Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

SCITECH
RESEARCH ORGANISATION|

Volume 5, Issue 5
Published online: November 04, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Some results for the generalized Beta function using *N*- fractional calculus

Abdul Malik H. AL – Hashemi [1], Mohammed Fadel Mohammed [2]
[1] Department of Mathematics , Faculty of Education Saber, Aden University , Yemen
[2] Department of Mathematics , Faculty of Education Saber, Aden University , Yemen

Abstract.

In this paper, some results for the generalized Beta function are derived by using N-fractional calculus of the logarithm function . Also, some results associated with the usual Beta function are obtained as special cases of the main results.

Keywords: N- Fractional Calculus Operator; Generalized Beta function; Logarithm function.

1. Introduction

We adopt the following definition of fractional calculus :

Definition 1.1. (by K. Nishimoto [2])

Let $D=\{D_-, D_+\}$, $C=\{C_-, C_+\}$, C_- be a curve along the cut joining two points z and $-\infty$, C_+ be a curve along the cut joining two points z and $-\infty + i\,Im(z)$, D_- be a domain surrounded by C_- , D_+ is a domain surrounded by C_+ (Here D contains the points over the curve C) .

Moreover , let $f = f(z)$ be a regular function in $D(z \in D)$

$$f_v(z) = (f)_v = {}_c(f)_v = \frac{\Gamma(v+1)}{2\pi i}\int_c \frac{f(\zeta)d\zeta}{(\zeta - z)^{v+1}} \quad (v \notin Z^-), \qquad (1.1)$$

and

$$(f)_{-m} = \lim_{v \to -m} (f)_v \quad (m \in Z^+), \qquad (1.2)$$

where

$$-\pi \leq arg(\zeta - z) \leq \pi \text{ for } C_- \text{ and } 0 \leq arg(\zeta - z) \leq 2\pi \text{ for } C_+ ,$$

$(\zeta \neq z)$, $z \in C, v \in R$; Γ : Gamma function ,

then $(f)_v$ is the fractional differintegration of arbitrary order v (derivatives of order v for $v > 0$ and integrals of order $-v$ for $v < 0$) with respect to z of the function f , if $|(f)_v| < \infty$.

On the fractional calculus operator N^v**,** we recall the following theorems [3] :

Theorem 1.1. Let fractional calculus operator **(**Nishimoto's Operator**)** N^v be

$$N^v = \left(\frac{\Gamma(v+1)}{2\pi i}\int_c \frac{d\zeta}{(\zeta - z)^{v+1}}\right) \quad (v \notin Z^-) , \qquad \text{(Refer to [2])} \quad (1.3)$$

$$\text{With} \quad N^{-m} = \lim_{v \to -m} N^v \qquad (m \in Z^+), \qquad (1.4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha \ f = N^\beta \ N^\alpha f = N^\beta \ (N^\alpha f) \qquad (\alpha, \beta \in R) , \qquad (1.5)$$

then the set $\{N^v\} = \{N^v \ ; v \in R\}$ (1.6)

is an A belian product group(having continuous index v) which has the inverse transform operator $(N^v)^{-1} = N^{-v}$ to the fractional calculus operator N^v, for the function f such that
$f \in \mathrm{F} = \{ f; 0 \ \neq |f_v| < \infty, v \in R \}, \text{where } f = f\,(z)$

and $z \in \mathbb{C}.(\text{vis.} -\infty < v < \infty)$.

(For our convenience , we call $N^\beta \circ N^\alpha$ as product of N^β and N^α.

Theorem 1.2. The set " F.O.G. $\{N^v : v \in R\}$ " is an action product group which has continuous index v " for the set of F . (F.O.G. Fractional calculus operator group).

Theorem 1.3.

$$\text{Let} \quad S := \{ {}_{\pm}N^v \} \cup \{0\} = \{N^v\} \cup \{ {}_{-}N^v \} \cup \{0\} \quad (v \in R) \qquad (1.7)$$

then the set S is a commutative ring for the function $f \in \mathrm{F}$ when the identity

$$N^\alpha + N^\beta = N^\gamma \qquad (N^\alpha, N^\beta, N^\gamma \in S) \qquad (1.8)$$

holds[4].

