SCITECH
RESEARCH ORGANISATION|

Volume 5, Issue 3

Published online: October 3, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Another weighted approximation of functions with singularities by combinations of Bernstein operators

Wen-ming Lu[1], Jin-ping Jia[2]

[1] School of Science, Hangzhou Dianzi Unviersity, Hangzhou, 310018 P.R. China.

[2] School of Mathematics and Statistics, Tianshui Normal University.

Abstract

We give direct and converse results for the weighted approximation of functions with inner singularities by a new type of Bernstein operators.

Keywords: Combinations of modified Bernstein polynomials; Functions with singularities; Weighted approximation; Direct and inverse results.

1. Introduction

The present work continues to study modified Bernstein operators following [11]. Here, the notation is referred to [11], for convenience, these notations will be listed. The set of all continuous functions, defined on the interval *I*, is denoted by *C(I)*. For any $f \in C([0,1])$, the corresponding Bernstein operators are defined as follows:

$$B_n(f,x) := \sum_{k=0}^{n} f(\frac{k}{n}) p_{nk}(x),$$

where

$$p_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \;\; k = 0,1,2,\ldots,n, \;\; x \in [0,1].$$

Let

$$\bar{w}(x) = |x-\xi|^{\alpha}, \; 0 < \xi < 1, \; \alpha > 0$$

and

$$C_{\bar{w}} := \{f \in C([0,1] \setminus \xi) : \lim_{x \longrightarrow \xi} (\bar{w}f)(x) = 0\}.$$

The norm $C_{\bar{w}}$ is defined as $\|f\|_{C_{\bar{w}}} := \|\bar{w}f\| = \sup_{0 \leqslant x \leqslant 1} |(\bar{w}f)(x)|$, and

$$W_{\bar{w}}^r := \{f \in C_{\bar{w}} : f^{(r-1)} \in A.C.((0,1)),\ \|\bar{w}\varphi^r f^{(r)}\| < \infty\}.$$

For $f \in C_{\bar{w}}$, we define the weighted modulus of smoothness by

$$\omega_\varphi^r(f,t)_{\bar{w}} := \sup_{0<h\leqslant t}\{\|\bar{w}\Delta_{h\varphi}^r f\|_{[16h^2,1-16h^2]} + \|\bar{w}\overrightarrow{\Delta}_h^r f\|_{[0,16h^2]} + \|\bar{w}\overleftarrow{\Delta}_h^r f\|_{[1-16h^2,1]}\},$$

where

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f(x+(\frac{r}{2}-k)h\varphi(x)),$$

$$\overrightarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f(x+(r-k)h),$$

$$\overleftarrow{\Delta}_h^r f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f(x-kh),$$

and $\varphi(x) = \sqrt{x(1-x)}$. The weighted *K*-function is given by

$$K_{r,\varphi}(f,t^r)_{\bar{w}} := \inf_g\{\|\bar{w}(f-g)\| + t^r\|\bar{w}\varphi^r g^{(r)}\| : g \in W_{\bar{w}}^r\}.$$

It was shown in [5] that $K_\varphi(f,t^r)_{\bar{w}} \sim \omega_\varphi^r(f,t)_{\bar{w}}$. Della Vecchia et al. firstly introduced $B_n^*(f,x)$ and $\bar{B}_n(f,x)$ in [3], where the properties of $B_n^*(f,x)$ and $\bar{B}_n(f,x)$ are studied.
Among others, they prove that

$$\|w(f - B_n^*(f))\| \leqslant C\omega_\varphi^2(f, n^{-1/2}),\ f \in C_w,$$

$$\|\bar{w}(f - \bar{B}_n(f))\| \leqslant \frac{C}{n^{3/2}}\sum_{k=1}^{[\sqrt{n}]} k^2\omega_\varphi^2(f,\frac{1}{k})_{\bar{w}}^*,\ f \in C_{\bar{w}},$$

Where $w(x) = x^\alpha(1-x)^\beta,\ \alpha,\beta \geqslant 0,\ \alpha+\beta > 0,\ 0 \leqslant x \leqslant 1.$

