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Some new equilibrium existence theorems for pair of abstract economies

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Abstract.

In this paper, we prove some new common equilibrium existence theorems for pair of non-compact abstract economies with an uncountable number of agents.

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Key words: Abstract economies; upper semicontinuous; lower semicontinuous; Hausdorff locally convex topological vector space.

1. Introduction and Preliminaries

The existence of equilibria is an abstract economy with compact strategy sets in $\mathbb{R}^n$ was proved by G. Debreu [3]. Since then many generalization of Debreu's theorem appeared in many directions (see [4],[5].[12],[13],[14],[15],[16],[17],[18], and the references therein).

The purpose of this paper is to give some new common equilibrium existence theorems for pair of non-compact abstract economies with an uncountable number of agents with an general constraint correspondences and preference correspondences. Our results improve and generalize some known results in literature[4,11,16,18].

Now we give some notations and definitions that are needed in the sequel.

Let A be a subset of a topological space. We shall denote by 2^A and $\bar{A}$ the family of all subsets of A and the closure of A in X, respectively. If A is a subset of a topological vector space X, we shall denote by coA and $\overline{co}A$ the convex hull of A and the closed convex hull of A, respectively.

Let X, Y be two topological spaces and $T: X \to 2^Y$ be a multivalued mapping.T is said to be upper semicontinuous (respectively, almost upper semicontinuous) if for any $x \in X$ and any open set V in Y with $T(x) \subset V$, there exists an open neighborhood $U \; of \; x$ in X such that $T(z) \subset V$ (respectively, $T(z) \subset \bar{V}$) for $z \in U$. Obviously, an upper semicontinuous multi- valued mapping is almost upper semicontinuous (see [12],[16]). T is said to be lower semicontinuous if for any open set V in Y, the set $\{x \in X: T(x) \cap V \neq \emptyset\}$ is open in X. It is clear that T is upper semicontinuous (respectively, lower semicontinuous), if and only if for any open set (respectively, closed set) M in Y, the set $\{x \in X : T(x) \subset M\}$ is open (respectively, closed) in X. T is said to have open graph in $X \times Y$ if the set $\{(x,y) : x \in X, y \in T(x)\}$ is open in $X \times Y$.

An abstract and socio-economy are a family of quadruples $\Gamma_1 = (X_i; A_i, B_i; P_{2i+1})_{i \in I}$ and $\Gamma_2 = (X_i; A_i, B_i; P_{2i+2})_{i \in I}$ respectively, where I is a finite or an infinite set of agents, X_i is a nonempty topological space (a choice set), $A_i, B_i : X = \prod_{j \in I} X_j \to 2^{X_i}$ are constraint correspondences and $P_{2i+1}, P_{2i+2} : X \to 2^{X_i}$ are preference correspondences. A common equilibrium of Γ_1 and Γ_2 is a point $\hat{x} \in X$, such that for each $i \in I, \hat{x}_i \in \overline{B_i(\hat{x})}$ and $P_{2i+1}(\hat{x}) \cap A_i(\hat{x}) = \emptyset; P_{2i+2}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$.

$\Gamma_1 = (X_i; P_{2i+1})_{i \in I}$ and $\Gamma_2 = (X_i; P_{2i+2})_{i \in I}$ are said to be a pair of qualitative game if for any $i \in I, X_i$ is a strategy set of player i, and $P_{2i+1}, P_{2i+2} : X = \prod_{j \in I} X_j \to 2^{X_i}$ are preference correspondences of player i. A common maximal element of Γ_1 and Γ_2 is a point $\hat{x} \in X$, such that $P_{2i+1}(\hat{x}) \cap P_{2i+2}(\hat{x}) = \emptyset$ for all $i \in I$.

Lemma 1.1.[11]

Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, D_i a nonempty compact subset of X_i and $S_i, T_i : X = \prod_{K \in I} X_K \to 2^{D_i}$ are two multivalued mappings with the following conditions:

(1) for any $x \in X, \emptyset \neq \overline{co} S_i(x) \subset T_i(x)$,

(2) S_i is almost upper semicontinuous.

Then there exists a point $\hat{x} \in D = \prod_{K \in I} D_K$, such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$.

