Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 2, Issue 1
February 26, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com

Operational Images and Relations of Two and Three Variable Hypergeometric Series

Maged G. Bin-Saad [1], Maisoon A. Hussein [2]

[1] Department of Mathematics, Aden University, Aden, P.O. Box 6014 ,Yemen.
mgbinsaad@yahoo.com.

[2] Department of Mathematics, Aden University, Aden, P.O. Box 6014, Yemen.
mgbinsaad@Hotamil.com.

Abstract

Based upon the classical derivative and integral operators we introduce a new symbolic operational images for hypergeometric functions of two and three variables. By means of these symbolic operational images a number of operational relations among the hypergeometric functions of two and three variables are then found. Other closely-related results are also considered.

Keywords: Formal operators Operational images; Appell's and Lauricella's hypergeometric functions; operational relations.

1. Introduction

The subject of operational calculus has gained importance and popularity during the past three decades, due mainly to its demonstrated applications in numerous seemingly diverse fields of science and engineering. One of the most recent development on the use of operational calculus is the finding of operational representations and relations for hypergeometric functions and polynomials which play an important role in the investigation of various useful properties of the hypergeometric function and polynomials. Operational representations and relations involving one and more variables hypergeometric series have been given considerable in the literature, see for example, Chen and Srivastava [2] ,Goyal ,Jain and Gaur([3],[4]) Kalla ([5],[6]),Kalla and Saxena([7] ,[8]),Kant and Koul [9] ,Tu, Chyan and Srivastava [13]. In this work we will deal with operational definitions ruled by the operators D_x and D_x^{-1} where D_x denotes the derivative operator and D_x^{-1} defines the inverse of the derivative [1]. zero. The following two formulas are well-know consequences of the derivative operator D_x and the integral operator D_x^{-1} (see, Ross [10]):

$$D_x^m x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)} x^{\lambda-m}, \quad \lambda > -1, \tag{1. 1}$$

$$D_x^{-m} x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m+1)} x^{\lambda+m}, \quad \lambda > -1, \tag{1. 2}$$

$$, \quad m \in N \cup \{0\}, \quad \lambda \in C - \{-1,-2,\ldots\}$$

Based on the operational relations (1. 1) and (1. 2) the action of the set of operators

$$\left\{D_x^m D_x^{-m}\right\}, \qquad (m = 0,1,2,3,\ldots),$$

is given by [1]

$$D_t^m D_u^{-m}\left\{t^{a+m-1}u^{b-1}\right\}=\frac{(a)_m}{(b)_m}\left\{t^{a-1}u^{b+m-1}\right\}. \quad (1.\,3)$$

By introducing the constricted notation

$$D^m[t;u]=\left(D_t u^{-1} D_u^{-1} t\right)^m=D_t^m u^{-m} D_u^{-m} t^m, m=0,1,2,\ldots . \quad (1.4)$$

it is easily verified that

$$D^m[t;u]\left\{t^{a-1}u^{c-1}\right\}=\left\{t^{a-1}u^{c-1}\right\}\frac{(a)_m}{(b)_m},\quad (m=0,1,2,\ldots). \quad (1.\,5)$$

Moreover, the multinomial expansion of algebra

$$\left(1-x_1-x_2\cdots-x_n\right)^{-a}=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}}{m_1!\ldots m_n!}x_1^{m_1}\ldots x_n^{m_n}, \quad (1.\,6)$$

has its analogue the operator multinomial expansion

$$\left(1-D[t_1;u_1]-\cdots-D[t_n;u_n]\right)^{-a}=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}}{m_1!\ldots m_n!}D^{m_1}[t_1;u_1]\cdots D^{m_n}[t_n;u_n]. \quad (1.\,7)$$

The aim of this paper is to obtain operational representations of binomial-types and exponential type for Appells functions of two variables F_1, F_2, F_3 and F_4 and Lauricella's functions (see [11,p.22-23]) of three variables $F_E, F_F, \ldots, F_T$ (see [11,p.42-43]). Also we aim here to derive operational relations between the above said Appell's and Lauricella's functions. Section 2 deals with the derivation of operational representations of binomial-type for Appell's double and Lauricella's triple hypergeometric series. In Section 3 we establish operational representation of exponential-type for Appell's double and Lauricella's triple hypergeometric series. Section 4 aims at presenting operational relations between Appell's functions F_1, F_2, F_3 and F_4 and Lauricella's functions $F_E, F_F, \ldots, F_T$.