In [11], for any $\alpha,\beta > 0,\ n \geqslant 2r+\alpha+\beta$, there hold

$$\|wB_{n,r}^*(f)\| \leqslant C\|wf\|,\ f \in C_w,$$

$$\|w(B^*_{n,r}(f)-f)\| \leqslant \begin{cases} \frac{C}{n^r}(\|wf\|+\|w\varphi^{2r}f^{(2r)}\|), & f\in W^{2r}_w, \\ C(\omega^{2r}_\varphi(f,n^{-1/2})_w+n^{-r}\|wf\|), & f\in C_w, \end{cases}$$

$$\|w\varphi^{2r}B^{*(2r)}_{n,r}(f)\| \leqslant \begin{cases} Cn^r\|wf\|, & f\in C_w, \\ C(\|wf\|+\|w\varphi^{2r}f^{(2r)}\|), & f\in W^{2r}_w. \end{cases}$$

and for any $0<\gamma<2r,$

$$\|w(B^*_{n,r}(f)-f)\| = O(n^{-\gamma/2}) \Longleftrightarrow \omega^{2r}_\varphi(f,t)_w = O(t^r).$$

The main purpose of the present paper is to give another new type of combinations of Bernstein operators so as to obtain higher approximation order. Throughout the paper, *C* denotes a positive constant independent of n and $x,$ which may be different in different cases.

2. Main Results

For any positive integer *r*, we consider the determinant

$$A_r := \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2r+1 & 2r+2 & 2r+3 & \cdots & 4r+1 \\ (2r)(2r+1) & (2r+1)(2r+2) & (2r+2)(2r+3) & \cdots & (4r)(4r+1) \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 2\cdots(2r+1) & 3\cdots(2r+2) & 4\cdots(2r+3) & \cdots & (2r+2)\cdots(4r+1) \end{vmatrix}$$

We obtain $A_r=\prod_{j=2}^{2r} j!$. Thus, there is a unique solution for the system of nonhomogeneous linear equations:

$$\begin{cases} a_1 + a_2 + \cdots + a_{2r+1} = 1, \\ (2r+1)a_1 + (2r+2)a_2 + \cdots + (4r+1)a_{2r+1} = 0, \\ (2r+1)(2r)a_1 + (2r+1)(2r+2)a_2 + \cdots + (4r)(4r+1)a_{2r+1} = 0, \\ \vdots \\ (2r+1)!a_1 + 3\cdots(2r+2)a_2 + \cdots + (2r+2)\cdots(4r+1)a_{2r+1} = 0. \end{cases} \quad (2.1)$$

Let

$$\psi(x)=\begin{cases} a_1x^{2r+1}+a_2x^{2r+2}+\cdots+a_{2r+1}x^{4r+1}, & 0<x<1, \\ 0, & x\leqslant 0, \\ 1, & x=1. \end{cases}$$

with the coefficients $a_1,\ a_2,\ \cdots,\ a_{2r+1}$ satisfying (2.1). From (2.1), we see that $\psi(x)\in C^{(2r)}(-\infty,+\infty),\ 0\leqslant\psi(x)\leqslant 1$ for $0\leqslant x\leqslant 1.$ Moreover, it holds that $\psi(1)=1,\ \psi^{(i)}(0)=0,\ i=0,1,\cdots,2r.$ and $\psi^{(i)}(1)=0,\ i=1,2,\cdots,2r.$

Let

$$L_r(f,x) := \sum_{i=1}^{r+1} f(x_i) l_i(x),$$

Where

$$l_i(x) := \frac{\prod_{j=1, j\neq i}^{r+1}(x - x_j)}{\prod_{j=1, j\neq i}^{r+1}(x_i - x_j)},$$

$$x_i = \frac{[n\xi - ((r-1)/2 + i)\sqrt{n}]}{n}, \ i = 1, 2, \cdots r+1.$$

and

$$R_r(f,x) := \sum_{i=1}^{r+1} f(x_i^*) l_i^*(x),$$

Where

$$l_i^*(x) := \frac{\prod_{j=1, j\neq i}^{r+1}(x - x_j^*)}{\prod_{j=1, j\neq i}^{r+1}(x_i^* - x_j^*)},$$

$$x_i^* = \frac{n - [n\xi - ((r-1)/2 + i)\sqrt{n}]}{n}, \ i = 1, 2, \cdots r+1.$$

