**Journal of Progressive Research in Mathematics(JPRM)**
***ISSN: 2395-0218***

SCITECH
RESEARCH ORGANISATION|

Volume 6, Issue 2
Published online: December 29, 2015|

**Journal of Progressive Research in Mathematics**
www.scitecresearch.com/journals

# Subdirect Sum of Ternary Rings and Subdirectly Irreducible Ternary Rings

**Md.Salim** [1, 1], **Miss. P. Mondal** [2, 2] and **T.K.Dutta** [3, 3]

[1,2,3]Department of Pure Mathematics,

University of Calcutta,

35, Ballygunge Circular Road,Kolkata-700019,India

**Abstract.**

In this paper we introduce the notions of subdirect sum of a family of ternary rings and the representation of a ternary ring as a subdirect sum of a family of ternary rings. We also introduce the notion of subdirectly irreducible ternary ring and characterize it. Lastly we characterize subdirectly irreducible Boolean ternary rings.

**Keywords and phrases:** Ternary rings; Ternary fields; Subdirectly irreducible ternary rings; Boolean ternary rings.

**AMS Mathematics Subject Classification (2010):** 16Y 30, 16Y 60.

## 1 Introduction

The introduction of mathematical literature of ternary algebraic system dated back to 1924.The notion of ternary algebraic system was first introduced by H.Prüfer [5] by the name 'schar'. After that W.Dörnte[2] further studied this type of algebraic system. In 1932, D.H. Lehmer[6] investigated certain ternary algebraic systems called triplexes which turn out to be a commutative ternary groups. Ternary groups are the special case of polyadic groups(in other terminologies which are known as n-groups) introduced by E. L. Post [4]. In 1971, W. G. Lister [7] introduced the notion of ternary ring and study some important properties of it. According to Lister [7], a ternary ring is an algebraic system consisting of a nonempty set $R$ together with a binary operation, called addition and a ternary multiplication, which forms a commutative group relative to addition, a ternary semigroup relative to multiplication and left, right, lateral distributive laws hold.

The notion of subdirect sum of a family of rings has been introduced by N.H. McCoy [3]. He also introduced and characterized representation of a ternary ring as a subdirect sum of a family of rings. Following Brikhoff [1], he introduced the notion of subdirectly irreducible ring and characterize it. In this paper we introduce the notions of subdirect sum of a family of ternary rings and a representation of a ternary ring as a subdirect sum of family of ternary rings. We obtain that "A ternary ring $R$ has a representation as a subdirect sum of a family of ternary rings $\{R_i : i \in I\}$ if and only if for each $i \in I$, there exists homomorphism $\phi_i : R \xrightarrow[\text{onto}]{} R_i$ such that if $r(\neq 0) \in R$, then $\phi_i(r) \neq 0$, for all least one $i$". and "A ternary ring $R$ has a representation as a subdirect sum of a family of ternary rings

[1] smpmath746@yahoo.in
[2] pinkimondal1992@gmail.com
[3] duttatapankumar@yahoo.co.in

$\{R_i : i \in I\}$ if and only if for each $i \in I$, there exists in $R$ a two sided ideal $K_i$ such that $R/K_i$ is isomorphic to $R_i$ and moreover $\cap K_i = \{0\}$". We also introduce subdirectly irreducible ternary rings. We prove that "Every ternary ring $R$ is isomorphic to subdirect sum of subdirectly irreducible ternary rings which are homomorphic images of $R$". Lastly we characterize subdirectly irreducible Boolean ternary rings.

Some earlier work of the authors on ternary ring and multiplicative ternary hyperring may be found in [8] and [9].

## 2 Preliminaries

**Definition 2.1** *A nonempty set $R$ together with a binary operation, called addition and a ternary multiplication denoted by juxtaposition, is said to be a ternary ring if $R$ is an additive commutative group satisfying the following properties:*

(*i*) *(abc)de = a(bcd)e = ab(cde),*

(*ii*) *(a+b)cd = acd + bcd,*

(*iii*) *a(b+c)d = abd + acd,*

(*iv*) *ab(c+d) = abc + abd for all $a,b,c,d,e \in R$.*

**Definition 2.2** *A nonempty subset $S$ of a ternary ring R is called a ternary subring of $R$ if $(S,+)$ is a subgroup of $(R,+)$ and if $s_1s_2s_3 \in S$ for all $s_1,s_2,s_3 \in S$.*

**Definition 2.3** *A ternary ring $R$ admits an identity provided that there exist elements $\{(e_i, f_i) \in R \times R(i = 1,2,....,n)\}$ such that $\sum_{i=1}^{n} e_i f_i x = \sum_{i=1}^{n} e_i x f_i = \sum_{i=1}^{n} x e_i f_i = x$ for all $x \in R$. In this case the ternary ring $R$ is said to be a ternary ring with identity $\{(e_i, f_i) : i \in 1,2,....,n\}$. In particular, if there exists an element $e \in R$ such that $eex = exe = xee = x$ for all $x \in R$ then $e$ is called a unital element of the ternary ring $R$.*

It is obvious that $xye = (exe)ye = ex(eye) = exy$ and $xye = x(eye)e = xe(yee) = xey$ for all $x, y \in R$. Hence the following result follows.