2. Operational Representations of Binomial-Type

By means of the operator $D[t;u]$ defined by (1. 4), we aim in this section at establishing the following operational representations.

$$\left(1-xD[t_1;u_1]-yD[t_2;u_2]-zD[t_2;u_3]\right)^{-a_1}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}t_3^{b_3-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}F_E\left(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z\right), \quad (2.\,1)$$

$$\left(1-xD[t_1;u_1]-yD[t_2;u_2]-zD[t_1;u_2]\right)^{-a_1}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_F\left(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right), \quad (2.\,2)$$

$$\left(1-xD[t_1;u_1]-yD[t_2;u_2]-zD[t_3;u_2]\right)^{-a_1}\left\{t_1^{b_1-1}t_2^{b_2-1}t_3^{b_3-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}t_3^{b_3-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_G\left(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;x,y,z\right), \quad (2.\,3)$$

$$\left(1-xD[t_1;u_1]\right)^{-b_1}\left(1-yD[t_1;u_2]-zD[t_1;u_3]\right)^{-b_2}\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}$$
$$=\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}F_E\left(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z\right), \quad (2.\,4)$$

$$\left(1-xD[t_1;u_1]\right)^{-a_1}\left(1-yD[t_2;u_2]-zD[t_1;u_3]\right)^{-a_2}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}F_K\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z\right), \quad (2.\,5)$$

$$\left(1-xD[t_1;u_1]\right)^{-a_1}\left(1-yD[t_2;u_2]-zD[t_1;u_2]\right)^{-a_2}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$

$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_M\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right), \quad (2.6)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-a_1}\left(1-yD\left[t_2;u_1\right]-zD\left[t_3;u_1\right]\right)^{-a_2}\left\{t_1^{b_1-1}t_2^{b_2-1}t_3^{b_3-1}u_1^{c_1-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}t_3^{b_3-1}u_1^{c_1-1}\right\}F_S\left(a_1,a_2,a_2,b_1,b_2,b_3;c_1,c_1,c_1;x,y,z\right), \quad (2.7)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-a_1}\left(1-yD\left[t_2;u_1\right]-zD\left[t_1;u_1\right]\right)^{-a_2}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}\right\}F_T\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_1,c_1;x,y,z\right), \quad (2.8)$$

$$\left(1-xD\left[t_1;u_1\right]-zD\left[t_1;u_2\right]\right)^{-b_1}\left(1-yD\left[t_1;u_2\right]\right)^{-b_2}\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_F\left(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right), \quad (2.9)$$

$$\left(1-xD\left[t_1;u_1\right]-zD\left[t_2;u_3\right]\right)^{-b_1}\left(1-yD\left[t_2;u_2\right]\right)^{-b_2}\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}$$
$$=\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}F_K\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z\right), \quad (2.10)$$

$$\left(1-xD\left[t_1;u_1\right]-zD\left[t_2;u_2\right]\right)^{-b_1}\left(1-yD\left[t_2;u_2\right]\right)^{-b_2}\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_M\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right), \quad (2.11)$$

$$\left(1-xD\left[t_1;u_1\right]-zD\left[t_3;u_2\right]\right)^{-b_1}\left(1-yD\left[t_2;u_2\right]\right)^{-b_2}\left\{t_1^{a_1-1}t_2^{a_2-1}t_3^{a_3-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{a_1-1}t_2^{a_2-1}t_3^{a_3-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_N\left(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right), \quad (2.12)$$

$$\left(1-xD\left[t_1;u_1\right]-zD\left[t_2;u_1\right]\right)^{-b_1}\left(1-yD\left[t_2;u_1\right]\right)^{-b_2}\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}\right\}$$
$$=\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}\right\}F_T\left(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_1,c_1;x,y,z\right), \quad (2.13)$$

