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Solutions for the fractional differential coupled sine-Gordon equation with Homotopy analysis method and the modified decomposition method

A. M. Yousef [*], S. Z. Rida and H. R. Ibrahim

Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt.

Abstract

In this paper, we solved the coupled sine-Gordon equation of fractional order. The fractional derivatives are described in the Caputo sense. The methods are homotopy analysis method (HAM) and modified decomposition method (ADM). We use the numerical simulation to compare these solutions.

Keywords: Coupled sine-Gordon; Modified decomposition method; Fractional calculus; Homotopy analysis method.

1. Introduction

It is difficult to solve nonlinear problems actually, but we can solve it analytically. Homotopy analysis method (HAM) is applied for analytic solution. The Homotopy analysis method (HAM) was first proposed by Liao in his Ph.D. thesis in 1992 [1, 2]. This method used to solve many types of homogeneous or non homogeneous equations, nonlinear partial differential equations (PDE) in engineering and science [3-7]. Also, many types of PDE with HAM by others [8-22]. And the HAM solutions for systems of nonlinear fractional differential equations were presented by Bataineh et al [23]. HAM applied to linear, homogeneous one and two-dimensional fractional heat-like subject to the Neumann boundary conditions [24]. The HAM contains a certain auxiliary parameter $\hbar$ which give us with a simple way to control the convergence region and the rate of convergence series of solution. In this paper, we will use Homotopy analysis method and modified decomposition method (in short MDM) to obtain the fractional solution of the coupled sine-Gordon equation.

Also the decomposition method provides an effective procedure for analytical solution of a wide and general class of the dynamical systems representing real physical problems [25-30]. This method efficiently works for initial-value or boundary-value problems and for partial differential equations and for linear or nonlinear. A reliable modification of Adomian decomposition method has been done by Wazwaz.

2. Preliminaries

In this paper, we shall consider the fractional order coupled sine-Gordon equations in the form

$$\begin{aligned} u_{tt} - u_{xx} &= -\delta^2 \sin(u-w), \\ w_{tt} - c^2 w_{xx} &= \sin(u-w), \end{aligned} \tag{2.1}$$

Which was introduced by Khusnutdinova and Pelinovsky [31]. The coupled sine-Gordon equations generalize the Frenkel-Kontorova model [32, 33]. The system (2.1) by c=1 was used to describe the open states in DNA [34]. In order to solve these system we will use the homotopy analysis method (HAM), It is one of the most effective method to obtain the exact solution. HAM contains an auxiliary parameter $\hbar$, which help us to adjust and control the convergence region of series solution. Also the decomposition method for solving coupled sine-Gordon equation has been implemented. By using a number of initial values, The solution is calculated in the form of convergent power series with easily computable components. The method performs in terms of simplicity, efficiency, stability and accuracy.

3. Fractional calculus

In the fractional calculus the basic mathematical ideas (integral and differential operations of noninteger order) were developed long ago by mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and the attention of the engineering world by Oliver Heaviside in the 1890s. The first book on this theme was posted by Oldham and Spanier. Here we state some of definitions of fractional calculus.

Definition 3.1. The operator I_a^α defined on the usual space $L_1[a,b]$ by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (z-t)^{\alpha-1} f(t)\,dt,$$
$$I_a^0 f(x) = f(x)$$

is called the Riemann-Liouville fractional integral operator of order α, we mention the following properties for $f \in L_1[a,b],\ \alpha,\beta \geq 0,\ and\ \gamma > -1$:

(1) $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$

(2) $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x)$

(3) $I^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Definition 3.3. The Caputo sense is defined as

$$D^\alpha f(x) = I^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha+1} f^{(m)}(t)\,dt$$

Definition 3.4. The Caputo derivative is defined as

$$D^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha+1} \frac{\partial^m u(x,t)}{\partial \tau^m}, & for\ m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & for\ \alpha = m \in N \end{cases}$$

Definition 3.5. For a homotopy –Maclaurin series $\phi=\sum_{k=0}^{\infty} u_k q^k$

It holds the recursion formulas

(1) $D_0(e^{\alpha\phi})=e^{\alpha\, u_0}$

(2) $D_m(e^{\alpha\phi})=\alpha\sum_{k=0}^{m-1}(1-\frac{k}{m})u_{m-k}D_k(e^{\alpha\phi}),$

Where $m\geq 1$ is an integer, and $\alpha\neq 0$ is independent of the homotopy-parameter q.

