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Oscillatory Behavior of Even Order Quasilinear Delay Difference Equations

R. Arul[1] , K. Alagesan[2] and G. Ayyappan[3]
[1&2]Department of Mathematics, Kandaswami Kandar's College, Velur-638182, Namakkal Dt., Tamil Nadu, India. Email: rarulkkc@gmail.com
[3]Department of Mathematics, Periyar University College of Arts and Science, Pappireddipatti – 636905, Tamil Nadu, India. Email: ayyapmath@gmail.com

Abstract. In this paper, we consider the quasilinear delay difference equation of the form

$$\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^{\alpha}\right)+q_n x_{\tau_n}^{\beta}=0,\ n\in\mathbb{N}_0$$

and obtained sufficient conditions for the oscillation for all solutions. Examples are given to illustrate the main results.

Keywords and Phrases. Oscillation; even order; delay difference equations.

1. Introduction

In this paper, we study the oscillatory behavior of even order quasilinear delay difference equation of the form

(1.1) $$\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^{\alpha}\right)+q_n x_{\tau_n}^{\beta}=0,\ n\in\mathbb{N}_0$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1}-x_n$ and $\mathbb{N}_0=\{n_0,n_0+1,n_0+2,...\}$ and n_0 is a nonnegative integer, subject to the following conditions:

(C_1) α and β are ratios of odd positive integers;

(C_2) $\{q_n\}$ and $\{a_n\}$ are positive real sequences with $\sum_{n=n_0}^{\infty}\frac{1}{a_n^{1/\alpha}}<\infty$ and $\Delta a_n\geq 0$ for all $n\in\mathbb{N}_0$;

(C_3) $\{\tau_n\}$ is a nondecreasing sequence of integers and $\tau_n\leq n$ with $\tau_n\to\infty$ as $n\to\infty$.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined and satisfying equations (1.1) for all $n\in\mathbb{N}_0$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In [2-5, 9-11, 13, 14, 16, 17], the authors investigated the oscillatory behavior or higher order difference equations.

In [11], the authors studied the nonoscillatory solutions fo higher order linear difference equation of the form

$$\Delta^m x_n+\delta a_n x_{n+1}=0$$

where $m\geq 2$ and $\delta=\pm 1$.

Consider the following equation

(1.2) $$\Delta\left(4^n\Delta^3 x_n\right)+4^{n-3}x_{n-3}=0.$$

Using matlab we found that equation (1.2) has two nonoscillatory solutions $x_n=\left\{(0.5)^n\right\}$, $x_n=\left\{(0.3831)^n\right\}$ and two oscillatory solutions $x_n=r^n\sin n\theta$, $x_n=r^n\cos n\theta$ where $r=1.2116$ and $\theta=12.39^\circ$. Now the question is: Can one obtain conditions under which all solutions of equation (1.1) are oscillatory?. The purpose of this work is to give an affirmative answer to this question. Therefore in this paper, we obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1).

In Section 2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.1) and in Section 3, we present some examples to illustrate the main results. The results established in this paper are discrete analogue of that in [15].

2. Main Results

To obtain our main results, we need the following lemmas.

Lemma 2.1. Let $\{x_n\}$ be defined for $n\geq n_0$ and $x_n>0$ with $\Delta^m x_n$ of constant sign for $n\geq n_0$ and not identically zero. Then, there exists an integer k, $0\leq k\leq m$ with $(m+k)$ odd for $\Delta^m x_n\leq 0$ and $(m+k)$ even for $\Delta^m x_n\geq 0$ such that

(i) $k\leq m-1$ implies $(-1)^{m+i}\Delta^i x_n>0$ for all $n\geq n_0$, $k\leq i\leq m-1$;

(ii) $k\geq 1$ implies $\Delta^i x_n>0$ for all large $n\geq n_0$, $1\leq i\leq k-1$.

Lemma 2.2. Let $\{x_n\}$ be defined for $n\geq n_0$ and $x_n>0$ with $\Delta^m x_n\leq 0$ and not identically zero. Then, there exists an integer $n_1\geq n_0$ such that

$$x_n\geq \frac{1}{(m-1)!}(n-n_1)^{m-1}\Delta^{m-1}x_{2^{m-k-1}n},\ n\geq n_1$$

where k is defined as in Lemma 2.1. Further, if x_n is increasing, then

$$x_n\geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x_n,\ n\geq 2^{m-1}n_1.$$

The proof of Lemma 2.1 and 2.2 can be found in [1].