Further, we recall that the generalized Beta function of n $(n \in Z^+ \geq 2)$ elements (n-dimensional or n-variables) is defined by [1]

$$_n\boldsymbol{B}(\alpha_k) = \boldsymbol{B}(\alpha_1, \alpha_2, \ldots, \alpha_n) := \frac{\prod_{k=1}^{n} \Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^{n} \alpha_k)}, \qquad (1.9)$$

$$(|\Gamma(\alpha_k)| , |\Gamma(\textstyle\sum_{k=1}^{n} \alpha_k)| < \infty) .$$

where α_k $(k = 1,2, \ldots, n \ (n \in Z^+ \geq 2)$) are variables with order number k .

We note the following special case :

$$_2\boldsymbol{B}(\alpha_k) = \boldsymbol{B}(\alpha_1, \alpha_2) = B(\alpha_1, \alpha_2) \qquad (1.10)$$

where $B(\alpha_1, \alpha_2)$ is the usual Beta functions [5].

Recently , Miyakoda and Nishimoto [1] derived some identities of the generalized Beta function by using N- fractional calculus of the power function $(z-c)^{-1}$. This paper is a further attempt in introducing certain results of the generalized Beta function by employing the technique of the fractional differintegration to the logarithm function $\log(z-c)$.

2. Main results

In the following

$$\varphi = \log(z-c) \qquad (z-c \neq 0), \tag{2.1}$$

we prove the following results by using N- fractional calculus :

Theorem 2.1.

We have the identity

$$\frac{\prod_{k=1}^{n}(\varphi_{{}_1\alpha_k} \cdot \varphi_{{}_2\alpha_k} \cdots \varphi_{{}_r\alpha_k})}{\varphi_{\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)}} = (-1)^{rn-1}\ {}_n\boldsymbol{B}({}_1\alpha_k)\,{}_n\boldsymbol{B}({}_2\alpha_k) \cdots {}_n\boldsymbol{B}({}_r\alpha_k)$$

$$\times \boldsymbol{B}\left(\sum_{k=1}^{n} {}_1\alpha_k, \sum_{k=1}^{n} {}_2\alpha_k, \ldots, \sum_{k=1}^{n} {}_r\alpha_k\right), \tag{2.2}$$

where

$$|\Gamma({}_1\alpha_k)|, |\Gamma({}_2\alpha_k)|, \ldots, |\Gamma({}_r\alpha_k)| < \infty, \left|\Gamma\left(\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)\right)\right| < \infty,$$

$|\Gamma(\sum_{k=1}^{n} {}_1\alpha_k)|, |\Gamma(\sum_{k=1}^{n} {}_2\alpha_k)|, \ldots, |\Gamma(\sum_{k=1}^{n} {}_r\alpha_k)| < \infty$ and ${}_1\alpha_k, {}_2\alpha_k, \ldots, {}_r\alpha_k$

$(r, k = 1,2,\ldots, n \in \mathbb{Z}^+, n \geq 2)$ are variables (constants for special case) with order number k .

Proof

Operating $N^{{}_1\alpha_k}$ to the both sides of (2.1) , we have

$$N^{{}_1\alpha_k}\varphi = N^{{}_1\alpha_k}\log(z-c),$$

that is

$$\varphi_{{}_1\alpha_k} = (\log(z-c))_{{}_1\alpha_k} \tag{2.3}$$

which on using the following identity [2]:

$$(\log(z-c))_\alpha = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \ (|\Gamma(\alpha)| < \infty), \tag{2.4}$$

in the r. h. s. of (2.3) , gives

$$\varphi_{{}_1\alpha_k} = -e^{-i\pi\,{}_1\alpha_k}\ \Gamma({}_1\alpha_k)(z-c)^{-{}_1\alpha_k} \ (|\Gamma({}_1\alpha_k)| < \infty), \tag{2.5}$$