**Proposition 2.4** *If $e$ is a unital element of a ternary ring $R$ then $exy = xey = xye$, for all $x, y \in R$.*

We now define left(right, lateral) ideal of a ternary ring.

**Definition 2.5** *An additive subgroup $I$ of a ternary ring $R$ is called a left(right, lateral) ideal of $R$ if $r_1r_2i$ (respectively $ir_1r_2, r_1ir_2) \in I$ for all $r_1, r_2 \in R$ and $i \in I$. If $I$ is a left, a right and a lateral ideal of $R$ then $I$ is called an ideal of $R$.*

**Definition 2.6** *Let $R$ and $R'$ be two ternary rings and $f$ be a mapping which maps $R$ into $R'$. Then the mapping $f : R \to R'$ is called a homomorphism of $R$ into $R'$ if the following conditions hold:*

$$f(a+b) = f(a) + f(b).$$

$$f(abc) = f(a)f(b)f(c).$$

for all $a,b,c \in R$.

**Definition 2.7** *A ternary ring $R$ is called commutative if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$, where $\sigma$ is a permutation of $\{1,2,3\}$ for all $x_1, x_2, x_3 \in R$.*

**Definition 2.8** *A non-trivial ternary ring $R$ with a unital element $e$ is said to be a division ternary ring if for every element $a(\neq 0) \in R$ there exists an element $b \in R$ such that $abx = x$ and $xba = x$ for all $x \in R$.*

**Definition 2.9** *Let $R$ be a commutative ternary ring with a unital element $e$. Then $R$ is called a ternary field if for every element $a(\neq 0)\in R$ there exists an element $b\in R$ such that $abx=x$ for all $x\in R$.*

**Proposition 2.10** *A ternary field does not contain divisors of zero.*

**Definition 2.11** *An element $x$ of a ternary ring $R$ is called idempotent if $x^3=x$.*

**Definition 2.12** *A ternary ring $R$ is called a simple ternary ring if $R^3\neq(0)$ and if it contains no nonzero proper ideal i.e $\{0\}$ and $R$ are only ideals of $R$.*

**Theorem 2.13** *A commutative ternary ring $R$ with a unital element $e$ is a ternary field if and only if $(0)$ and $R$ are the only ideals of $R$.*

*Proof.* Let $R$ be a ternary field. Let $e(\neq 0)$ be a unital element of $R$. Let $I(\neq 0)$ be any ideal of $R$ and $a(\neq 0)\in I$. Since $R$ is a ternary field, there exists an element $b\in R$ such that $abx=x$ for all $x\in R$. Now $a\in I\Rightarrow x=abx\in I$ for all $x\in R$. So $I=R$. Hence $R$ contains only two ideals $(0)$ and $R$. Conversely let the condition hold. Let $a(\neq 0)$ be an element of $R$. Consider the ideal $(a)$ of $R$. Since $(a)\neq(0)$, it follows that $(a)=R$. So $e\in(a)$. Since $R$ is commutative, $(a)=aRR$; Then $e=\sum_{i=1}^{n} ar_is_i$ for some $r_i,s_i\in R, i=1,2,\ldots$. Now $x=eex=(\sum_{i=1}^{n} ar_is_i)ex=a(\sum_{i=1}^{n} r_is_ie)x=abx$ where $b=\sum_{i=1}^{n} r_is_ie$. Thus there exists an element $b\in R$ such that $abx=x$ $\forall x\in R$. So $R$ is a ternary field.