In the sequel, we use the following notations:

$$\delta_n=\sum_{s=n}^{\infty}\frac{1}{a_s^{1/\alpha}},\quad R_n=\sum_{s=n}^{\infty}\frac{(s+1-n)^{(m-3)}}{(m-3)!}\delta_s,\ \text{and}\ Q_n=\sum_{s=n}^{\infty}\frac{(s+1-n)^{(m-4)}}{(m-4)!}\delta_s.$$

Theorem 2.1. Let $m\geq 4,$ be even and $(C_1)-(C_3)$ hold. Assume that the difference equation

(2.1) $$\Delta y_n+q_n\left(\frac{\tau_n^{m-1}}{(m-1)!a_{\tau_n}^{1/\alpha}}\right)^{\beta}y_{\tau_n}^{\beta/\alpha}=0$$

is oscillatory. If

(2.2) $$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[M^{\beta-\alpha}q_s\frac{2^{(4-2m)\beta}\tau_s^{\beta(m-2)}}{\left((m-2)!\right)^{\beta}}\delta_s^{\alpha}+\frac{\Delta\delta_s^{\alpha}}{\delta_s^{\alpha}}\right]>1,$$

and

(2.3) $$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[M^{\beta-\alpha}q_sR_s^{\alpha}-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{(\Delta R_s)^{\alpha+1}}{R_sQ_s^{\alpha}}\right]>1,$$

hold for every constant $M>0$, then every solution of equation (1.1) is oscillatory.

Proof: Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. It follows from equation (1.1) and Lemma 2.1 that there exists three possible cases:

(i) $x_n>0,\ \Delta x_n>0,\ \Delta^{m-1}x_n>0,\ \Delta^m x_n\le 0$ and $\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^{\alpha}\right)\le 0$;

(ii) $x_n>0,\ \Delta x_n>0,\ \Delta^{m-2}x_n>0,\ \Delta^{m-1}x_n\le 0$ and $\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^{\alpha}\right)\le 0$;

(iii) $x_n>0,\ \Delta^j x_n<0,\ \Delta^{j+1}x_n>0$ for every odd integer $j=1,2,\ldots,n-s$ and $\Delta^{m-1}x_n<0$, $\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^{\alpha}\right)\le 0$ for all $n\ge n_1$.

Assume that Case(i) holds. From Lemma 2.2, we have

(2.4) $$x_n=\frac{1}{(m-1)!a_n^{1/\alpha}}\left(\frac{n}{2^{m-1}}\right)^{m-1}a_n^{1/\alpha}\Delta^{m-1}x_n,$$

where $n\ge n_2=2^{m-1}n_1$. Hence by equation (1.1), we see that $y_n=a_n\left(\Delta^{m-1}x_n\right)^{\alpha}$ is a positive solution of the difference inequality

$$\Delta y_n+q_n\left(\frac{\tau_n^{m-1}}{(m-1)!a_{\tau_n}^{1/\alpha}}\right)^{\beta}y_{\tau_n}^{\beta/\alpha}\le 0,\quad n\ge n_2.$$

Therefore by Lemma 5 of Section 2 in [9], the difference equation

$$\Delta y_n+q_n\left(\frac{\tau_n^{m-1}}{(m-1)!a_{\tau_n}^{1/\alpha}}\right)^{\beta}y_{\tau_n}^{\beta/\alpha}=0$$

has an eventually positive solution for $n\ge n_2$. This contradicts the fact that (2.1) is oscillatory.

Assume Case(ii) holds. Define the function w_n by

(2.5) $$w_n=\frac{a_n(\Delta^{m-1}x_n)^{\alpha}}{\left(\Delta^{m-2}x_n\right)^{\alpha}},\ n\ge n_1.$$

Then $w_n<0$ for $n\ge n_1$. Taking into consideration that $a_n\left(\Delta^{m-1}x_n\right)^{\alpha}$ is decreasing, we have

$$a_s^{1/\alpha}\Delta^{m-1}x_s \le a_n^{1/\alpha}\Delta^{m-1}x_n,\ \ s \ge n \ge n_1.$$

Dividing the above inequality by $a_s^{1/\alpha}$ and summing from n to $\ell-1$, we obtain

$$\Delta^{m-2}x_\ell \le \Delta^{m-2}x_n + a_n^{1/\alpha}\Delta^{m-1}x_n\sum_{s=n}^{\ell-1}\frac{1}{a_s^{1/\alpha}}.$$

