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Nonlinear Differential Equations and Mixture of Tarig Transform and Differential Transform Method

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Abstract

In this paper, we apply a new integral transform "Tarig transform" with the differential transform method to solve some nonlinear differential equations .The method is based on Tarig transform and differential transform methods. The nonlinear terms can be easily handled by the use of differential transform method.

Keywords: Tarig transform; Differential transform method; Nonlinear differential equations.

1. Introduction

Many problems of physical interest are described by ordinary or partial differential equations with appropriate initial or boundary conditions, these problems are usually formulated as initial value problems or boundary value problems, Tarig transform method [8-13] is particularly useful for finding solutions for these problems.

Tarig transform is a useful technique for solving linear Differential equations but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. This paper is using differential transforms method [5, 6, 7,] to decompose the nonlinear term, so that the solution can be obtained by iteration procedure. This means that we can use both Tarig transform and differential transform methods to solve many nonlinear problems. The main thrust of this technique is that the solution which is expressed as an infinite series converges fast to exact solutions.

2. Tarig transform

Consider functions in the set A defined by:

$$A=\left\{ f(t):\exists M, k_1, k_2>0, \ |f(t)|<M e^{|t|/k_j}, \qquad if \quad t\in(-1)^j\times[0,\infty) \right\}$$

Where M a constant is must be finite number and k_1, k_2 can be finite or infinite.

Tarig transform denoted by the operator $T(\cdot)$, is defined by the integral equation:

$$T[f(t)]=F(u)=\frac{1}{u}\int_0^\infty f(t)e^{-\frac{t}{u^2}}dt \ , \qquad u\neq 0, \qquad t>0 \tag{1}$$

Theorem (1)

Let $F(u)$ be Tarig transform of $f(t)$ $\quad [T(f(t))=F(u)]$ then:

(i) $T\left[f'(t)\right]=\frac{T(u)}{u^{2}}-\frac{1}{u}f(0)$ (ii) $T\left[f''(t)\right]=\frac{F(u)}{u^{4}}-\frac{1}{u^{3}}f(0)-\frac{1}{u}f'(0)$

Proof

(i) By the definition we have:

$T\left[f'(t)\right]=\frac{1}{u}\int_{0}^{\infty}f'(t)e^{-\frac{t}{u^{2}}}dt$, Integrating by parts, we get:

$$T\left[f'(t)\right]=\frac{T(u)}{u^{2}}-\frac{1}{u}f(0)$$

Let $g(t)=f'(t)$, Then: $T\left[g'(t)\right]=\frac{1}{u^{2}}T\left[g(t)\right]-\frac{1}{u}g(0)$, using (i) to find that:

$$T\left[f''(t)\right]=\frac{F(u)}{u^{4}}-\frac{1}{u^{3}}f(0)-\frac{1}{u}f'(0)$$

3. Differential Transform

Differential transform of the function $y(x)$ is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^{k}y(x)}{dx^{k}}\right]_{x=0} \tag{2}$$

And the inverse differential transform of $Y(k)$ is defined as:

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^{k}$$

The main theorems of the one – dimensional differential transform are.

Theorem (2): If $w(x)=y(x)\pm z(x)$, then $W(k)=Y(k)\pm Z(x)$

Theorem (3): If $w(x)=cy(x)$, Then $W(k)=cY(k)$

Theorem (4): If $w(x)=\frac{dy(x)}{dx}$, then $W(k)=(k+1)Y(k+1)$

Theorem (5): If $w(x)=\frac{d^{n}y(x)}{dx^{n}}$, then $W(k)=\frac{(k+n)!}{k!}Y(k+n)$

Theorem (6): If $w(x)=y(x)z(x)$, then $W(x)=\sum_{r=0}^{k}Y(r)Z(k-r)$

Theorem (7): If $w(x)=x^{n}$, then $W(k)=\delta(k-n)=\begin{cases}1 & , \quad k=n\\ 0 & , \quad k\neq n\end{cases}$

Note that c is a constant and n is a nonnegative integer.