$$\left(1-xD\left[t_1;u_1\right]-zD\left[t_3;u_2\right]\right)^{-a_1}\left(1-yD\left[t_1;u_2\right]\right)^{-a_2}\left\{t_1^{b_1-1}t_3^{b_3-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_3^{b_3-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_P\left(a_1,a_2,a_1,b_1,b_1,b_3;c_1,c_2,c_2;x,y,z\right), \quad (2.14)$$

$$\left(1-xD\left[t_1;u_1\right]-yD\left[t_2;u_1\right]\right)^{-a_1}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}\right\}F_1\left[a_1,b_1,b_2;c_1;x,y\right], \quad (2.15)$$

$$\left(1-xD\left[t_1;u_1\right]-yD\left[t_2;u_2\right]\right)^{-a_1}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_2\left[a_1,b_1,b_2;c_1,c_2;x,y\right], \quad (2.16)$$

$$\left(1-xD\left[t_1;u_1\right]-yD\left[t_1;u_2\right]\right)^{-a_1}\left\{t_1^{b_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{b_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_4\left[a_1,b_1;c_1,c_2;x,y\right], \quad (2.17)$$

$$\left(1-xD\left[t_1;u_1\right]-zD\left[t_1;u_2\right]\right)^{-a_1}\left(1-yD\left[t_2;u_2\right]\right)^{-a_2}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_R\left(a_1,a_2,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right), \quad (2.18)$$

$$\left(1-xD\left[t_1;u_1\right]-yD\left[t_2;u_2\right]\right)^{-b_1}\left(1-zD\left[t_1;u_2\right]\right)^{-b_2}\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_P\left(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_2;x,y,z\right), \quad (2.19)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-b_1}\left(1-yD\left[t_1;u_2\right]\right)^{-b_2}\left(1-zD\left[t_1;u_2\right]\right)^{-b_3}\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_G\left(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;x,y,z\right), \quad (2.20)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-b_1}\left(1-yD\left[t_2;u_1\right]\right)^{-b_2}\left(1-zD\left[t_2;u_1\right]\right)^{-b_3}\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}\right\}$$

$$=\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}\right\}F_S\left(a_1,a_2,a_2,b_1,b_2,b_3;c_1,c_1,c_1;x,y,z\right), \quad (2.\ 21)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-a_1}\left(1-yD\left[t_2;u_2\right]\right)^{-a_2}\left(1-zD\left[t_1;u_2\right]\right)^{-a_3}\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_N\left(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z\right), \quad (2.\ 22)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-b_1}\left(1-yD\left[t_1;u_1\right]\right)^{-b_2}\left\{t_1^{a_1-1}u_1^{c_1-1}\right\}$$
$$=\left\{t_1^{a_1-1}u_1^{c_1-1}\right\}F_1\left[a_1,b_1,b_2;c_1;x,y\right], \quad (2.\ 23)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-b_1}\left(1-yD\left[t_1;u_2\right]\right)^{-b_2}\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}$$
$$=\left\{t_1^{a_1-1}u_1^{c_1-1}u_2^{c_2-1}\right\}F_2\left[a_1,b_1,b_2;c_1,c_2;x,y\right], \quad (2.\ 24)$$

$$\left(1-xD\left[t_1;u_1\right]\right)^{-b_1}\left(1-yD\left[t_2;u_1\right]\right)^{-b_2}\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}\right\}$$
$$=\left\{t_1^{a_1-1}t_2^{a_2-1}u_1^{c_1-1}\right\}F_3\left[a_1,a_2,b_1,b_2;c_1;x,y\right]. \quad (2.\ 25)$$

Derivation of the results (2. 1) to (2. 25)

To prove formula (2. 1), let I denotes the left-hand side of equation (2. 1), then in view of the multinomial expansion (1. 7) one gets

$$I=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}}{m!n!p!}\left(xD\left[t_1,u_1\right]\right)^m\left(yD\left[t_2,u_2\right]\right)^n\left(zD\left[t_2,u_3\right]\right)^p\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\},$$

$$=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}}{m!n!p!}x^mD^m\left[t_1,u_1\right]y^nD^n\left[t_2,u_2\right]z^pD^p\left[t_2,u_3\right]\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}.$$

Now, by using the operator $D\left[t;u\right]$ defined by (1.14.5), we get

$$I=\left\{t_1^{b_1-1}t_2^{b_2-1}u_1^{c_1-1}u_2^{c_2-1}u_3^{c_3-1}\right\}\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}}{m!n!p!}\frac{(b_1)_m(b_2)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p}x^my^nz^p,$$

which in view of the definition of Lauricella's functions F_E (see [11,p.42(1)]) yields the right-hand side of equation (2.1) and thereby (2. 1) is proved. The proofs of formulas (2. 2) to (2. 25) run parallel to that of formula (2. 1), so are skipped details .