Lemma 1.2.[18]

Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, D_i a nonempty compact metrizable subset of X_i and $S_i, T_i : X = \prod_{K \in I} X_K \to 2^{D_i}$ are two multivalued mappings with the following conditions:

(1) for any $x \in X, \emptyset \neq \overline{co} S_i(x) \subset T_i(x)$,

(2) S_i is lower semicontinuous.

Then there exists a point $\hat{x} \in D = \prod_{K \in I} D_K$, such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$.

2. Common Equilibrium Existence Theorems

In this section, we give some new common equilibrium existence theorems for pair of abstract economies.

Theorem 2.1. Let $\Gamma_1 = (X_i; A_i, B_i; P_{2i+1})_{i \in I}$ and $\Gamma_2 = (X_i; A_i, B_i; P_{2i+2})_{i \in I}$ be a pair of generalized games (abstract economy), where I be any index set such that for each $i \in I$:

(1) X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i and D_i is a nonempty compact subset of X_i.

(2) For all $x \in X = \Pi_{i \in I} X_i, P_{2i+1}(x) \subset D_i$ and $P_{2i+2}(x) \subset D_i, A_i(x) \subset B_i(x) \subset D_i$, and $B_i(x)$ is nonempty convex.

(3) The set $W_i = \{x \in X : A_i(x) \cap P_{2i+1}(x) \neq \emptyset$ and $A_i(x) \cap P_{2i+2}(x) \neq \emptyset\}$ is open in X.

(4) The mappings $H_i, G_i : X \to 2^{D_i}$, defined by

$$H_i(x) = A_i(x) \cap P_{2i+1}(x)$$

and

$$G_i(x) = A_i(x) \cap P_{2i+2}(x), \forall x \in X$$

are upper semicontinuous and $B_i : X \to 2^{D_i}$ is upper semicontinuous.

(5) For each $x \in W_i, x_i \notin \overline{co}(A_i(x) \cap P_{2i+1}(x))$ and also $x_i \notin \overline{co}\big(A_i(x) \cap P_{2i+2}(x)\big)$.

Then Γ_1 and Γ_2 have a common equilibria point, i.e, there exists a point $\hat{x} \in D = \Pi_{i\in I} D_i$, such that $\hat{x}_i \in \overline{B_i(\hat{x})}$; $P_{2i+1}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ and $P_{2i+2}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ for all $i \in I$.

Proof. For each $i \in I$ and $x \in X$, let

$$S_i(x) = \begin{cases} A_i(x) \cap P_{2i+1}(x), & if\ x \in W_i, \\ B_i(x), & if\ x \notin W_i, \end{cases}$$

and

$$T_i(x) = \begin{cases} \overline{co}\ (A_i(x) \cap P_{2i+1}(x)), & if\ x \in W_i, \\ \overline{B_i(x)}, & if\ x \notin W_i. \end{cases}$$

Then, $S_i, T_i: X \to 2^{D_i}$ are two multivalued mappings with nonempty values and $\overline{co}S_i(x) \subset T_i(x)$ for all $x \in X$.

Now, we prove that S_i is upper semicontinuous. In fact, for each open set V in D_i, the set

$$\{x \in X: S_i(x) \subset V\} = \{x \in W_i: A_i(x) \cap P_{2i+1}(x) \subset V\} \cup$$
$$\{x \in X\backslash W_i: B_i(x) \subset V\}$$
$$\subset \{x \in W_i: H_i(x) \subset V\} \cup \{x \in X: B_i(x) \subset V\}.$$

On the other hand, when $x \in W_i$ *and* $H_i(x) \subset V$, we have $S_i(x) = H_i(x) \subset V$. When $x \in X$ and $B_i(x) \subset V$, since $H_i(x) \subset B_i(x)$, we know that $S_i(x) \subset V$ and so

$$\{x \in W_i: H_i(x) \subset V\} \cup \{x \in X: B_i(x) \subset V\} \subset \{x \in X: S_i(x) \subset V\}.$$

Therefore,

$$\{x \in X: S_i(x) \subset V\} = \{x \in W_i: H_i(x) \subset V\} \cup \{x \in X: B_i(x) \subset V\}$$
$$= W_i \cap \{x \in X: H_i(x) \subset V\} \cup \{x \in X: B_i(x) \subset V\}.$$

Since H_i and B_i are upper semicontinuous, the sets $\{x \in x: H_i(x) \subset V\}$ and $\{x \in X: B_i(x) \subset V\}$ are open. It follows that $\{x \in X: S_i(x) \subset V\}$ is open and so the mapping $S_i: X \to 2^{D_i}$ is upper semicontinuous.