Hence

$$\prod_{k=1}^{n} \varphi_{{}_1\alpha_k} = \prod_{k=1}^{n} (-1)^k \; . \prod_{k=1}^{n} e^{-i\pi\, {}_1\alpha_k} \prod_{k=1}^{n} (z-c)^{-\,{}_1\alpha_k} \prod_{k=1}^{n} \Gamma\left({}_1\alpha_k\right)$$

$$= (-1)^n \; e^{-i\pi \sum_{k=1}^{n} {}_1\alpha_k} \; . \; (z-c)^{-\sum_{k=1}^{n} {}_1\alpha_k} \prod_{k=1}^{n} \Gamma\left({}_1\alpha_k\right) \; (n \in Z^+, n \geq 2) . \qquad (2.6)$$

In the same way , we obtain

$$\prod_{k=1}^{n} \varphi_{{}_2\alpha_k} = \prod_{k=1}^{n} (-1)^k \; . \prod_{k=1}^{n} e^{-i\pi\, {}_2\alpha_k} \prod_{k=1}^{n} (z-c)^{-\,{}_2\alpha_k} \prod_{k=1}^{n} \Gamma\left({}_2\alpha_k\right)$$

$$= (-1)^n \; e^{-i\pi \sum_{k=1}^{n} {}_2\alpha_k} \; . \; (z-c)^{-\sum_{k=1}^{n} {}_2\alpha_k} \prod_{k=1}^{n} \Gamma\left({}_2\alpha_k\right) (n \in Z^+, n \geq 2) . \qquad (2.7)$$

Continuing this process r times, we get

$$\prod_{k=1}^{n} \varphi_{{}_r\alpha_k} = \prod_{k=1}^{n} (-1)^k \; . \prod_{k=1}^{n} e^{-i\pi\, {}_r\alpha_k} \prod_{k=1}^{n} (z-c)^{-\,{}_r\alpha_k} \prod_{k=1}^{n} \Gamma\left({}_r\alpha_k\right)$$

$$= (-1)^n \; e^{-i\pi \sum_{k=1}^{n} {}_r\alpha_k} \; . \; (z-c)^{-\sum_{k=1}^{n} {}_r\alpha_k} \prod_{k=1}^{n} \Gamma\left({}_r\alpha_k\right) \; (n \in Z^+, n \geq 2) . \qquad (2.8)$$

Again, operating $N^{\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \dots + {}_r\alpha_k)}$ to the both sides of (2.1) and using (2.4), we have

$$\varphi_{\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)} = -e^{-i\pi \sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)}$$

$$\times \Gamma\left(\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)\right) (z-c)^{1-\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)}$$

$$\left(\left|\Gamma\left(\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)\right)\right| < \infty \right) . \qquad (2.9)$$

Therefore, we obtain

$$\frac{\prod_{k=1}^{n}(\varphi_{{}_1\alpha_k} \cdot \varphi_{{}_2\alpha_k} \cdots \varphi_{{}_r\alpha_k})}{\varphi_{\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \dots + {}_r\alpha_k)}} = \frac{\prod_{k=1}^{n}\varphi_{{}_1\alpha_k} . \prod_{k=1}^{n}\varphi_{{}_2\alpha_k} \cdots \prod_{k=1}^{n}\varphi_{{}_r\alpha_k}}{\varphi_{\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \dots + {}_r\alpha_k)}}, \qquad (2.10)$$

which on using (2.6) ,(2.7) , (2.8) and (2.9) in the r. h. s., becomes

$$\frac{\prod_{k=1}^{n}(\varphi_{{}_1\alpha_k} \cdot \varphi_{{}_2\alpha_k} \cdots \varphi_{{}_r\alpha_k})}{\varphi_{\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \dots + {}_r\alpha_k)}} = \frac{(-1)^n \; e^{-i\pi \sum_{k=1}^{n} {}_1\alpha_k} \, . (z-c)^{-\sum_{k=1}^{n} {}_1\alpha_k} \prod_{k=1}^{n}\Gamma\left({}_1\alpha_k\right)}{-e^{-i\pi \sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)}}$$