(3) $D_0(Sin\,\phi)=Sin\,u_0$

(4) $D_m(Sin\,\phi)=\sum_{k=0}^{m-1}(1-\frac{k}{m})u_{m-k}(\phi)\,D_k Cos\phi$

4. (I) Fundamentals of the Homotopy analysis method

The basic idea of HAM, we will consider the following differential equation

$$N[u(x,t)]=0 \qquad (4.1)$$

where N is nonlinear operators, x and t refer to the independent variables and u is known function. Liao constructs the so-called zero-order deformation equation

$$(1-p)\,L[\phi(x,t;p)-u_0(x,t)]\,p\,\hbar\,H(x,t)\,N[\phi(x,t;p)] \qquad (4.2)$$

Where $p\in[0,1]$ is embedding parameter, $\hbar\neq 0$ is a non-zero auxiliary parameter, L is an auxiliary parameter, $H(x,t)\neq 0$ denotes a nonzero auxiliary function, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t,p)$ is an unknown function. When p=0 and p=1, it holds $\phi(x,t,0)=u_0(x,t),\ \phi(x,t,1)=u(x,t)$. Thus as p increase from 0 to 1, the solution u(x, t) varies from the initial guess $u_0(x,t)$ Expand $\phi(x,t,p)$ in Taylor series with respect to p, one has

$$\phi(x,t,p)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\,p^m \qquad (4.3)$$

Where

$$u_m(x,t)=\frac{1}{m!}\left.\frac{\partial^m\phi(x,t,q)}{\partial q^m}\right|_{p=0}$$

If the auxiliary linear operator, auxiliary parameter $\hbar$, the initial guess and the auxiliary function are chosen, then the series () convergence at p=1 so we have

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)$$

It must be one of the solutions of the nonlinear equation, as proved by Liao. Define the vector

$$u_n^{\rightarrow} = (u_0(x,t), u_1(x,t), u_2(x,t), \ldots\ldots\ldots\ldots, u_n(x,t)).$$

Differentiate the zeroth-order deformation equation (4.2) m-times with respect to p then dividing them by m! and setting p=0 so we get

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(u_{m-1}^{\rightarrow}), \tag{4.4}$$

Where

$$R_m(u_{m-1}^{\rightarrow}) = \frac{1}{m!} \frac{\partial^{m-1} N[\phi(x,t,p)]}{\partial p^{m-1}}\bigg|_{p=0}$$

And

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

After we applying the Riemann-liouville integral operator I^{α} on both side of Eq (4.4) we have

$$u_m = \chi_m u_{m-1}(x,t) + \hbar I^{\alpha}[R_m(u_{m-1}^{\rightarrow})]$$

(II) The analysis of modified decomposition method

We consider the coupled sine-Gordon equations (1) in the operator form

$$\begin{aligned} L_{tt}u &= L_{xx}u - \delta^2 N(u,w), \\ L_{tt}w &= c^2 L_{xx} w + N(u,w), \end{aligned} \tag{4.5}$$

Where $L_{tt}u \equiv \frac{\partial^2}{\partial t^2}, L_{xx} \equiv \frac{\partial^2}{\partial x^2}$ represent the linear differential operators and $N(u,w) = \sin(u-w)$ represent the nonlinear operator.