3. Operational Relations

In the present section, we shall establish certain operational relations involving Appell's functions of two variables defined in [11,p.22-23] and Lauricella's functions of three variables defined in [11,p.42-43]. For convenience, let $\Delta=\left(1-zD\left[t;u\right]\right)$. Consider

$$\left(1-zD[t;u]\right)^{-b_1}F_1\left[a_1,b_1,b_2;c_1;\frac{x}{\Delta},y\right]\left\{t^{a_2-1}u^{c_2-1}\right\}$$

$$=\sum_{m,n=0}^{\infty}\frac{(a_1)_{m+n}(b_1)_m(b_2)_n}{m!n!}x^my^n\left(1-zD[t;u]\right)^{-b_1-m}\left\{t^{a_2-1}u^{c_2-1}\right\}$$

$$=\left\{t^{a_2-1}u^{c_2-1}\right\}\sum_{m,n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_p(b_1)_{m+p}(b_2)_n}{(c_1)_{m+n}(c_2)_p\,m!n!p!}x^my^nz^p.$$

Therefore

$$(1-zD[t;u])^{-b_1} F_1\left[a_1,b_1,b_2;c_1;\frac{x}{\Delta},y\right]\{t^{a_2-1}u^{c_2-1}\}$$

$$=\{t^{a_2-1}u^{c_2-1}\}F_M(a_2,a_1,a_1,b_1,b_2,b_1;c_2,c_1,c_1;z,y,x). \quad (3.1)$$

Applying the same techniques and making slight adjustment in interchanging of variables, we derived the following operational relationships.

$$(1-zD[t;u])^{-b_1} F_1\left[a_1,b_1,b_2;c_1;\frac{xt}{\Delta},yt\right]\{t^{a_1-1}u^{c_2-1}\}$$

$$=\{t^{a_1-1}u^{c_2-1}\}F_F(a_1,a_1,a_1,b_1,b_2,b_1;c_2,c_1,c_1;z,yt,xt), \quad (3.2)$$

$$(1-zD[t;u])^{-b_1} F_1\left[a_1,b_1,b_2;c_1;\frac{xu}{\Delta},yu\right]\{t^{a_2-1}u^{c_1-1}\}$$

$$=\{t^{a_2-1}u^{c_1-1}\}F_T(a_2,a_1,a_1,b_1,b_2,b_1;c_1,c_1,c_1;z,yu,xu), \quad (3.3)$$

$$(1-zD[t;u])^{-b_2} F_1\left[a_1,b_1,b_2;c_1;x,\frac{y}{\Delta}\right]\{t^{a_2-1}u^{c_2-1}\}$$

$$=\{t^{a_2-1}u^{c_2-1}\}F_M(a_2,a_1,a_1,b_2,b_1,b_2;c_2,c_1,c_1;z,x,y), \quad (3.4)$$

$$(1-zD[t;u])^{-b_2} F_1\left[a_1,b_1,b_2;c_1;xt,\frac{yt}{\Delta}\right]\{t^{a_1-1}u^{c_2-1}\}$$

$$=\{t^{a_1-1}u^{c_2-1}\}F_F(a_1,a_1,a_1,b_2,b_1,b_2;c_2,c_1,c_1;z,xt,yt), \quad (3.5)$$

$$(1-zD[t;u])^{-b_2} F_1\left[a_1,b_1,b_2;c_1;xu,\frac{yu}{\Delta}\right]\{t^{a_2-1}u^{c_1-1}\}$$

$$=\{t^{a_2-1}u^{c_1-1}\}F_T(a_2,a_1,a_1,b_2,b_1,b_1;c_1,c_1,c_1;z,xu,yu), \quad (3.6)$$