4. Analysis of Differential Transform

In this section, we will introduce a reliable and efficient algorithm to calculate the differential transform of nonlinear functions.

I / Exponential nonlinearity: $f(y)=e^{ay}$.

From the definition of transform

$$F(0)=\left[e^{ay(x)}\right]_{x=0}=e^{ay(0)}=e^{aY(0)} \tag{3}$$

Taking a differential of $f(y)=e^{ay}$ with respect to x ,we get:

$$\frac{df(y)}{dx}=ae^{ay}\frac{dy(x)}{dx}=af(y)\frac{dy(x)}{dx} \tag{4}$$

Application of the differential transform to Eq (4) gives:

$$(k+1)F(k+1)=a\sum_{m=0}^{k}(m+1)Y(m+1)F(k-m) \tag{5}$$

Replacing $k+1$ by k gives

$$F(k)=a\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-1-m),\ k\geq 1 \tag{6}$$

Then from Eqs (3) and (6), we obtain the recursive relation

$$F(k)=\begin{cases} e^{aY(0)} & ,\quad k=0 \\ a\sum_{m=0}^{k-1}\frac{m+1}{k}Y(m+1)F(k-1-M) & ,\quad k\geq 1 \end{cases} \tag{7}$$

II / Logarithmic nonlinearity: $f(y)=\ln(a+by),\ a+by>0.$

Differentiating $f(y)=\ln(a+by)$, with respect to x ,we get:

$$\frac{df(y(x))}{dx}=\frac{b}{a+by}\frac{dy(x)}{dx}\text{, or } a\frac{df(y)}{dx}=b\left[\frac{dy(x)}{dx}-y\frac{df(y)}{dx}\right] \tag{8}$$

By the definition of transform:

$$F(0)=\left[\ln(a+by(x))\right]_{x=0}=\ln\left[a+by(0)\right]=\ln\left[a+bY(0)\right] \tag{9}$$

Take the differential transform of Eq.(8) to get:

$$aF(k+1)=b\left[Y(k+1)-\sum_{m=0}^{k}\frac{m+1}{k+1}F(m+1)Y(k-m)\right] \tag{10}$$

Replacing $k+1$ by k yields:

$$aF(k)=b\left[Y(k)-\sum_{m=0}^{k-1}\frac{m+1}{k}F(m+1)Y(k-1-m)\right],\ k\geq 1 \tag{11}$$

Put $k=1$ into Eq.(11) to get:

$$F(1)=\frac{b}{a+bY(0)}Y(1). \tag{12}$$

For $k \geq 2$, Eq. (11) can be rewritten as

$$F(k)=\frac{b}{a+bY(0)}\left[Y(k)-\sum_{m=0}^{k-2}\frac{m+1}{k}F(m+1)Y(k-1-m)\right] \tag{13}$$

Thus the recursive relation is:

$$F(k)=\begin{cases} \ln\left[a+bY(0)\right] & , \quad k=0 \\ \dfrac{b}{a+bY(0)}Y(1) & , \quad k=1 \\ \dfrac{b}{a+bY(0)}\left[Y(k)-\sum_{m=0}^{k-2}\dfrac{m+1}{k}F(m+1)Y(k-1-m)\right] & , \quad k\geq 2 \end{cases}$$

5. Application

In this section we solve some nonlinear differential equation by combine Tarig transform and differential transform method

Example (1)

Consider the simple nonlinear first order differential equation.

$$y'=y^2 \quad , \qquad y(0)=1 \tag{14}$$

First applying Tarig transform on both sides to find:

$$\frac{Y(u)}{u^2}-\frac{1}{u}y(0)=T\left[y^2\right]\Rightarrow Y(u)=u+u^2T\left[y^2\right] \tag{15}$$

$Y(u)$ is the Tarig transform of $y(t)$,

The standard Tarig transformation method defines the solution $y(t)$ by the series.