Applying the integration inverse operator $L_{tt}^{-1} \equiv \int_0^t () \int_0^t (\bullet)\, dt\, dt$ to the Equ (4.5) and using the initial conditions

$$\begin{aligned} u(x,t) &= u(x,0) + t\, u_t(x,0) + L_{tt}^{-1} L_{xx} u - \delta^2 L_{tt}^{-1} N(u,w) \\ w(x,t) &= w(x,0) + t\, w_t(x,0) + c^2 L_{tt}^{-1} L_{xx} w + L_{tt}^{-1} N(u,w) \end{aligned} \tag{4.6}$$

The Adomain decomposition method [29, 30] assumes an infinite series solution for the function u(x, t) and w(x, t) as

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t) \end{aligned} \tag{4.7}$$

And $N(u,v)=\sum_{n=0}^{\infty}A_n(u_0,u_1,u_2,u_3,........u_n,w_0,w_1,w_2,.........w_n)$, where A_n is the Adomian's polynomial, that we can defined it in the general formula as

$$A_n(u_0,u_1,u_2,........u_n,w_0,w_1,.......w_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{\infty}\lambda^k u_k,\sum_{k=0}^{\infty}\lambda^k w_k)]|_{\lambda=0},\quad n\geq 0 \quad (4.8)$$

We can give the first few Adomian polynomials for $N(u,w)=\sin(u-w)$ as

$$A_0=\sin(u_0-w_0)$$
$$A_1=(u_1-w_1)\cos(u_0-w_0)$$
$$A_2=(u_2-w_2)\cos(u_0-w_0)+u_1w_1\sin(u_0-w_0)-\frac{u_1{}^2}{2}\sin(u_0-w_0)-\frac{w_1{}^2}{2}\sin(u_0-w_0)$$
$$A_3=(u_3-w_3)\cos(u_0-w_0)-(u_1-w_1)(u_2-w_2)\sin(u_0-w_0)-\frac{1}{6}(u_1-w_1)^3\cos(u_0-w_0)$$

and so on, the ret polynomial can be calculate in similar manner. Substituting the initial conditions into (3) and select the zeroth components u_0 and w_0, so we obtain the subsequent components as

$$\begin{aligned} u_{n+1}&=L_{tt}{}^{-1}L_{xx}\,u_n-\delta^2 L_{tt}{}^{-1}(A_n),\quad n\geq 0\\ w_{n+1}&=c^2 L_{tt}{}^{-1}L_{xx}\,w_n+L_{tt}{}^{-1}(A_n),\quad n\geq 0\end{aligned}\qquad (4.9)$$

Wazwaz [31] proposed the construction of the zeroth component of the decomposition series in a different way. In [35], he assumed that if the zeroth component $u_0=g$ where the function g is divided into two parts such as g_1 and g_2 so we will have the modified recursive scheme as

$$\begin{aligned} u_0&=g_1\\ u_1&=g_2+L_{tt}{}^{-1}L_{xx}\,u_0-\delta^2L_{tt}{}^{-1}(A_0)\\ u_{n+1}&=L_{tt}{}^{-1}L_{xx}\,u_n-\delta^2 L_{tt}{}^{-1}(A_n),\ n\geq 1\end{aligned}\qquad (4.10)$$

Similarly, if the zeroth components $w_0=g'$ and the function g' can be divide in to two parts such as $g_1{}'$ and $g_2{}'$ so we have the modified recursive scheme as

$$\begin{aligned} w_0&=g_1{}'\\ w_1&=g_2{}'+L_{tt}{}^{-1}L_{xx}\,w_0-\delta^2L_{tt}{}^{-1}(A_0)\\ w_{n+1}&=L_{tt}{}^{-1}L_{xx}\,w_n-\delta^2 L_{tt}{}^{-1}(A_n),\ n\geq 1\end{aligned}\qquad (4.11)$$

This type of modification helps us to solve complicate nonlinear differential equations. It avoids the unnecessary computation especially in calculation of the Adomian polynomials. The decomposition series (4.7) solutions are generally converge very rapidly in real physical problems [36]. This rapidity of this convergence means that few terms are required.