**Corollary 2.14** *Let $T_3=\{0,f,-f\}$ Then $T_3$ is a ternary field in which `$+$' and ternary multiplication is defined by*

| + | 0 | f | -f |
|---|---|---|---|
| 0 | 0 | f | -f |
| f | f | -f | 0 |
| -f | -f | 0 | f |

| $a$ | $b$ | $c$ | $abc$ |
|---|---|---|---|
| $f$ | $f$ | $f$ | $f$ |
| $-f$ | $-f$ | $-f$ | $-f$ |
| $f$ | $f$ | $-f$ | $-f$ |
| $f$ | $-f$ | $f$ | $-f$ |
| $-f$ | $f$ | $-f$ | $f$ |
| $-f$ | $f$ | $f$ | $-f$ |
| $-f$ | $-f$ | $f$ | $f$ |
| $f$ | $-f$ | $-f$ | $f$ |

and the product three elements with at least one zero is zero and $f$ and $-f$ are unitals elements of $T_3$.

**Definition 2.15** *Let $R$ be a ternary ring and $I$ be an ideal of $R$. Define the sets $a+I=\{a+x : x\in I\}$ for each $a\in R$ and $R/I=\{a+I : a\in R\}$. Then $R/I$ forms a ternary ring with addition and multiplication defined by*

$$(a+I)+(b+I)=(a+b)+I \quad \text{and}$$

$$(a+I)(b+I)(c+I)=abc+I$$

for all $a,b,c\in R$. This ternary ring $R/I$ is called the quotient ternary ring of $R$ by $I$.

**Definition 2.16** *Let $R$ be a ternary ring such that $R\neq\{0\}$. A proper ideal $I$ of $R$ is called maximal if $I$ is not contained in any other proper ideal of $R$. i.e for any ideal $J$ of $R$, $I\subseteq J\subseteq R$ implies that either $I=J$ or $J=R$.*

**Theorem 2.17** *Let $R$ be a commutative ternary ring with a unital element $e$. Then an ideal $M$ of $R$ is maximal if and only if $R/M$ is a ternary field.*

*Proof.* Let $R$ be a ternary ring with a unital element $e$. Let $M$ be a maximal ideal of $R$. Since $R$ is commutative with unital element $e$, $R/M$ is also commutative with unital element $e+M$. Let $a+M\in R/M$ be such that $a+M\neq 0+M$. Then $a\notin M$. Hence the ideal $M+aRR$ properly contains $M$. Since $M$ is a maximal ideal, we have $M+aRR=R$. This implies that there exists $m\in M$ and $r_i,s_i\in R, i=1,2,\ldots$ such that $m+\sum_{i=1}^{n} ar_is_i=e$. Then $e+M=\sum_{i=1}^{n}(a+M)(r_i+M)(s_i+M)$. Now $x+M=(e+M)(e+M)(x+M)=(\sum_{i=1}^{n}(a+M)(r_i+M)(s_i+M))(e+M)(x+M)=(a+M)$ $(\sum_{i=1}^{n}(r_i+M)(s_i+M)(e+M))(x+M)$. Thus there exists an elements $b+M\in R/M$ such that $(x+M)=(a+M)(b+M)(x+M)$ where $b+M=\sum_{i=1}^{n}(r_i+M)(s_i+M)(e+M)$. So $R/M$ is ternary field. Conversely, suppose that $R/M$ is a ternary field. Since $R/M$ is a ternary field, $R\neq M$. Let $I$ be an ideal of $R$ such that $M\subset I\subseteq R$. Then there exists $a\in I$ such that $a\notin M$. Then $a+M\neq 0+M$. Since $R/M$ is a ternary field, there exists an elements $b+M\in R/M$ and such that

$(a+M)(b+M)(x+M)=x+M$ for all $x+M \in R/M$. So in particular $(a+M)(b+M)(e+M)=e+M$ which implies $e-abe \in M$. This implies $e \in I$. Hence $I=R$. Therefore $M$ is maximal.

## 3 Subdirect Sum of Ternary Rings and Subdirectly Irreducible Ternary Rings

**Definition 3.1** *Let $\{R_i : i \in I\}$ be a family of ternary rings indexed by the set $I$. Let $R=\{f : I \to \cup R_i$ such that $f(i) \in R_i, \forall i \in I\}$. We define addition and multiplication on $R$ by*

$$(f+g)(i) = f(i)+g(i) \quad \text{and}$$

$$(fgh)(i) = f(i)g(i)h(i).$$

for all $i \in I$. Then $R$ forms a ternary ring. This ternary ring $R$ is called the complete direct sum of the family of ternary rings $\{R_i : i \in I\}$. Let $R' = \{f : I \to \cup R_i$ such that $f(i)=0$ for all most all $i\}$. Then $R'$ is a ternary subring of $R$. This subring is called the discrete direct sum of the family of ternary rings $\{R_i : i \in I\}$.