$$(1-zD[t;u])^{-a_1} F_1\left[a_1,b_1,b_2;c_1;\frac{x}{\Delta},\frac{y}{\Delta}\right]\{t^{b_3-1}u^{c_2-1}\}$$

$$=\{t^{b_3-1}u^{c_2-1}\}F_G(a_1,a_1,a_1,b_3,b_1,b_2;c_2,c_1,c_1;z,x,y), \quad (3.7)$$

$$(1-zD[t;u])^{-a_1} F_1\left[a_1,b_1,b_2;c_1;\frac{x}{\Delta},\frac{yt}{\Delta}\right]\{t^{b_2-1}u^{c_2-1}\}$$

$$=\{t^{b_2-1}u^{c_2-1}\}F_F(a_1,a_1,a_1,b_2,b_1,b_2;c_2,c_1,c_1;z,x,yt), \quad (3.8)$$

$$(1-zD[t;u])^{-b_1} F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{x}{\Delta},y\right]\{t^{a_2-1}u^{c_3-1}\}$$

$$=\{t^{a_2-1}u^{c_3-1}\}F_K(a_2,a_1,a_1,b_1,b_2,b_1;c_3,c_2,c_1;z,y,x), \quad (3.9)$$

$$(1-zD[t;u])^{-b_1} F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{xt}{\Delta},yt\right]\{t^{a_1-1}u^{c_3-1}\}$$

$$=\left\{t^{a_1-1}u^{c_3-1}\right\}F_E\left(a_1,a_1,a_1,b_2,b_1,b_1;c_2,c_1,c_3;yt,xt,z\right), \quad (3.10)$$

$$\left(1-zD[t;u]\right)^{-b_1}F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{xt}{\Delta},ytu\right]\left\{t^{a_1-1}u^{c_2-1}\right\}$$

$$=\left\{t^{a_1-1}u^{c_2-1}\right\}F_F\left(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;xt,ytu,z\right), \quad (3.11$$

$$\left(1-zD[t;u]\right)^{-b_1}F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{x}{\Delta},yu\right]\left\{t^{a_2-1}u^{c_2-1}\right\}$$

$$=\left\{t^{a_2-1}u^{c_2-1}\right\}F_P\left(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_2;x,z,yu\right), \quad (3.12)$$

$$\left(1-zD[t;u]\right)^{-b_2}F_2\left[a_1,b_1,b_2;c_1,c_2;x,\frac{y}{\Delta}\right]\left\{t^{a_2-1}u^{c_3-1}\right\}$$

$$=\left\{t^{a_2-1}u^{c_3-1}\right\}F_K\left(a_2,a_1,a_1,b_2,b_1,b_2;c_3,c_1,c_2;z,x,y\right), \quad (3.13)$$

$$\left(1-zD[t;u]\right)^{-b_2}F_2\left[a_1,b_1,b_2;c_1,c_2;xt,\frac{yt}{\Delta}\right]\left\{t^{a_1-1}u^{c_3-1}\right\}$$

$$=\left\{t^{a_1-1}u^{c_3-1}\right\}F_E\left(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;xt,yt,z\right), \quad (3.14)$$

$$\left(1-zD[t;u]\right)^{-b_2}F_2\left[a_1,b_1,b_2;c_1,c_2;xtu,\frac{yt}{\Delta}\right]\left\{t^{a_1-1}u^{c_1-1}\right\}$$

$$=\left\{t^{a_1-1}u^{c_1-1}\right\}F_F\left(a_1,a_1,a_1,b_2,b_1,b_2;c_2,c_1,c_1;yt,xtu,z\right), \quad (3.15)$$

$$\left(1-zD[t;u]\right)^{-b_2}F_2\left[a_1,b_1,b_2;c_1,c_2;xu,\frac{y}{\Delta}\right]\left\{t^{a_2-1}u^{c_1-1}\right\}$$

$$=\left\{t^{a_2-1}u^{c_1-1}\right\}F_P\left(a_1,a_2,a_1,b_2,b_2,b_1;c_2,c_1,c_1;y,z,xu\right), \quad (3.16)$$

$$\left(1-zD[t;u]\right)^{-a_1}F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{x}{\Delta},\frac{y}{\Delta}\right]\left\{t^{b_3-1}u^{c_3-1}\right\}$$