By Lemma 1.1, there exists a point $\hat{x} \in D = \prod_{i\in I} D_i$, such that, $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$. By Condition (5), we have $\hat{x}_i \in \overline{B_i(\hat{x})}$ and $P_{2i+1}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ for all $i \in I$.

Similarly, it can be established that for each $i \in I$, $\hat{x}_i \in \overline{B_i(\hat{x})}$ and $P_{2i+2}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$, i.e., Γ_1 and Γ_2 have a common equilibria point. This completes the proof of Theorem.

Theorem 2.2. Let $\Gamma_1 = (X_i; A_i, B_i; P_{2i+1})_{i\in I}$ and $\Gamma_2 = (X_i; A_i, B_i; P_{2i+2})_{i\in I}$ be a pair of generalized games (abstract economy), such that for each $i \in I$, the following conditions are satisfied.

(1) X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E_i and D_i is a nonempty compact metrizable subset of X_i.

(2) For all $x \in X = \prod_{i\in I} X_i, P_{2i+1}(x) \subset D_i$ and $P_{2i+2}(x) \subset D_i$, $A_i(x) \subset B_i(x) \subset D_i$ and $B_i(x)$ is nonempty convex.

(3) The set $W_i = \{x \in X: A_i(x) \cap P_{2i+1}(x) \neq \emptyset$ and $A_i(x) \cap P_{2i+2}(x) \neq \emptyset\}$ is closed in X.

(4) The mappings $A_i: X \to 2^{D_i}$ (respectively, $P_{2i+1}, P_{2i+2}: X \to 2^{D_i}$) is lower semicontinuous, P_{2i+1}, P_{2i+2} (respectively, A_i) have open graph in $X \times D_i$, and $B_i: X \to 2^{D_i}$ is lower semicontinuous.

(5) For each $x \in W_i, x_i \notin \overline{co}(A_i(x) \cap P_{2i+1}(x))$ and also $x_i \notin \overline{co}\big(A_i(x) \cap P_{2i+2}(x)\big)$.

Then Γ_1 and Γ_2 have a common equilibria point, i.e, there exists a point $\hat{x} \in D = \Pi_{i\in I} D_i$, such that $\hat{x}_i \in \overline{B_i(\hat{x})}$; $P_{2i+1}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ and $P_{2i+2}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ for all $i \in I$.

Proof. For each $i \in I$ and $x \in X$, let

$$S_i(x) = \begin{cases} A_i(x) \cap P_{2i+1}(x), & if\ x \in W_i, \\ B_i(x), & if\ x \notin W_i, \end{cases}$$

and

$$T_i(x) = \begin{cases} \overline{co}\ (A_i(x) \cap P_{2i+1}(x)), & if\ x \in W_i, \\ \overline{B_i(x)}, & if\ x \notin W_i. \end{cases}$$

Then, $S_i, T_i: X \to 2^{D_i}$ are two multivalued mappings with nonempty values and $\overline{co}S_i(x) \subset T_i(x)$ for all $x \in X$. From Condition (4) and [19, Lemma 4.2], we know that the mapping $H_i: X \to 2^{D_i}$ defined by

$$H_i(x) = A_i(x) \cap P_{2i+1}(x), \forall x \in X$$

is lower semicontinuous.

Now, we prove that S_i is lower semicontinuous. In fact, for each closed set V in D_i, as in the proof of Theorem 2.1, we have

$$\begin{aligned} \{x \in X: S_i(x) \subset V\} &= \{x \in W_i: A_i(x) \cap P_{2i+1}(x) \subset V\} \cup \\ &\quad \{x \in X \backslash W_i: B_i(x) \subset V\} \\ &= \{x \in W_i: H_i(x) \subset V\} \cup \{x \in X: B_i(x) \subset V\} \\ &= W_i \cap \{x \in W_i: H_i(x) \subset V\} \cup \{x \in X: B_i(x) \subset V\}. \end{aligned}$$

Since H_i and B_i are lower semicontinuous, the sets $\{x \in X: H_i(x) \subset V\}$ and $\{x \in X: B_i(x) \subset V\}$ are closed. It follows that $\{x \in X: S_i(x) \subset V\}$ is closed and so the mapping $S_i: X \to 2^{D_i}$ is lower semicontinuous.