**Remark 3.2** *For a finite set of ternary rings the notions of complete direct sum and that of discrete direct sum coincide.*

**Definition 3.3** *Let $\{R_i : i \in I\}$ be a family of ternary rings indexed by the set $I$ and $R$ be their direct sum. For each $i \in I$, we define a mapping $\theta_i$ from $R$ into $R_i$ by $\theta_i(f) = f(i)$. This mapping $\theta_i$ is called projection on $R$.*

**Proposition 3.4** *For each $i \in I$, $\theta_i : R \to R_i$ is an epimorphism of ternary rings.*

*Proof.* Let $f, g \in R$. Now $\theta_i(f+g) = (f+g)(i) = f(i)+g(i) = \theta_i(f)+\theta_i(g)$ and $\theta_i(fgh) = (fgh)(i)$ $= f(i)g(i)h(i) = \theta_i(f)\theta_i(g)\theta_i(h)$. Thus $\theta_i$ is a ternary ring morphism. Let $t \in R_i$. We now define a mapping $f : I \to \cup R_i$ by

$$f(j) = \begin{cases} t & if \quad j = i \\ 0 & if \quad j \neq i \end{cases}$$

Then, $\theta_i(f) = t$. So $\theta_i$ is surjective. Thus $\theta_i$ is a ternary ring epimorphism.

**Definition 3.5** *Let $\{R_i : i \in I\}$ be a family of ternary rings and $R$ be their complete direct sum. A ternary subring $R'$ of $R$ is called a subdirect sum of $\{R_i : i \in I\}$ if $\theta_i(R') = R_i$, $\forall i \in I$, where $\theta_i : R \to R_i$ is the projection map.*

**Remark 3.6** *For a given family of ternary rings $\{R_i : i \in I\}$, there may be many subdirect sums for the family of ternary rings $\{R_i : i \in I\}$.*

For example, the complete direct sum and the discrete direct sum are subdirect sum of ternary rings $\{R_i : i \in I\}$.

**Definition 3.7** *If a ternary ring $R$ isomorphic to a subdirect sum $T$ of a family of ternary rings $\{R_i : i \in I\}$, then $T$ is called a representation of $R$ as a subdirect sum of the family of ternary rings $\{R_i : i \in I\}$.*

In this case if $\alpha$ is the isomorphism of $R$ onto $T$ and $\theta_i$ is the projection map then $\phi_i = \theta_i \circ \alpha$ is a homomorphism from $R$ onto $R_i$. This homomorphism $\phi_i$ is called the natural homomorphism of $R$ onto $R_i$.

**Theorem 3.8** *A ternary ring $R$ has a representation as a subdirect sum of a family of ternary rings*

*$\{R_i : i \in I\}$ if and only if for each $i \in I$, there exists homomorphism $\phi_i : R \xrightarrow[\text{onto}]{} R_i$ such that if $r(\neq 0) \in R$, then $\phi_i(r) \neq 0$, for at least one i.*

*Proof.* Suppose that $R$ has a representation $T$ as a subdirect sum of the family of ternary rings $\{R_i : i \in I\}$. Then there exists an isomorphism $\alpha$ from $R$ onto $T$. Let $\theta_i$ be the projection map. Let $\phi_i = \theta_i \circ \alpha$. Then $\phi_i$ is a homomorphism from $R$ onto $R_i$ for each $i \in I$. Let $r(\neq 0) \in R$. Then $\alpha(r) \neq 0$ [as $\alpha$ is an isomorphism]. Since $\alpha(r) \in T$ ,there exists at least one $i \in I$ such that $\alpha(r)(i) \neq 0$. i.e $\theta_i(\alpha(r)) \neq 0$ i.e $(\theta_i \circ \alpha)(r) \neq 0$ i.e $\phi_i(r) \neq 0$ for at least one i. Conversely assume the condition stated in the theorem. For each $r \in R$, we define a mapping $f_r : I \to \cup_{i\in I} R_i$ by $f_r(i) = \phi_i(r)$. Then $f_r \in S$, the complete direct sum of $\{R_i : i \in I\}$. Let $T = \{f_r : r \in R\}$. Let $f_{r_1}, f_{r_2} \in T$, where $r_1, r_2 \in R$. Now $(f_{r_1} + f_{r_2})(i) = f_{r_1}(i) + f_{r_2}(i) = \phi_i(r_1) + \phi_i(r_2) = \phi_i(r_1 + r_2)$ [as $\phi_i$ is a homomorphism] $= f_{r_1+r_2}(i)$ for all $i \in I$. Thus $f_{r_1} + f_{r_2} = f_{r_1+r_2} \in T$.

