltonian.

4.15.Example.

Consider the wheel W_4

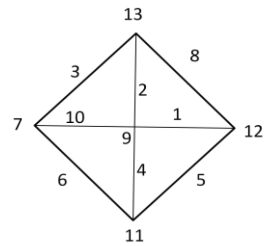

Fig.7.Wheel W_4

The Hamiltonian cycles here are 7-3-13-8-12-5-11-6-7,7-3-13-2-9-1-12-5-11-4-9-10-7,...

The vertex magic total labeling here is obtained as follows keeping x_1 as the hub vertex,

Let

$f(x_1) = 9, f(x_2) = 11, f(x_3) = 7,$

$f(x_4) = 13, f(x_5) = 12,$

$f(x_1x_2) = 4, f(x_2x_3) = 6, f(x_3x_4) = 3, f(x_4x_1) = 2,$

$f(x_1x_5) = 1, f(x_2x_5) = 5, f(x_3x_1) = 10, f(x_4x_5) = 8.$

Then,

$f(x_1) + f(x_1x_2) + f(x_1x_3) + f(x_1x_4) + f(x_1x_5)$

$= f(x_2) + f(x_2x_1) + f(x_2x_3) + f(x_2x_5)$

$= f(x_3) + f(x_3x_2) + f(x_3x_4) + f(x_3x_1)$

$= f(x_4) + f(x_4x_1) + f(x_4x_3) + f(x_4x_5)$

$= f(x_5) + f(x_5x_1) + f(x_5x_2) + f(x_5x_4)$

$= 29$

This graph has vertex magic total labeling with constant k = 29

On making use the theorem 4.13,

here $n = 4, v = n + 1 = 5, e = 2n = 8, v + e = 13$

$$k \geq wt(c) \geq \frac{(n+1)(n+2)}{2} = 15$$

$k \geq 15$

Also,

$wt(x_1) + \cdots + wt(x_n) \leq n(7n + 6) = 136$,

we have

$k \leq 7n + 6 = 34$

Therefore the wheel W_4 is Hamiltonian and has vertex magic total labeling.

5. Conclusion

In this paper we have discussed that some of the vertex magic total labeling graphs are Hamiltonian graphs. Such relationships can be studied for other labeling graphs and other graphs like ladders, sun, fan, friendship graphs.

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