$$=\left\{t^{b_3-1}u^{c_3-1}\right\}F_A^{(3)}\left(a_1,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right), \quad (3.17)$$

$$\left(1-zD[t;u]\right)^{-a_1}F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{x}{\Delta},\frac{yt}{\Delta}\right]\left\{t^{b_2-1}u^{c_3-1}\right\}$$

$$=\left\{t^{b_2-1}u^{c_3-1}\right\}F_E\left(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,yt,z\right), \quad (3.18)$$

$$\left(1-zD[t;u]\right)^{-a_1}F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{xt}{\Delta},\frac{yu}{\Delta}\right]\left\{t^{b_1-1}u^{c_2-1}\right\}$$

$$=\left\{t^{b_1-1}u^{c_2-1}\right\}F_F\left(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;xt,yu,z\right), \quad (3.19)$$

$$\left(1-zD[t;u]\right)^{-a_1}F_2\left[a_1,b_1,b_2;c_1,c_2;\frac{x}{\Delta},\frac{yu}{\Delta}\right]\left\{t^{b_3-1}u^{c_2-1}\right\}$$

$$=\left\{t^{b_3-1}u^{c_2-1}\right\}F_G\left(a_1,a_1,a_1,b_1,b_1,b_3;c_1,c_2,c_2;x,yu,z\right), \quad (3.20)$$

$$(1-zD[t;u])^{-b_1} F_3\left[a_1,a_2,b_1,b_2;c_1;\frac{x}{\Delta},y\right]\{t^{a_3-1}u^{c_2-1}\}$$

$$=\{t^{a_3-1}u^{c_2-1}\}F_N(a_3,a_2,a_1,b_1,b_2,b_1;c_2,c_1,c_1;z,y,x), \quad (3.21)$$

$$(1-zD[t;u])^{-b_1} F_3\left[a_1,a_2,b_1,b_2;c_1;\frac{xt}{\Delta},y\right]\{t^{a_1-1}u^{c_2-1}\}$$

$$=\{t^{a_1-1}u^{c_2-1}\}F_R(a_1,a_2,a_1,b_1,b_2,b_1;c_2,c_1,c_1;z,y,xt), \quad (3.22)$$

$$(1-zD[t;u])^{-b_1} F_3\left[a_1,a_2,b_1,b_2;c_1;\frac{xu}{\Delta},yut\right]\{t^{a_2-1}u^{c_1-1}\}$$

$$=\{t^{a_2-1}u^{c_1-1}\}F_T(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_1,c_1;xu,yut,z), \quad (3.23)$$

$$(1-zD[t;u])^{-b_1} F_3\left[a_1,a_2,b_1,b_2;c_1;\frac{x}{\Delta},yt\right]\{t^{a_2-1}u^{c_2-1}\}$$

$$=\{t^{a_2-1}u^{c_2-1}\}F_P(a_2,a_1,a_2,b_1,b_1,b_2;c_2,c_1,c_1;z,x,yt), \quad (3.24)$$

$$(1-zD[t;u])^{-b_2} F_3\left[a_1,a_2,b_1,b_2;c_1;x,\frac{y}{\Delta}\right]\{t^{a_3-1}u^{c_2-1}\}$$

$$=\{t^{a_3-1}u^{c_1-1}\}F_N(a_3,a_1,a_2,b_2,b_1,b_2;c_2,c_1,c_1;z,x,y), \quad (3.25)$$

$$(1-zD[t;u])^{-b_2} F_3\left[a_1,a_2,b_1,b_2;c_1;xt,\frac{y}{\Delta}\right]\{t^{a_1-1}u^{c_2-1}\}$$

$$=\{t^{a_1-1}u^{c_2-1}\}F_P(a_1,a_2,a_1,b_2,b_2,b_1;c_2,c_1,c_1;z,y,xt), \quad (3.26)$$

$$(1-zD[t;u])^{-b_2} F_3\left[a_1,a_2,b_1,b_2;c_1;x,\frac{yt}{\Delta}\right]\{t^{a_2-1}u^{c_2-1}\}$$

$$=\{t^{a_2-1}u^{c_2-1}\}F_R(a_2,a_1,a_2,b_2,b_1,b_2;c_2,c_1,c_1;z,x,yt), \quad (3.27)$$