Let $r_1, r_2, r_3 \in R$. $(f_{r_1} f_{r_2} f_{r_3})(i) = f_{r_1}(i) f_{r_2}(i) f_{r_3}(i) = \phi_i(r_1)\phi_i(r_2)\phi_i(r_3) = \phi_i(r_1 r_2 r_3)$ [as $\phi$ is a homomorphism] $= f_{r_1 r_2 r_3}(i)$, $\forall i \in I$. Therefore $f_{r_1} f_{r_2} f_{r_3} = f_{r_1 r_2 r_3} \in T$. Again $(-f_{r_1})(i) = -f_{r_1}(i) = -\phi_i(r_1) = \phi_i(-r_1)$ [as $\phi$ is a homomorphism] $= f_{-r_1}(i)$, $\forall i \in I$. Therefore $-f_{r_1} = f_{-r_1} \in T$. Thus $T$ is a ternary subring of $S$. Let $f_r \in T$. Now $\theta_i(f_r) = f_r(i) = \phi_i(r) \in R_i$ for $f_r \in T$. So, $\theta_i(T) \subseteq R_i$. Let $r_i \in R_i$. Since $\phi_i$ is onto, there exists $r \in R$ such that $\phi_i(r) = r_i$. i.e $f_r(i) = r_i$ i.e $\theta_i(f_r) = r_i$. Thus $r_i = \theta_i(f_r) \in \theta_i(T)$. So $R_i \subseteq \theta_i(T)$. Therefore $R_i = \theta_i(T)$. Thus $T$ is the subdirect sum of the family of ternary subrings $\{R_i : i \in I\}$. We now define a mapping $\alpha : R \to T$ by $\alpha(r) = f_r$. Let $r_1, r_2, r_3 \in R$. Then $\alpha(r_1 + r_2) = f_{r_1+r_2} = f_{r_1} + f_{r_2} = \alpha(r_1) + \alpha(r_2)$ and $\alpha(r_1 r_2 r_3) = f_{r_1 r_2 r_3} = f_{r_1} f_{r_2} f_{r_3} = \alpha(r_1)\alpha(r_2)\alpha(r_3)$. Therefore $\alpha$ is a ternary ring morphism. Let $r \in$ Ker $\alpha$. Therefore $\alpha(r) = 0 \Rightarrow (\alpha(r))(i) = 0$, $\forall i \in I$ $\Rightarrow f_r(i) = 0$ $\Rightarrow \phi_i(r) = 0$, $\forall i \in I \Rightarrow r = 0$ (by the given condition). Therefore $\alpha$ is injective. Obviously $\alpha$ is surjective. Hence $\alpha$ is an isomorphism. Thus $R$ has a representation $T$ as a subdirect sum of the family of ternary rings $\{R_i : i \in I\}$.

**Remark 3.9** *Since $(\theta_i \circ f)(r) = \theta_i(f(r)) = \theta_i(f_r) = f_r(i) = \phi_i(r)$, $\forall r \in R$, $\theta_i \circ f = \phi_i$. Thus the homomorphism $\phi_i$ in the above theorem is nothing but the natural homomorphism.*

**Theorem 3.10** *A ternary ring $R$ has a representation as a subdirect sum of a family of ternary rings $\{R_i : i \in I\}$ if and only if for each $i \in I$, there exists in $R$ a two sided ideal $K_i$ such that $R/K_i$ is isomorphic to $R_i$ and moreover $\cap K_i = (0)$.*

*Proof.* Suppose that $R$ has a representation as a subdirect sum of a family of ternary rings $\{R_i : i \in I\}$. Then for each $i \in I$ there exists a homomorphism $\phi_i : R \xrightarrow[\text{onto}]{} R_i$ such that if $r(\neq 0) \in R$ then $\phi_i(r) \neq 0$, for at least one i. Let $K_i = Ker\phi_i$, $i \in I$. Then for each $i \in I$, $K_i$ is a two sided ideal of $R$. Again by the “First Isomorphism Theorem" on ternary ring $R/Ker\phi_i \cong R_i$, $\forall i \in I$ i.e $R/K_i \cong R_i$, $\forall i \in I$. Let $r \in \cap K_i \Rightarrow r \in K_i = Ker\phi_i$, $\forall i \in I \Rightarrow \phi_i(r) = 0$, $\forall i \in I \Rightarrow r = 0$. Thus $\cap K_i = (0)$. Conversely suppose that for each $i \in I$, there exists a two sided ideal $K_i$ in $R$ such that $R/K_i \cong R_i$ and $\cap K_i = 0$. Let $\pi_i : R \to R/K_i$ be natural epimorphism for each $i \in I$ and $\alpha_i : R/K_i \to R_i$ be the isomorphism, $\forall i \in I$. Let $\phi_i = \alpha_i \circ \pi_i$. Then for each $i \in I$ there exists a homomorphism $\phi_i$ from $R$ onto $R_i$. Now suppose that $r(\neq 0) \in R$. Then $r \notin (0) = \bigcap_{i\in I} K_i \Rightarrow r \notin K_i = Ker\pi_i$, for at least one $i$. $\Rightarrow \pi_i(r) \neq 0 \Rightarrow (\alpha_i \circ \pi_i)(r) \neq 0 \Rightarrow \phi_i(r) \neq 0$, for at least one i(since $\alpha_i$ is an isomorphism). Then $R$ has a representation as a subdirect sum of a family of ternary rings $\{R_i : i \in I\}$.