Letting $\ell \to \infty$, we have

$$0 \le \Delta^{m-2}x_n + a_n^{1/\alpha}\Delta^{m-1}x_n\delta_n.$$

or

$$-\frac{a_n^{1/\alpha}\Delta^{m-1}x_n}{\Delta^{m-2}x_n}\delta_n \le 1.$$

Then, by (2.5), we obtain

(2.6) $$-\delta_n^\alpha w_n \le 1.$$

In view of (2.5), we have

$$\Delta w_n = \frac{\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^\alpha\right)\left(\Delta^{m-2}x_n\right)^\alpha - a_n\left(\Delta^{m-1}x_n\right)^\alpha\Delta\left(\Delta^{m-2}x_n\right)^\alpha}{\left(\Delta^{m-2}x_n\right)^\alpha\left(\Delta^{m-2}x_{n+1}\right)^\alpha}$$

$$= \frac{\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^\alpha\right)}{\left(\Delta^{m-2}x_{n+1}\right)^\alpha} - \frac{a_n\left(\Delta^{m-1}x_n\right)^\alpha\left(\left(\Delta^{m-2}x_{n+1}\right)^\alpha - \left(\Delta^{m-2}x_n\right)^\alpha\right)}{\left(\Delta^{m-2}x_n\right)^\alpha\left(\Delta^{m-2}x_{n+1}\right)^\alpha}$$

$$= \frac{\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^\alpha\right)}{\left(\Delta^{m-2}x_{n+1}\right)^\alpha} - \frac{a_n\left(\Delta^{m-1}x_n\right)^\alpha}{\left(\Delta^{m-2}x_n\right)^\alpha} + \frac{a_n\left(\Delta^{m-1}x_n\right)^\alpha}{\left(\Delta^{m-2}x_{n+1}\right)^\alpha}$$

(2.7) $$\le -q_n\frac{x_{\tau_n}^\beta}{\left(\Delta^{m-2}x_{n+1}\right)^\alpha} - w_n + w_n\frac{\left(\Delta^{m-2}x_n\right)^\alpha}{\left(\Delta^{m-2}x_{n+1}\right)^\alpha},$$

we observe that since $\Delta^{m-1}x_n < 0$, we deduce that $\Delta^{m-2}x_n$ is decreasing. Therefore $\Delta^{m-2}x_n \ge \Delta^{m-2}x_{n+1} > 0$ and

$$w_n\frac{\left(\Delta^{m-2}x_n\right)^\alpha}{\left(\Delta^{m-2}x_{n+1}\right)^\alpha} \le w_n \text{ for all } n \ge n_1.$$

Hence, (2.7) becomes

$$\Delta w_n \le q_n\frac{x_{\tau_n}^\beta}{\left(\Delta^{m-2}x_n\right)^\alpha}.$$

On the other hand, by Lemma 2.2, we get

$$x_n \geq \frac{1}{(m-2)!}\frac{n^{m-2}}{2^{2m-4}}\Delta^{m-2}x_n, \quad n \geq n_2 = 2^{m-2}n_1.$$

Thus we have

$$x_{\tau_n} \geq \frac{1}{(m-2)!}\tau_n^{m-2}2^{4-2m}\Delta^{m-2}x_{\tau_n},$$

for sufficiently large $n \geq n_3 \geq n_2$. Then, there exists a constant $M > 0$ such that

$$\Delta w_n \leq -q_n\left(\frac{2^{(4-2m)}}{(m-1)!}\tau_n^{m-2}\right)^{\beta}\frac{\left(\Delta^{m-2}x_{\tau_n}\right)^{\beta}}{\left(\Delta^{m-2}x_n\right)^{\alpha}}$$

$$\leq -q_n\left(\frac{2^{(4-2m)}}{(m-1)!}\tau_n^{m-2}\right)^{\beta}M^{\beta-\alpha}, \quad n \geq n_3.$$

Multiplying the above inequality by δ_n^{α} and summing from n_3 to $n-1$, we obtain

$$\delta_n^{\alpha}w_n - \delta_{n_3}^{\alpha}w_{n_3} - \sum_{s=n_3}^{n-1}w_s\Delta\delta_s^{\alpha} + \sum_{s=n_3}^{n-1}M^{\beta-\alpha}q_s\frac{2^{(4-2m)\beta}}{((m-1)!)^{\beta}}\tau_s^{(m-2)\beta}\delta_s^{\alpha} \leq 0$$

or

$$\sum_{s=n_3}^{n-1}\left(M^{\beta-\alpha}q_s\frac{2^{(4-2m)\beta}}{((m-1)!)^{\beta}}\tau_s^{(m-2)\beta}\delta_s^{\alpha} + \frac{\Delta\delta_s^{\alpha}}{\delta_s^{\alpha}}\right) \leq \delta_{n_3}^{\alpha}w_{n_3} + 1 \leq 1.$$

By using (2.5) and fact that $\Delta\delta_s^{\alpha} < 0$, which contradicts to (2.2).