5. Implementation of the methods

Example 1. Consider the following fractional coupled sine-Gordon equation

$$\begin{aligned} &D_t^{\alpha} u - u_{xx} + \delta^2 \sin(u-w) = 0 \\ &D_t^{\alpha} w - c^2 w_{xx} - \sin(u-w) = 0 \qquad 1 \le \alpha < 2, \ -\infty < x < \infty, \ \ t > 0 \end{aligned} \tag{5.1}$$

Subject to initial condition

$$u(x,0) = A\cos kx, \qquad w(x,0) = 0$$
$$u_t(x,0) = 0 \qquad\qquad w_t(x,0) = 0$$

We adopt modified decomposition method for solving Eq. (5.1). In the light of this method we can write

$$u_0(x,t) = 0$$

$$w_0(x,t) = 0$$

$$u_1(x,t) = u(x,o) + L_{tt}^{-1} L_{xx}(u_0) - \ \delta^2 L_{tt}^{-1}(A_0) = A\cos kx$$

$$w_1(x,t) = c^2 L_{tt}^{-1} L_{xx}(w_0) + L_{tt}^{-1}(A_0) = 0$$

$$u_2(x,t) = L_{tt}^{-1} L_{xx}(u_1) - \delta^2 L_{tt}^{-1}(A_1) = \frac{-A(k^2+\delta^2)\cos kx}{\Gamma(\alpha+1)} t^{\alpha}$$

$$w_2(x,t) = c^2 L_{tt}^{-1} L_{xx}(w_1) + L_{tt}^{-1}(A_1) = \frac{A\cos kx}{\Gamma(\alpha+1)} t^{\alpha}$$

$$u_3(x,t) = L_{tt}^{-1} L_{xx}(u_2) - \delta^2 L_{tt}^{-1}(A_2) = \frac{Ak^2(k^2+\delta^2)\cos kx}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{A\delta^2(1+k^2+\delta^2)\cos kx}{\Gamma(2\alpha+1)} t^{2\alpha}$$

$$w_3(x,t) = c^2 L_{tt}^{-1} L_{xx}(w_2) + L_{tt}^{-1}(A_2) = \frac{-Ac^2 k^2 \cos kx}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{A(1+k^2+\delta^2)\cos kx}{\Gamma(2\alpha+1)} t^{2\alpha}$$

Also

And so on, in this manner the other components of the decomposition series can be easily obtained of which u(x, t) and w(x, t) were evaluated in a series form

$$u(x,t) = A\cos kx \frac{-A(k^2+\delta^2)\cos kx}{\Gamma(\alpha+1)} t^{\alpha} \frac{Ak^2(k^2+\delta^2)\cos kx}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{A\delta^2(1+k^2+\delta^2)\cos kx}{\Gamma(2\alpha+1)} t^{2\alpha} + \ldots\ldots,$$

$$w(x,t) = \frac{A\cos kx}{\Gamma(\alpha+1)} t^{\alpha} \frac{-Ac^2 k^2 \cos kx}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{A(1+k^2+\delta^2)\cos kx}{\Gamma(2\alpha+1)} t^{2\alpha} + \ldots\ldots\ldots,$$

Follow immediately with the aid of Mathematics.

Also with Homotopy analysis method and by using Eqs. (4.2- 4.4) we could be able to calculate some of the terms of the homotopy series as:-

$$u_0(x,t) = u(x,0) = A\cos kx$$

$$w_0(x,t) = w(x,0) = 0$$

$$u_1(x,t) = \hbar_1[u_0 - I^{\alpha}u_{0xx} + \delta^2 I^{\alpha}\sin(u_0 - w_0)]$$
$$= \hbar_1[A\cos kx + \frac{t^{\alpha}}{\Gamma(\alpha+1)}(Ak^2\cos kx + \delta^2\sin(A\cos kx))]$$

$$w_1(x,t) = \hbar_2[w_0 - c^2 I^{\alpha}w_{0xx} - I^{\alpha}\sin(u_0 - w_0)] = \frac{-\hbar_2 t^{\alpha}}{\Gamma(\alpha+1)}\sin[A\cos kx]$$