**Definition 3.11** *A ternary ring $R$ is said to subdirectly irreducible if for every representation $T$ of $R$ as a subdirect sum of a family of ternary rings $\{R_i : i \in I\}$, there exists an $i \in I$ such that the homomorphism $\phi_i$ from $R$ onto $R_i$ is an isomorphism where $\phi_i = \theta_i \circ \alpha$, $\theta_i$ is the projection map and $\alpha$ is the isomorphism from $R$ onto $T$.*

Trivial ternary rings i.e the ternary rings consisting of zero element only are assumed to be subdirectly irreducible.

**Theorem 3.12** *A nonzero ternary ring $R$ is subdirectly irreducible if and only if the intersection of all nonzero ideals of $R$ is a nonzero ideal.*

*Proof.* Suppose that the nonzero ternary ring $R$ is subdirectly irreducible. Let $\{K_i : i \in I\}$ be the family of all nonzero ideals of $R$. If possible, let $\bigcap_{i\in I} K_i = (0)$. Let $R_i = R/K_i$. Then $\{R_i : i \in I\}$ is a family of ternary rings. Now for each $i \in I$, there exists a homomorphism $\phi_i$ from $R$ onto $R_i$ (natural epimorphism). Now suppose that $r(\neq 0) \in R$. Then $r \notin (0) = \bigcap_{i\in I} K_i \Rightarrow r \notin K_i \Rightarrow \phi_i(r) \neq 0$ for at least one $i \in I$. So $R$ has a representation $T$ as subdirect sum of family of ternary rings $\{R_i : i \in I\}$. Since for any $i \in I$, $\phi_i : R \to R_i$ is not an isomorphism, it follows that $R$ is not subdirectly irreducible, a contradiction. So intersection of all nonzero ideals of $R$ is a nonzero ideal. Conversely suppose that intersection of all nonzero ideals of $R$ is a nonzero ideal. Let $T$ be a representation of $R$ as a subdirect sum of a family of ternary rings $\{R_i : i \in I\}$. Then for each $i \in I$ there exists an onto homomorphism $\phi_i : R \to R_i$ such that for $r(\neq 0) \in R$, $\phi_i(r) \neq 0$ for at least one i. Let $K$ be the intersection of all nonzero ideals of $R$, then $K \neq (0)$. Let $r(\neq 0) \in K$. So there exists an onto homomorphism $\phi_i : R \to R_i$ such that $\phi_i(r) \neq 0$. So $r \notin$ Ker $\phi_i$. But $K$ is the smallest nonzero ideal of $R$. So, this is possible only when Ker $\phi_i = (0)$, which implies that $\phi_i$ is a monomorphism. Also $\phi_i$ is an epimorphism. Thus $\phi_i$ is an isomorphism. Thus there exists an $i \in I$ such that $\phi_i$ from $R$ onto $R_i$ is an isomorphism. So $R$ is subdirectly irreducible.

**Corollary 3.13** (1) *Every division ternary ring or ternary field is subdirectly irreducible.*

*Proof.* Let $R$ be a division ternary ring or a field. Then $\{0\}$ and $R$ are only ideals. Here $R$ is the only nonzero ideal. Hence the result.