By Lemma 1.2, there exists a point $\hat{x} \in D = \prod_{i \in I} D_i$, such that, $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$. By Condition (5), we have $\hat{x}_i \in \overline{B_i(\hat{x})}$ and $P_{2i+1}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ for all $i \in I$.

Similarly, it can be established that for each $i \in I$, $\hat{x}_i \in \overline{B_i(\hat{x})}$ and $P_{2i+2}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$, i.e., Γ_1 and Γ_2 have a common equilibria point. This completes the proof of Theorem.

Theorem 2.3. Let $\Gamma_1 = (X_i; A_i, B_i; P_{2i+1})_{i \in I}$ and $\Gamma_2 = (X_i; A_i, B_i; P_{2i+2})_{i \in I}$ be a pair of generalized games (abstract economy), such that for each $i \in I$, the following conditions are satisfied.

(1) X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i and D_i is a nonempty compact subset of X_i.

(2) For all $x \in X = \Pi_{i \in I} X_i, P_{2i+1}(x) \subset D_i$ and $P_{2i+2}(x) \subset D_i, A_i(x) \subset B_i(x) \subset D_i$, $P_{2i+1}(x)$ and $P_{2i+2}(x)$ are convex and $B_i(x)$ is nonempty convex.

(3) The set $W_i = \{x \in X: A_i(x) \cap P_{2i+1}(x) \neq \emptyset$ and $A_i(x) \cap P_{2i+2}(x) \neq \emptyset\}$ is open in X.

(4) The mappings $B_i, P_{2i+1}, P_{2i+2}: X \to 2^{D_i}$ are almost upper semicontinuous.

(5) For each $x \in W_i, x_i \notin \overline{B_i(x)} \cap \overline{P_{2i+1}(x)}$ and also $x_i \notin \overline{B_i(x)} \cap \overline{P_{2i+2}(x)}$.

Then Γ_1 and Γ_2 have a common equilibria point, i.e, there exists a point $\hat{x} \in D = \Pi_{i \in I} D_i$, such that $\hat{x}_i \in \overline{B_i(\hat{x})}$; $P_{2i+1}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ and $P_{2i+2}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ for all $i \in I$.

Proof. For each $i \in I$ and $x \in X$, let

$$T_i(x) = \begin{cases} \overline{B_i(x)} \cap \overline{P_{2i+1}(x)}, & if\ x \in W_i, \\ \overline{B_i(x)}, & if\ x \notin W_i. \end{cases}$$

Then, $T_i: X \to 2^{D_i}$ is a multivalued mapping with nonempty closed convex values. Since B_i and P_{2i+1} are two almost upper semicontinuous multivaled mappings with convex values, by [12, Lemmas 1 and 2], we know that $\overline{B_i}$ and $\overline{P_{2i+1}}$ are upper semicontinuous. Hence, $\overline{B_i} \cap \overline{P_{2i+1}}$ is upper semicontinuous by [1, Proposition 3.1.7 and Theorem 3.1.8]. As in the proof the Theorem 2.1, we know that T_i is upper semicontinuous.

Define mapping $T: X \to 2^D$ by $T(x) = \prod_{i\in I} T_i(x), \forall\, x \in X,$

then T is an upper semicontinuous multivalued mapping with nonempty closed convex valued by [9, Lemma 3]. Therefore, by applying Himmelberg's fixed-point theorem [10], there exists a point $\hat{x} \in D = \prod_{i\in I} D_i$, such that, $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$. By Condition (5), we have $\hat{x}_i \in \overline{B_i(\hat{x})}$ and $P_{2i+1}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ for all $i \in I$.

Similarly, it can be established that for each $i \in I$, $\hat{x}_i \in \overline{B_i(\hat{x})}$ and $P_{2i+2}(\hat{x}) \cap A_i(\hat{x}) = \emptyset$, i.e., Γ_1 and Γ_2 have a common equilibria point. This completes the proof of Theorem.

Remark: In Theorem 2.1-2.3, when $A_i(x) = B_i(x) = X_i$ for all $x \in X$ and $i \in I$, we can obtain some new common existence theorems of maximal element for qualitative games.

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