Assume Case(iii) hold. Taking into consideration that $a_n\left(\Delta^{m-1}x_n\right)^{\alpha}$ is nonincreasing, we have

$$a_s^{1/\alpha}\Delta^{m-1}x_s \leq a_n^{1/\alpha}\Delta^{m-1}x_n, \quad s \geq n \geq n_1$$

or

$$\Delta^{m-1}x_s \leq \frac{a_n^{1/\alpha}\Delta^{m-1}x_n}{a_s^{1/\alpha}}.$$

Summing the last inequality from n to $\ell - 1$, we have

$$\Delta^{m-2}x_{\ell} - \Delta^{m-2}x_n \leq a_n^{1/\alpha}\Delta^{m-1}x_n\sum_{s=n}^{\ell-1}\frac{1}{a_s^{1/\alpha}}.$$

Letting $\ell \to \infty$, we get

$$0-\Delta^{m-2}x_n \le a_n^{1/\alpha}\Delta^{m-1}x_n\sum_{s=n}^{\infty}\frac{1}{a_s^{1/\alpha}}$$

or

$$-\Delta^{m-2}x_n \le a_n^{1/\alpha}\Delta^{m-1}x_n\delta_n.$$

Summing the above inequality from n to ∞, we have

$$-\Delta^{m-3}x_n \ge -a_n^{1/\alpha}\Delta^{m-1}x_n\sum_{s=n}^{\infty}\delta_s.$$

Again summing the last inequality from n to ∞ a total of $(n-4)$ times, we have

$$-\Delta x_n \ge -a_n^{1/\alpha}\Delta^{m-1}x_n\sum_{s=n}^{\infty}\frac{(s+1-n)^{(m-4)}}{(m-4)!}\delta_s. \tag{2.8}$$

Summing the above inequality from n to ∞ implies that

$$x_n \ge -a_n^{1/\alpha}\Delta^{m-1}x_n\sum_{s=n}^{\infty}\frac{(s+1-n)^{(m-3)}}{(m-3)!}\delta_s. \tag{2.9}$$

Define

$$u_n=\frac{a_n\left(\Delta^{m-1}x_n\right)^{\alpha}}{x_n^{\alpha}}. \tag{2.10}$$

Then $u_n<0$ for all $n\ge n_1$, we have

$$\Delta u_n=\frac{\Delta\left(a_n\left(\Delta^{m-1}x_n\right)^{\alpha}\right)}{x_n^{\alpha}}-\frac{a_{n+1}\left(\Delta^{m-1}x_{n+1}\right)^{\alpha}}{x_n^{\alpha}x_{n+1}^{\alpha}}\Delta x_n^{\alpha}.$$

Using (1.1) and (2.10) in the last inequality, we have

$$\Delta u_n \le -q_n\frac{x_{\tau_n}^{\alpha}}{x_n^{\alpha}}-u_{n+1}\frac{\Delta x_n^{\alpha}}{x_n^{\alpha}}. \tag{2.11}$$

By Mean Value Theorem, we have

$$\Delta x_n^{\alpha}=\alpha t^{\alpha-1}\Delta x_n.$$

where $x_n \le t \le x_{n+1}$. Since $\alpha \ge 1$, we have

$$\Delta x_n^{\alpha} \ge \alpha x_n^{\alpha-1}\Delta x_n. \tag{2.12}$$

Again using the inequality (2.12) in (2.11), we obtain

$$\Delta u_n \le -q_n\frac{x_{\tau_n}^{\alpha}}{x_n^{\alpha}}-\alpha u_{n+1}\frac{\Delta x_n}{x_n}.$$

Using the inequality (2.8) in the last inequality, we have

$$\Delta u_n \le -q_n \frac{x_{\tau_n}^{\alpha}}{x_n^{\alpha}} - \alpha u_{n+1}^{1+1/\alpha} \sum_{s=n}^{\infty} \frac{(s+1-n)^{(m-4)}}{(m-4)!}\delta_s.$$

We observe that $\Delta x_n < 0$ and $\tau_n < n$, then there exist a constant $M > 0$, such that

$$\Delta u_n \le -q_n \frac{x_{\tau_n}^{\alpha}}{x_n^{\alpha}} x_{\tau_n}^{\beta-\alpha} - \alpha u_{n+1}^{1+1/\alpha} \sum_{s=n}^{\infty} \frac{(s+1-n)^{(m-4)}}{(m-4)!}\delta_s$$
$$\le -q_n M^{\beta-\alpha} - \alpha u_{n+1}^{1+1/\alpha} \sum_{s=n}^{\infty} \frac{(s+1-n)^{(m-4)}}{(m-4)!}\delta_s.$$