$$\times \frac{(-1)^n \; e^{-i\pi \sum_{k=1}^{n} {}_2\alpha_k} \, . \, (z-c)^{-\sum_{k=1}^{n} {}_2\alpha_k} \prod_{k=1}^{n}\Gamma\left({}_2\alpha_k\right)}{\Gamma\left(\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)\right)}$$

$$\times \frac{\cdots \; (-1)^n \, e^{-i\pi \sum_{k=1}^{n} {}_r\alpha_k} \, . \, (z-c)^{-\sum_{k=1}^{n} {}_r\alpha_k} \prod_{k=1}^{n}\Gamma\left({}_r\alpha_k\right)}{(z-c)^{1-\sum_{k=1}^{n}({}_1\alpha_k + {}_2\alpha_k + \cdots + {}_r\alpha_k)}}$$

$$= (-1)^{rn-1} \frac{\prod_{k=1}^{n} \Gamma(\,_1\alpha_k) \times \prod_{k=1}^{n} \Gamma(\,_2\alpha_k) \times \ldots \times \prod_{k=1}^{n} \Gamma(\,_r\alpha_k)}{\Gamma(\sum_{k=1}^{n}(\,_1\alpha_k + \,_2\alpha_k + \cdots + \,_r\alpha_k))} \qquad (2.11)$$

Multiplying the r. h. s. of (2.11) by

$$\frac{\Gamma(\sum_{k=1}^{n} \,_1\alpha_k)\,\Gamma(\sum_{k=1}^{n} \,_2\alpha_k) \ldots \Gamma(\sum_{k=1}^{n} \,_r\alpha_k)}{\Gamma(\sum_{k=1}^{n} \,_1\alpha_k)\,\Gamma(\sum_{k=1}^{n} \,_2\alpha_k) \ldots \Gamma(\sum_{k=1}^{n} \,_r\alpha_k)} \quad ,$$

we obtain

$$\frac{\prod_{k=1}^{n}(\varphi_{\,_1\alpha_k} \cdot \varphi_{\,_2\alpha_k} \cdots \varphi_{\,_r\alpha_k})}{\varphi_{\sum_{k=1}^{n}(\,_1\alpha_k + \,_2\alpha_k + \ldots + \,_r\alpha_k)}} = (-1)^{rn-1} \frac{\prod_{k=1}^{n}\Gamma(\,_1\alpha_k)}{\Gamma(\sum_{k=1}^{n} \,_1\alpha_k)} \frac{\prod_{k=1}^{n}\Gamma(\,_2\alpha_k)}{\Gamma(\sum_{k=1}^{n} \,_2\alpha_k)} \cdots \frac{\prod_{k=1}^{n}\Gamma(\,_r\alpha_k)}{\Gamma(\sum_{k=1}^{n} \,_r\alpha_k)}$$

$$\times \frac{\Gamma(\sum_{k=1}^{n} \,_1\alpha_k)\Gamma(\sum_{k=1}^{n} \,_2\alpha_k) \ldots \Gamma(\sum_{k=1}^{n} \,_r\alpha_k)}{\Gamma(\sum_{k=1}^{n}(\,_1\alpha_k + \,_2\alpha_k + \cdots + \,_r\alpha_k))} \quad , \qquad (2.12)$$

which on using definition (1.9) in the r. h. s., yields assertion (2.2) of Theorem 2.1, under the conditions.

Remark 2.1.

Operating $N^{\alpha_k - 1}$ to the both sides of (2.1) and using identity (2.4) and then proceeding on the same lines of proof of Theorem (2.1),we get the following result :

Theorem 2.2.