Further, let

$$x_1' = \frac{[n\xi - 2\sqrt{n}]}{n}, \ x_2' = \frac{[n\xi - \sqrt{n}]}{n}, \ x_3' = \frac{[n\xi + \sqrt{n}]}{n}, \ x_4' = \frac{[n\xi + 2\sqrt{n}]}{n}.$$

Set

$$\bar{F}_n(f,x) := \bar{F}_n(x) = (1 - \psi(nx-1))L_r(f,x) + (1 - \psi(nx - n + 2))\psi(nx-1)f(x)$$

$$+\psi(nx - n + 2)R_r(f,x).$$

$$= \begin{cases} L_r(f,x), & x \in [0, 1/n], \\ (1-\psi(nx-1))L_r(f,x) + \psi(nx-1)f(x), & x \in [1/n, 2/n], \\ f(x), & x \in [2/n, 1-2/n], \\ (1-\psi(nx-n+2))f(x) + \psi(nx-n+2)R_r(f,x), & x \in [1-2/n, 1-1/n], \\ R_r(f,x), & x \in [1-1/n, 1]. \end{cases}$$

$\bar{F}_n(f,x)$ is a linear is a linear polynomial of degree *r*, and $\bar{F}_n(f,x) \in C^{(2r)}([0,1])$, provided that $f \in C^{(2r)}([0,1])$.

We define our combinations of Bernstein operators as follows:

$$\bar{B}_{n,r}(f,x) := B_{n,r}(\bar{F}_n, x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(\bar{F}_n, x),$$

where $C_i(n)$ satisfy the conditions (a)-(d).

$(a) n = n_0 < n_1 < \cdots < n_{r-1} \leqslant Cn,$
$(b) \sum_{i=0}^{r-1} |C_i(n)| \leqslant C,$
$(c) \sum_{i=0}^{r-1} C_i(n) = 1,$
$(d) \sum_{i=0}^{r-1} C_i(n) n_i^{-k} = 0, \text{ for } k = 1, \ldots, r-1.$

We have our main results as follows:

Theorem. For any $\alpha > 0,\ 0 \leqslant \lambda \leqslant 1,$ we have

$$\|\bar{w}\bar{B}_{n,r}^{(2r)}(f)\| \leqslant Cn^{2r}\|\bar{w}f\|,\ f \in W_{\bar{w}}^{2r}, \tag{2.2}$$

$$|\bar{w}(x)\varphi^{2r\lambda}(x)\bar{B}_{n,r}^{(2r)}(f,x)| \leqslant \begin{cases} Cn^r\{\max\{n^{r(1-\lambda)}, \varphi^{2r(\lambda-1)}\}\}\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|), & f \in W_{\bar{w}}^{2r}, \end{cases} \tag{2.3}$$

$$\|\bar{w}\bar{B}_{n,r}(f)\| \leqslant C\|\bar{w}f\|,\ f \in C_{\bar{w}}, \tag{2.4}$$

$$\|\bar{w}(\bar{B}_{n,r}(f) - f)\| \leqslant \begin{cases} \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), & f \in W_{\bar{w}}^{2r}, \\ C(\omega_\varphi^{2r}(f, n^{-1/2})_{\bar{w}} + n^{-r}\|\bar{w}f\|), & f \in C_{\bar{w}}, \end{cases} \tag{2.5}$$

and for $0 < \gamma < 2r,$

$$\|\bar{w}(\bar{B}_{n,r}(f) - f)\| = O(n^{-\gamma/2}) \Longleftrightarrow \omega_\varphi^{2r}(f,t)_{\bar{w}} = O(t^r). \tag{2.6}$$

3. Lemmas

Lemma 1. ([13]) For any non-negative real u and v, we have

$$\sum_{k=1}^{n-1}(\frac{k}{n})^{-u}(1-\frac{k}{n})^{-v}p_{nk}(x) \leqslant Cx^{-u}(1-x)^{-v}. \tag{3.1}$$

Lemma 2. For any positive real $\alpha,$ and $f \in W_{\bar{w}}^{2r}$, we have

$$\|\bar{w}\varphi^{2r-2j}f^{(2r-j)}\| \leqslant C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|). \tag{3.2}$$