Multiplying the above inequality by R_n^{α} and summing the resulting inequality from n_1 to $n-1$, we have

$$R_n^{\alpha} u_n - R_{n_1}^{\alpha} \Delta u_{n_1} \le -\sum_{s=n_1}^{n-1} R_s^{\alpha} M^{\beta-\alpha} q_s + \alpha \sum_{s=n_1}^{n-1} u_s R_s^{\alpha-1}(\Delta R_s) - \alpha \sum_{s=n_1}^{n-1} R_s^{\alpha} u_{s+1}^{1+1/\alpha} Q_s$$

or

(2.13) $$R_n^{\alpha} u_n - R_{n_1}^{\alpha} \Delta u_{n_1} - \alpha \sum_{s=n_1}^{n-1} u_s R_s^{\alpha-1}(\Delta R_s) + \sum_{s=n_1}^{n-1} R_s^{\alpha} M^{\beta-\alpha} q_s + \alpha \sum_{s=n_1}^{n-1} R_s^{\alpha} u_{s+1}^{1+1/\alpha} Q_s \le 0.$$

By setting $A = R_n^{\alpha} Q_n$, $B = -(\Delta R_n) R_n^{\alpha-1}$ and $v = -u_n$. Using the inequality $Av^{\frac{\alpha+1}{\alpha}} - Bv \ge -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}$, $A > 0$ in (2.13), we have

$$\sum_{s=n_1}^{n-1} \left[M^{\beta-\alpha} q_s R_s^{\alpha} - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{(\Delta R_s)^{\alpha+1}}{R_s Q_s^{\alpha}} \right] \le R_{n_1}^{\alpha} u_{n_1} + 1 \le 1.$$

Letting the limit supreme in the last inequality, we get a contradiction to (2.3). This completes the proof.

From Theorem 2.1, we obtain the following corollaries.

Corollary 2.1. Let $m \ge 4$ be even, and $(C_1)-(C_3)$ hold. Further, assume that $\alpha = \beta$ and $\tau_n = n-k$, k is a positive integer. If

(2.14) $$\limsup_{n\to\infty} \sum_{s=\tau_n}^{n-1} q_s \frac{\left(\tau_s^{m-1}\right)^{\alpha}}{a_{\tau_s}} > \left((m-1)!\right)^{\alpha} \left(\frac{k}{k+1}\right)^{\alpha}.$$

(2.15) $$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[q_s \frac{2^{(4-2m)\beta} \tau_s^{\beta(m-2)}}{\left((m-2)!\right)^{\beta}} \delta_s^{\alpha} + \frac{\Delta \delta_s^{\alpha}}{\delta_s^{\alpha}} \right] > 1,$$

and

(2.16) $$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[q_s R_s^{\alpha}-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{(\Delta R_s)^{\alpha+1}}{R_s Q_s^{\alpha}}\right]>1,$$

hold, then every solution of equation (1.1) is oscillatory.

Corollary 2.2. Let $\alpha>\beta,\ m\geq 4$ be even, and $(C_1)-(C_3)$ hold. Assume that

(2.17) $$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1} q_s\left(\frac{\tau_s^{m-1}}{a_{\tau_s}^{1/\alpha}}\right)^{\beta}=\infty.$$

If (2.2) and (2.3) hold for every constant $M>0$, then every solution of equation (1.1) is oscillatory.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the fourth order difference equation

(3.1) $$\Delta\left(e^n\Delta^3 x_n\right)+(n+1)e^{n-1}x_{n-1}=0,\ n\geq 3.$$

Here $a_n=e^n, q_n=(n+1)e^{n-1},\ \tau_n=n-1,\ m=4$ and $\alpha=\beta=1$. It is easy to see that all conditions of Corollary 2.1 are satisfied. Hence every solution of equation (3.1) is oscillatory.

Example 3.2. Consider the fourth order difference equation

(3.2) $$\Delta\left(e^n\Delta x_n\right)^3+(n+1)e^{\frac{n-1}{3}}x_{n-1}=0,\ n\geq 3.$$

Here $a_n=e^n, q_n=(n+1)e^{\frac{n-1}{3}},\ \tau_n=n-1,\ m=4,\ \alpha=3$ and $\beta=1$. It is easy to see that all conditions of Corollary 2.2 are satisfied. Hence every solution of equation (3.2) is oscillatory.

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