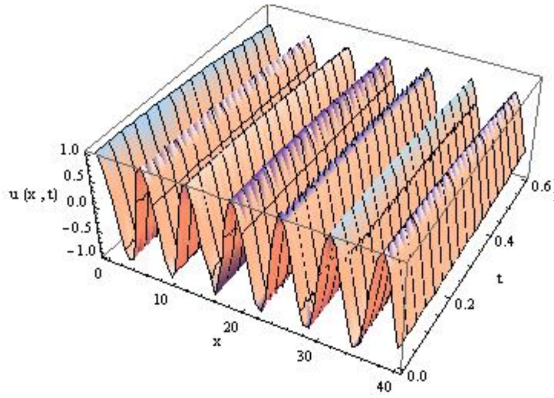
$$u_2(x,t) = (1+\hbar_1)u_1 - \hbar_1[I^{\alpha}u_{1xx} + \delta^2 I^{\alpha}(u_1 - w_1)\cos(u_0 - w_0)]$$
$$= \frac{\hbar_1}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}[t^{\alpha}\Gamma(2\alpha+1)(A\cos kx(k^2 + 2\hbar_1 k^2 - \delta^2\hbar_1\cos(A\cos kx)$$
$$+\delta^2(1+\hbar_1)\sin(A\cos kx) + \frac{1}{2}\Gamma(\alpha+1)(2A(1+\hbar_1)\cos kx\,\Gamma(2\alpha+1) + t^{2\alpha}(2Ak^2\hbar_1$$
$$(k^2\cos kx + A\delta^2\sin^2 kx\sin(A\cos kx + \delta^2(-\delta^2\hbar_1 + \hbar_2)\sin(2A\cos kx)))]$$

$$w_2(x,t) = (1+\hbar_2)w_1 - \hbar_2[c^2 I^{\alpha}w_{1xx} - (u_1 - w_1)\sin(u_0 - w_0)]$$

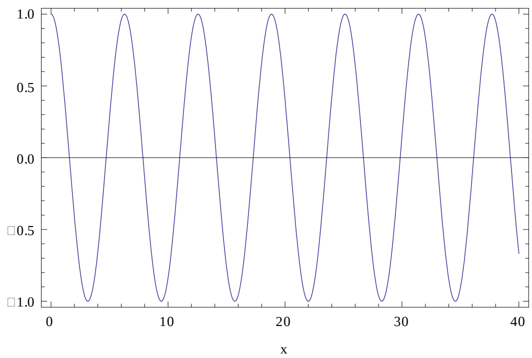
$$= -\hbar_2 t^{\alpha}[(1+\hbar_2 - A\hbar_1\cos kx)\Gamma(2\alpha+1)\sin(A\cos kx)]$$

6. Numerical evaluation and discussion

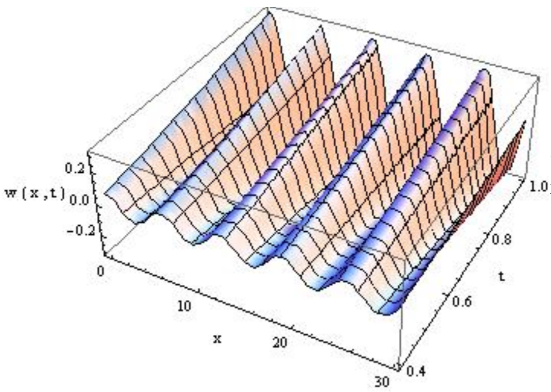
The fractional solution of the coupled sine-Gordon Eq.(5.1) have been shown in Fig. 1 with help of ϕ_4 and Ψ_4 for the decomposition series solution of u(x,t) and w(x,t). We have assumed $c = 0.5,\ A = 1,\ k = 1,\ \delta = 0.5,\ \alpha = 2$. We have drawn by using Mathematica.