**Corollary 3.14** *Every simple ternary ring is subdirectly irreducible.*

**Theorem 3.15** *Every ternary ring $R$ is isomorphic to a subdirect sum of subdirectly irreducible ternary rings which are homomorphic images of $R$.*

*Proof.* Obviously we may restrict ourselves to the case in which $R$ has nonzero elements. Let $a(\neq 0) \in R$. Let $\mathcal{F} = \{I : I$ is an ideal of $R$ such that $a \notin I\}$. Since $(0) \in \mathcal{F}$, $\mathcal{F} \neq \phi$. Now applying Zorns lemma, we can find a maximal element $M_a$ in $\mathcal{F}$. Then $M_a$ is an ideal of $R$, maximal with respect to the properly that $a \notin M_a$. i.e if $N$ is an ideal of $R$ such that $M_a \subsetneqq N$ then $a \in N$. Let $R_a = R/M_a$. Then $\{R_a : a(\neq 0) \in R\}$ is a family of ternary rings. Let $N/M_a$ be a nonzero ideal of $R/M_a$. Then $M_a \subsetneqq N$. This implies that $a \in N$. Now $a + M_a \neq 0 + M_a$ and $a + M_a \in N/M_a$. This is true for all nonzero ideals $N/M_a$ of $R/M_a$. Thus the intersection of all nonzero ideals of $R/M_a$ is nonzero. Consequently $R_a = R/M_a$ is subdirectly irreducible. Now we consider the family of subdirectly irreducible ternary rings $\{R_a : a(\neq 0) \in R\}$ where $R_a = R/M_a$. Now for each $a \in R \setminus \{0\}$, there exists an ideal $M_a$ in $R$ such that $R/M_a \cong R_a$ [actually $R/M_a = R_a$]. If possible let $b(\neq 0) \in \bigcap_{a\in R\setminus\{0\}} M_a$. Then $b \in M_b$, a contradiction. So $\bigcap_{a\in R\setminus\{0\}} M_a = (0)$. Consequently $R$ has a representation $T$ as subdirect sum of the family of subdirectly irreducible ternary rings $\{R_a : a \in R \setminus \{0\}\}$. Thus $R$ is isomorphic to the subdirect sum of the family of subdirectly irreducible ternary rings $\{R_a : a \in R \setminus \{0\}\}$, which are homomorphic images of $R$.

**Theorem 3.16** *A subdirectly irreducible commutative ternary ring with a unital element $f$ and with more than one element and with no nonzero nilpotent elements is a ternary field.*

*Proof.* Let $R$ be a subdirectly irreducible commutative ternary ring with a unital element $f$ and with more than one element and with no nonzero nilpotent element. Let $e$ be an idempotent element of $R$. Consider the ideals $eRR$ and $A = \{r - eer : r \in R\}$. Now let $x \in eRR \cap A$. Then $x = \sum_{i=1}^{n} er_i s_i = r - eer$, where $r_i, s_i, r \in R, i = 1,2.....n$. Now $eex = x$ [as $e$ is an idempotent element]. Again $eex = eer - eeeer = eer - eer = 0$. So $eRR \cap A = (0)$. Since $R$ is subdirectly irreducible either $eRR = (0)$ or $A = (0)$. If $eRR = (0)$, then $e = eee \in eRR = (0)$; so $e = 0$. If $A = (0)$ then $r = eer$ for all $r \in R$. So $e$ is a unital element of $R$. Let $z(\neq 0) \in$ intersection of all non-zero ideals of $R$. Consider the ideal $z^2R$. Then $z^2R \neq (0)$, for $R$ contains no non-zero nilpotent elements. Now $z \in z^2R$. So $z = z^2t$ for some $t \in R$. Then $ztf = z^2ttf = ztztf$ [as $R$ is commutative] $= zt.z^2ttf = z^3t^3f^3$ [as $R$ is commutative] $= (ztf)^3$. So, $ztf$ is an idempotent of $R$. So $ztf = 0$ or $ztf$ is a unital element of $R$. If $ztf = 0$, then $z = z^2tff = z(ztf)f = 0$, which is a contradiction. So $ztf$ is a unital element of $R$. Let $I(\neq (0))$ be an ideal of $R$. Then $z \in I \Rightarrow ztf \in I \Rightarrow x = x(ztf)(ztf) \in I$, $\forall x \in R$. So $I = R$. Thus $R$ is a commutative ternary ring with a unital element and $(0)$ and $R$ are the only ideals of $R$. So $R$ is a ternary field.