$$y=\sum_{n=0}^{\infty}y(n) \tag{16}$$

Operating with Tarig inverse on both sides of Eq (15) gives:

$$y(t)=1+T^{-1}\left[u^2T\left(y^2\right)\right] \tag{17}$$

Substituting Eq (16) into Eq (17) we find:

$$y(n+1)=T^{-1}\left\{u^2T\left[A_n\right]\right\} \quad , \quad n\geq 0 \tag{18}$$

Where $y(0)=1,\ A_n=\sum_{r=0}^{n}y(r)\,y(n-r)$, and $A_0=1$

For $n=0$, we have: $y(1)=T^{-1}\left\{u^2T\left[A_0\right]\right\}=T^{-1}\left\{u^3\right\}=t$

For $n=1$, we have: $A_1=2t$ and $y(2)=T^{-1}\left\{u^2T\left[2t\right]\right\}=t^2$

For $n=2$,we haves: $A_2 = 3t^2$ and $y(3) = T^{-1}\left\{u^2 T\left[3t^2\right]\right\} = t^3$

The solution in a series form is given by.

$$y(t) = y(0) + y(1) + y(2) + y(3) + \ldots\ldots \Rightarrow y(t) = 1 + t + t^2 + t^3 + \ldots\ldots\ldots = \frac{1}{1-t}$$

Example (2)

We consider the following nonlinear differential equation.

$$\frac{dy}{dt} = y - y^2 \quad , \quad y(0) = 2 \tag{19}$$

In a similar way we have:

$$\frac{Y(u)}{u^2} - \frac{2}{u} = T\left[y - y^2\right] \text{ or } Y(u) = 2u + u^2 T\left[y - y^2\right] \tag{20}$$

The inverse of Tarig transform implies that:

$$y(t) = 2 + T^{-1}\left\{u^2 T\left[y - y^2\right]\right\} \tag{21}$$

The recursive relation is given by:

$$y(n+1) = T^{-1}\left\{u^2 T\left[y(n) - A_n\right]\right\}, \; n \geq 0 \tag{22}$$

Where $y(0) = 2$, and $A_n = \sum_{r=0}^{n} y(r) y(n-r)$

The first few components of A_n are

$$A_0 = y^2(0) \; , \; A_1 = 2y(0)y(1) \; , \; A_2 = 2y(0)y(2) + y^2(1)$$
$$A_3 = 2y(0)y(3) + 2y(1)y(2) \, , \; \ldots\ldots$$

From the recursive relation we have:

$$y(0) = 2 \quad , \quad A_0 = 4$$
$$y(1) = T^{-1}\left\{u^2 T\left[y(0) - A_0\right]\right\} = T^{-1}\left\{u^2 T\left[-2\right]\right\} = -2t$$
$$y(2) = T^{-1}\left\{u^2 T\left[y(1) - A_1\right]\right\} = T^{-1}\left\{u^2 T\left[6t\right]\right\} = 3t^2$$
$$y(3) = T^{-1}\left\{u^2 T\left[y(2) - A_2\right]\right\} = T^{-1}\left\{u^2 T\left[-13t^2\right]\right\} = -\frac{13}{3}t^3$$

Then we have the following approximate solution to the initial problem.

$$y(t) = y(0) + y(1) + y(2) + \ldots.$$
$$y(t) = 2 - 2t + 3t^2 - \frac{13}{3}t^3 + \frac{25}{4}t^4 \ldots\ldots = \frac{2}{2 - e^{-t}}$$

Example (3)

Consider the nonlinear initial – value Problem

$$y''(x) = 2y + 4y \ln y \; , \; y > 0 \; , \; y(0) = 1 \; , \; y'(0) = 0 \tag{23}$$

Applying Tarig transform to Eq (23) and using the initial conditions, we obtain.