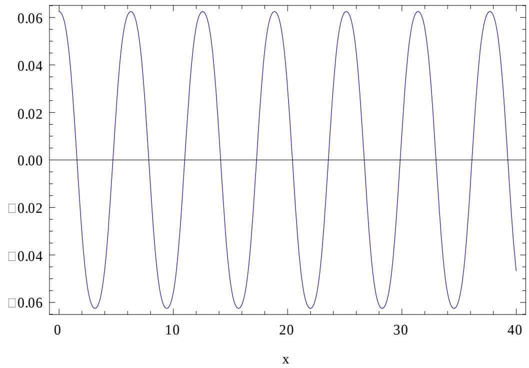
(a)



(a)



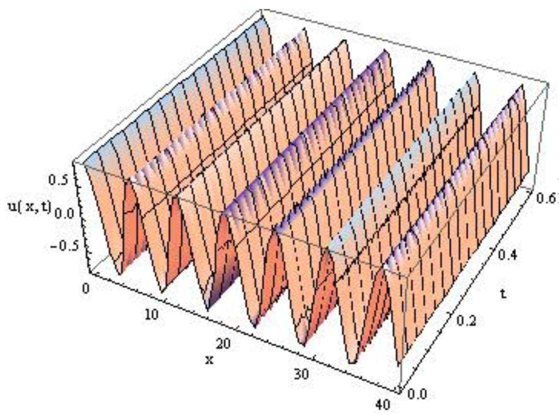
(b)



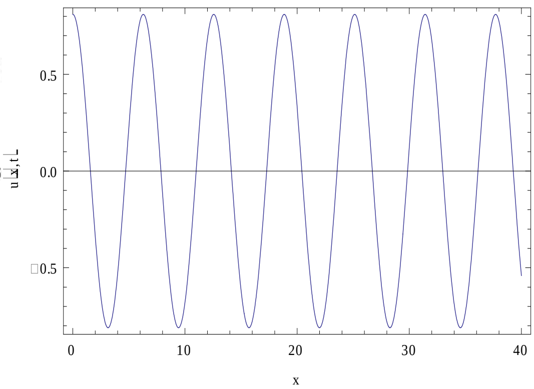
(b)

Fig. 1. The behavior of: (a) u(x, t) and (b) w(x, t) obtained by by ADM for coupled sine-Gordon at $A=1,\ c=0.5,\ k=1,\ \alpha=2,\ \delta=0.5$

Also we will represent the behavior by using the homotopy analysis method



(c)



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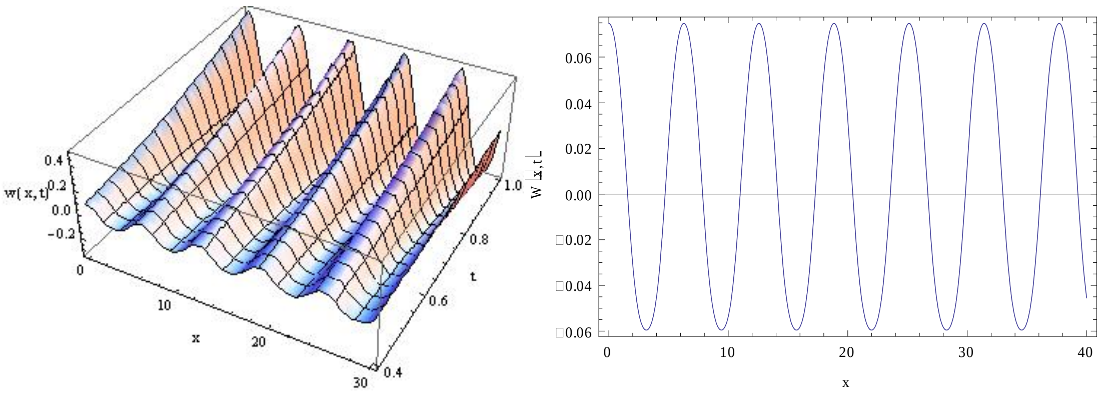
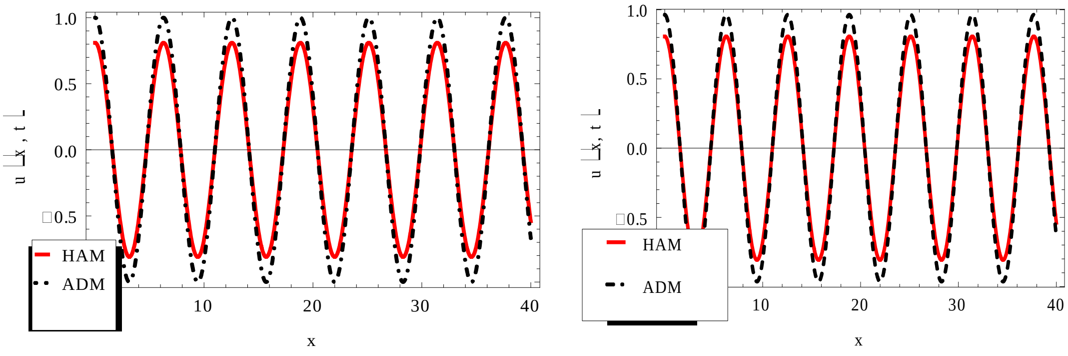