# 4 Subdirectly Irreducible Boolean Ternary Rings

**Definition 4.1** *A ternary ring in which every element is idempotent is called a Boolean ternary ring.*

**Theorem 4.2** *A commutative Boolean ternary ring $R$ is subdirectly irreducible if and only if $R \cong T_3$ .[defined in corollary 2.14]*

*Proof.* Suppose that the commutative Boolean ternary ring $R$ is subdirectly irreducible. Let $e \in R$. Now consider the ideals $eRR$ and $A = \{r - err : r \in R\}$ of $R$. Let $x \in eRR \cap A$. Then $x = \sum_{i=1}^{n} er_i s_i = r - eer$, where $r_i, s_i, r \in R, i = 1,2,..n$. Now, $eex = x$ [as $e$ is an idempotent element]. Again, $eex = eer - eeeer = eer - eer = 0 = x$. So $x = eex = 0$. Thus $eRR \cap A = (0)$. Since $R$ is subdirectly irreducible $eRR = (0)$ or $A = (0)$. If $eRR = (0)$ then $e = e^3 \in eRR = (0)$ i.e $e = 0$. If $A = (0)$ then $r = eer$ for all $r \in R$. So $e$ is a unital element of $R$. Thus every non zero element of $R$ is a unital element of $R$. Let $e(\neq 0), f(\neq 0) \in R$. Then $e + f \in R$. So $e + f = 0$ or $e + f$ is a unital element of $R$. If $e + f = 0$ then $e = -f$. Let $e + f \neq 0$. Then $e + f$ is a unital element of $R$. So $(e+f)(e+f)e = e$. This implies that $e^3 + efe + fee + ffe = e$ i.e $e + f + f + e = e$ or $2f = -e$. Similarly we get $2e = -f$ ., Thus $2e - e = 2f - f$ i.e $e = f$. Thus $R \cong T_3$. Conversely suppose that $R \cong T_3$. Since $T_3$ is a ternary field, so $T_3$ and hence $R$ is subdirectly irreducible.

**Theorem 4.3** *A ternary ring $R$ is isomorphic to a subdirect sum of ternary fields $\{R_i : i \in I\}$ where $R_i \cong T_3$ $\forall i \in I$ if and only if $R$ is a commutative Boolean ternary ring.*

*Proof.* Let $R$ be a commutative Boolean ternary ring. Then $R$ is isomorphic to a subdirect sum of subdirectly irreducible ternary ring $\{R_i : i \in I\}$ which are homomorphic images of $R$. Since $R$ is commutative Boolean, each homomorphic image $R_i$ of $R$ is also commutative Boolean. Also, each $R_i$ is subdirectly irreducible. So, each $R_i \cong T_3$. Then each $R_i$ is a ternary field. Thus the commutative Boolean ternary ring $R$ is isomorphic to a subdirect sum of ternary fields $\{R_i : i \in I\}$, where $R_i \cong T_3$,

$\forall i \in I$. Conversely suppose that $R$ is isomorphic to subdirect sum, say $T$ of ternary fields $\{R_i : i \in I\}$, where $R_i \cong T_3$, for each $i \in I$. Let $f \in T$ then $f(i) \in R_i$, for $i \in I$. Since $R_i \cong T_3$. $(f(i))^3 = f(i)$ i.e $f(i).f(i).f(i) = f(i)$ i.e $f^3(i) = f(i)$, for all $i \in I$. So $f^3 = f$. Thus each element of $T$ is idempotent. Again $f(i) \in R_i \cong T_3$. Hence, each element of $R$ is also idempotent(as $RT$). Again since each $R_i \cong T_3$, each $R_i$ is commutative. So the complete direct sum and hence the subdirect sum $T$ of ternary fields $\{R_i : i \in I\}$ is commutative. Thus $R$ is commutative. So $R$ is a commutative Boolean ternary ring.

## References

[1] Brikhoff, G. Subdirect unions in universal algebra, Bulletin of American Mathematical society, vol. 50(1944), pp. 764-768.

[2] Dörnte, W. : Untersuchungen uber einen Verallgemeinerten Gruppenbegriff; Math. Z. 29 (1928), 1-19.

[3] McCoy, N. H. The theory of rings, Chelsea publishing company, New York,1964.

[4] Post, E. L.: Polyadic groups; Trans. Amer. Math. Soc. 48 (1940), 208-350.

[5] Prüfer, H. : Theorie der Abelschen Gruppen, I. Grundeigenschaften; Math.Z. 20 (1924), 165-187.

[6] Lehmer, D. H.: A Ternary Analogue of Abelian Groups; Amer. J. Math., 54(2) (1932), 329-338.

[7] Lister, W. G.: Ternary Rings; Trans. Amer. Math. Soc. 154 (1971), 37-55.

[8] Salim, Md. and Dutta, T. K. - *A Class of General Type of Regularities And Generalized Semiprime Ideals In Ternary Ring* —- (Accepted for publication)—- **The Southeast Asian Bullatin of Mathematics.**

[9] Salim, Md. Chanda, T. and Dutta, T. K. - *Regular equivalence and strongly regular equivalence on multiplicative ternary hyperring.*—- (Accepted for publication)—- **Journal of Hyperstructure.**