$$\frac{\prod_{k=1}^{n}(\varphi_{\,_1\alpha_k - 1} \cdot \varphi_{\,_2\alpha_k - 1} \cdots \varphi_{\,_r\alpha_k - 1})}{\varphi_{\sum_{k=1}^{n}((\,_1\alpha_k + \,_2\alpha_k + \ldots + \,_r\alpha_k) - 1)}}$$

$$= (z-c)^{rn-1} \,_n\boldsymbol{B}(\,_1\alpha_k - 1)\,_n\boldsymbol{B}(\,_2\alpha_k - 1) \ldots \,_n\boldsymbol{B}(\,_r\alpha_k - 1)$$

$$\times \boldsymbol{B}(\sum_{k=1}^{n} \,_1\alpha_k - 1, \sum_{k=1}^{n} \,_2\alpha_k - 1, \ldots, \sum_{k=1}^{n} \,_r\alpha_k - 1), \qquad (2.13)$$

where

$|\Gamma(\,_1\alpha_k - 1)|, |\Gamma(\,_2\alpha_k - 1)|, \ldots, |\Gamma(\,_r\alpha_k - 1)| < \infty, |\Gamma(\sum_{k=1}^{n}(\,_1\alpha_k - 1 + \,_2\alpha_k - 1 + \cdots + \,_r\alpha_k - 1))| < \infty,$

$|\Gamma(\sum_{k=1}^{n} \,_1\boldsymbol{\alpha_k} - 1)|, |\Gamma(\sum_{k=1}^{n} \,_2\boldsymbol{\alpha_k} - 1)|, \ldots, |\Gamma(\sum_{k=1}^{n} \,_r\boldsymbol{\alpha_k} - 1)| < \infty$ and

$\,_1\alpha_k - 1 \;, \,_2\alpha_k - 1 \;, \ldots, \;\,_r\alpha_k - 1$

$(r, k = 1,2,\ldots,n \in Z^{+}, n \geq 2)$ are variables (constants for special case) with order number k .

In the next section ,we drive some results for the usual Beta functions as special case of main results (2.2) and (2.13).

3. Special cases

Taking $r = 2$ in results (2.2) and (2.13), we get the following results for the usual Beta function :

$$\frac{\prod_{k=1}^{n}(\varphi_{\,_1\alpha_k}\cdot\varphi_{\,_2\alpha_k})}{\varphi_{\sum_{k=1}^{n}\left(\varphi_{\,_1\alpha_k}+\varphi_{\,_2\alpha_k}\right)}} = (-1)^{2n-1}\ {}_n\boldsymbol{B}({}_1\alpha_k)\times {}_n\boldsymbol{B}({}_2\alpha_k)$$
$$\times\boldsymbol{B}\left(\sum_{k=1}^{n}{}_1\alpha_k,\sum_{k=1}^{n}{}_2\alpha_k\right) \qquad (3.1)$$

and

$$\frac{\prod_{k=1}^{n}(\varphi_{\,_1\alpha_k-1}\cdot\varphi_{\,_2\alpha_k-1})}{\varphi_{\sum_{k=1}^{n}((\,_1\alpha_k+\,_2\alpha_k)-1)}} = (z-c)^{2n-1}\,{}_n\boldsymbol{B}({}_1\alpha_k-1)\ {}_n\boldsymbol{B}({}_2\alpha_k-1)$$
$$\times\boldsymbol{B}\left(\sum_{k=1}^{n}{}_1\alpha_k-1,\sum_{k=1}^{n}{}_2\alpha_k-1\right) \qquad (3.2)$$

respectively.

Acknowledgment

The authors would like to thank Dr. Ahmed Al-Gonah of Aden University for his valuable comments and suggestions which improved the presentation of the paper.

References

[1] Miyakoda, T. and Nishimoto, K., Some applications of the multiply elements Beta functions obtained from the N- fractional calculus of a power function , Proceeding of the 2nd IFAC , Workshop on fractional differentiation and its applications , Porto , Portugal , July, 19 – 21, 2006 .

[2] Nishimoto , K., Fractional calculus vol.1 . Descartes press, Koriyama , Japan, 1984.

[3] Nishimoto, K., On Nishimoto's fractional calculus operator (on an acting group)**,** *J. Frac. Calc.* **4** (1993), 1-11.

[4] Nishimoto, K., Ring and field produced from the set of N- Fractional calculus operator , *J. Frac. Calc.* **24** (2003), 29-36 .

[5] Rainville, E. D., Special functions , Chelsea Pub. Com. Bronx, New York, 1960.