$$Y(u)=u+u^4T\left[2y+4y\ln y\right] \quad (24)$$

Take the inverse of Eq (24) to find:

$$y(x)=1+T^{-1}\left\{u^4T\left[2y+4y\ln y\right]\right\} \quad (25)$$

The recursive relation is given by:

$$y(n+1)=T^{-1}\left\{u^4T\left[2y(n)+4A_n\right]\right\} \quad (26)$$

Where $A_n=\sum_{m=0}^{n}y(m)F(n-m)$ and $y(0)=1$ (27)

And
$$F(n)=\begin{cases}\ln\left(y(0)\right) & , \quad n=0\\ \dfrac{y(1)}{y(0)} & , \quad n=1\\ \dfrac{y(n)}{y(0)}-\sum_{m=0}^{n-2}\dfrac{m+1}{ny(0)}F(m+1)y(n-1-m) & , \quad n\geq 2\end{cases} \quad (28)$$

Then we have:

$$F(0)=0\Rightarrow A_0=0 \text{ , and } y(1)=T^{-1}\left\{u^4T[2]\right\}=T^{-1}\left[2u^5\right]=x^2$$

$$F(1)=x^2\Rightarrow A_1=x^2 \text{ , and } y(2)=T^{-1}\left\{u^4T\left[6x^2\right]\right\}=\frac{x^4}{2}$$

$$F(2)=0\Rightarrow A_2=x^4\text{ , and } y(3)=T^{-1}\left\{u^4T\left[5x^4\right]\right\}=\frac{x^6}{6}$$

Then the exact solution is:

$$y(x)=1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+....=\sum_{k=0}^{\infty}\frac{1}{k!}\left(x^2\right)^k=e^{x^2}$$

Example (4)

Consider the initial –value problem of Bratu-type.

$$y''(x)-2e^y=0 \quad , \quad 0<x<1 \quad , \quad y(0)=y'(0)=0 \quad (29)$$

Take Tarig transform of this equation and use the initial condition to obtain:

$$Y(u)=u^4E\left[2e^y\right] \quad (30)$$

Take the inverse to obtain:

$y(x)=T^{-1}\left\{u^4T\left[2e^y\right]\right\}$. Then the recursive relation is given by:

$$y(n+1)=T^{-1}\left\{u^4E\left[2F(n)\right]\right\},\ y(0)=0 \quad (31)$$

Where $y(0)=0$, and $F(n)=\begin{cases} e^{y(0)} & , \quad n=0 \\ \sum_{m=0}^{n-1}\frac{m+1}{n}y(m+1)F(n-m-1) & , \quad n\geq 1 \end{cases}$ (32)

Then from Eqs (31) and (32) we have

$$F(0)=1 \ , \text{ and } \quad y(1)=T^{-1}\{u^4T[2]\}=T^{-1}[2u^5]=x^2$$

$$F(1)=x^2, \text{ and } \quad y(2)=T^{-1}\{u^4T[2x^2]\}=\frac{x^4}{6}$$

$$F(2)=\frac{2}{3}x^4, \text{ and } \quad y(3)=T^{-1}\left\{u^4T\left[\frac{4}{3}x^4\right]\right\}=\frac{2}{45}x^6$$

Then the series solution is

$$y(x)=x^2+\frac{1}{6}x^4+\frac{2}{45}x^6+......=-2\ln(\cos x)$$

Conclusions

In this paper, the exact solutions of nonlinear differential equations are obtained by using Tarig transform and differential transform methods. This technique is useful to solve linear and nonlinear differential equations.

Appendix

Tarig transform of some functions

S.N0.	$f(t)$	$F(u)$
1	1	u
2	t	u^3
3	e^{at}	$\frac{u}{1-au^2}$
4	t^n	$n!\,u^{2n+1}$
5	t^a	$\Gamma(a+1)u^{2a+1}$
6	$\sin at$	$\frac{au^3}{1+a^2u^4}$
7	$\cos at$	$\frac{u}{1+a^2u^4}$
8	$\sinh at$	$\frac{au^3}{1-a^2u^4}$
9	$\cosh at$	$\frac{u}{1-a^2u^4}$

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