$$(1-zD[t;u])^{-b_2} F_3\left[a_1,a_2,b_1,b_2;c_1;xtu,\frac{yu}{\Delta}\right]\{t^{a_1-1}u^{c_1-1}\}$$

$$=\{t^{a_1-1}u^{c_1-1}\}F_T(a_2,a_1,a_1,b_2,b_1,b_2;c_1,c_1,c_1;yu,xtu,z), \quad (3.28)$$

$$(1-zD[t;u])^{-a_1} F_4\left[a_1,b_1;c_1,c_2;\frac{x}{\Delta},\frac{y}{\Delta}\right]\{t^{b_2-1}u^{c_3-1}\}$$

$$=\{t^{b_1-1}u^{c_3-1}\}F_E(a_1,a_1,a_1,b_2,b_1,b_1;c_3,c_2,c_1;z,y,x), \quad (3.29)$$

$$(1-zD[t;u])^{-a_1} F_4\left[a_1,b_1;c_1,c_2;\frac{x}{\Delta},\frac{yu}{\Delta}\right]\{t^{b_2-1}u^{c_2-1}\}$$

$$=\{t^{b_2-1}u^{c_2-1}\}F_F(a_1,a_1,a_1,b_2,b_1,b_2;c_1,c_2,c_2;x,z,yu). \quad (3.30)$$

4. Conclusion

In this work we introduce a certain set of operators as the main working tools to develop a theory of operational representations and operational relations of hypergeometric functions of two , three and four variables. Indeed, we have obtained operational representations of binomial-types for Appell's functions of two variables F_1, F_2, F_3 and F_4 and Lauricella's functions of three variables $F_E, F_F, \ldots, F_T$. Also we derived operational relations between the above said Appell's and Lauricella's functions. In a forthcoming papers we will consider the problems of using the operational representations obtained in this work in order to derive a number of expansion and summation formulas involving the double functions F_1, F_2, F_3 and F_4, the triple functions $F_E, F_F, \ldots, F_T$,Kampé de Fériet's Function $F_{l:m;n}^{p:q;k}$ defined in [11,p.27(28)] , and the Generalized hypergeometric Function ${}_pF_q$ [11,p.19(23)].

References

[1] Bin-Saad, Maged G. (2011). Symbolic operational images and decomposition formulas of hypergeometric functions. *J. Math. Anal. Appl. ,376,* 451–468.

[2] Chen, M.P., Srivastava, H.M. (1997). Fractional calculus operators and their applications involving power functions and summation of series. *Appl. Math. Comput., 81*, 283-304.

[3] Goyal, S.P., Jain, R.M., Gaur, N.(1992). Fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials II. *Indian J. Pure Appl. Math. ,23,* 121-128.

[4] Goyal, S.P. ,Jain, R.M. ,Gaur, N.(1991). Fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials. *Indian J. Pure Appl. Math., 22,* 403-411.

[5] Kalla, S.L. (1970).Integral operators of fractional integration. *Mat. Notae 22,*89-93.

[6] Kalla, S.L.(1976) .Integral operators of fractional integration II .*Mat. Notae 25,* 29-35.

[7] Kalla, S.L., Saxena, R. K.(1969). Integral operators involving hypergeometric functions. *Math. Zeitschr.108* , 231-234.

[8] Kalla, S.L., Saxena, R. K.(1974). Integral operators involving hypergeometric functions II. *Univ. Nac. Tucuman Rev. Ser. A 24* , 31-36.

[9] Kant, S., Koul, C.L.(1991). On fractional integral operators. *J. Indian Math. Soc.(N.S) 56,*97-107.

[10] Ross, B., Fractional calculus and its applications(1975). Lecture Notes in Mathematics. Vol. 457, Springer-Verlag.

[11] Srivastava, H.M. and Karlsson, P.W. (1985).Multiple Gaussian hypergeometric series. Halsted Press, Brisbane, New York and Toronto .

[12] Tu, S.T. ,Chyan, D.-K., Srivastava, H.M.(1996). Certain operators of fractional calculus and their applications associated with logarithmic and Digamma functions. *J. Fractional calculus 10,* 67-73.