(d) (d)

Fig..**2**. The behavior of: (c) u(x, t) and (d) w(x, t) obtained by HAM
$\hbar_1 = -0.1, \hbar_2 = -1, \alpha = 2, A = 1, c = 0.5, \delta = 0.5, k = 1$



(a) (b)

Fig.3. The comparison of the u(x, t) obtained by HAM and ADM at $(a)\ t = 0,\ (b)\ t = 0.2$

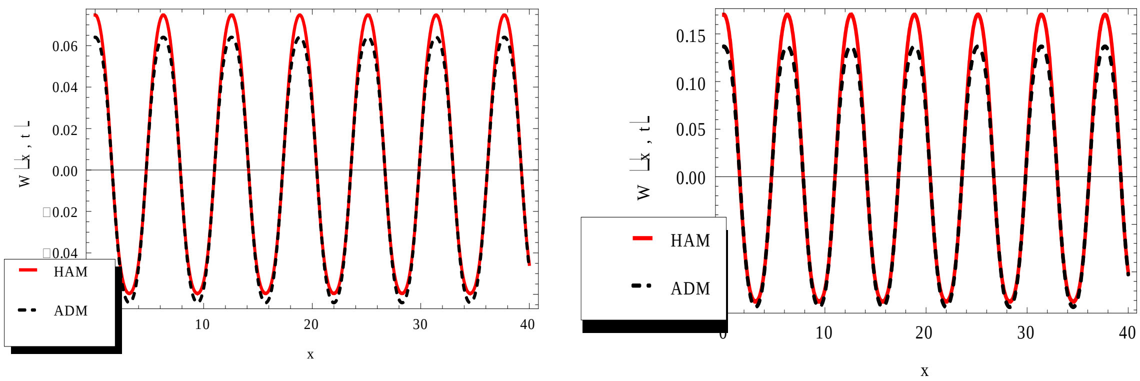


Fig.4. The comparison of the u(x, t) obtained by HAM and ADM at $(c)\ t = 0.4,\ (d)\ t = 0.6$

Conclusion

In this paper, we were used ADM and HAM for finding the fractional solution for the coupled sine-Gordon with the initial conditions. The approximate solution to the equations has been calculated without any need to linearization of the equations and any need to transformation techniques. This method eliminates the difficulties and huge computation work. The series solution can be easily computed using any mathematical symbolic package. The decomposition method is straightforward. It provides more realistic series solutions that is converge very rapidly in real physical problems. HAM is a very powerful and efficient technique for finding solutions for wide classes of nonlinear problems in the form of analytical expressions and displays a rapid convergence for solution. Many of the result achieved in this paper confirm the idea that is a powerful mathematical tool for solving different kinds of nonlinear problem emerging in various fields of science and engineering. HAM provides highly accurate numerical solutions for nonlinear problems, in a comparison with other methods. This method avoids linearization and physically unrealistic assumptions

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