Proof. It follows from Kolmogolov's inequality that

$$|f^{(2r-j)}(\frac{1}{2})| \leqslant C(\|f\|_{[1/4,3/4]} + \|f^{(2r)}\|_{[1/4,3/4]}),$$

Moreover,

$$|f^{(2r-j)}(\frac{1}{2})| \leqslant C(\|\bar{w}f\|_{[1/4,3/4]} + \|\bar{w}\varphi^{2r}f^{(2r)}\|_{[1/4,3/4]}). \tag{3.3}$$

When $0 \leqslant x \leqslant \frac{1}{2}$, if $|t-\xi| > \frac{1}{2\sqrt{n}}$, then $f^{(2r-j+1)}(t) \neq 0$. Morever we have $\bar{w}(x) \leqslant \bar{w}(t)(1+n^{\frac{\alpha}{2}}|t-x|)^{\alpha}$. Therefore

$$\begin{aligned} |(f^{(2r-j)}(x) - f^{(2r-j)}(\frac{1}{2}))| &\leqslant \int_x^{\frac{1}{2}} |f^{(2r-j+1)}(u)|du \\ &\leqslant C\frac{\|\bar{w}\varphi^{2r-2j+2}f^{(2r-j+1)}\|}{\bar{w}(x)} \int_x^{\frac{1}{2}} \frac{\bar{w}(x)du}{\bar{w}(u)\varphi^{2r-2j+2}(u)} \\ &\leqslant C\|\bar{w}\varphi^{2r-2j+2}f^{(2r-j+1)}\|\frac{x^{-r+j+1}(1+n^{\frac{\alpha}{2}}x^{\alpha})}{\bar{w}(x)}. \end{aligned}$$

Which, together with (3.3), gives that

$$|\bar{w}(x)\varphi^{2r-2j}(x)f^{(2r-j)}(x)| \leqslant C(\|\bar{w}\varphi^{2r-2j+2}f^{(2r-j+1)}\| + \|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

Similarly, we can prove that the above inequality also holds when $1/2 < x \leqslant 1$. Therefore we obtain that

$$|\bar{w}(x)\varphi^{2r-2j}(x)f^{(2r-j)}(x)| \leqslant C(\|\bar{w}\varphi^{2r-2j+2}f^{(2r-j+1)}\| + \|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|). \qquad (3.4)$$

Now, the result follows from (3.4) when $j = 1$, and thus the result can be deduced from (3.4) by induction when $1 < j \leqslant r$.

Lemma 3. $f \in W_{\bar{w}}^{2r}$, $\alpha > 0$, we have

$$\|\bar{w}(f - L_r(f))\|_{[0,\frac{2}{n}]} \leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), \qquad (3.5)$$

$$\|\bar{w}(f - R_r(f))\|_{[1-\frac{2}{n},1]} \leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|). \qquad (3.6)$$

The proof is similar to the lemma 3 of [?],we don't give a proof here.

Lemma 4. For any $f \in W_{\bar{w}}^{2r}$ and $\alpha > 0$, we have

$$\|\bar{w}\varphi^{2r}\bar{F}_n^{(2r)}\| \leqslant C(\|\bar{w}\varphi^{2r}f^{(2r)}\| + \|\bar{w}f\|). \qquad (3.7)$$

The proof is similar to the lemma 4 of [11], we don't give a proof here.

Lemma 5. ([3]) Let $A_n(x) := \bar{w}(x) \sum\limits_{|k-n\xi| \leqslant \sqrt{n}} p_{n,k}(x)$. Then $A_n(x) \leqslant Cn^{-\alpha/2}$ for $0 < \xi < 1$and $\alpha > 0$.

Lemma 6. For $0 < \xi < 1$, $\alpha, \beta > 0$,we have

$$\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^{\beta}p_{n,k}(x)\leqslant Cn^{\frac{\beta-\alpha}{2}}\varphi^{\beta}(x). \tag{3.8}$$

Proof. By (3.1) and the lemma 5, we have

$$\bar{w}(x)^{\frac{1}{2n}}(\bar{w}(x)\sum_{|k-n\xi|\leqslant\sqrt{n}}p_{n,k}(x))^{\frac{2n-1}{2n}}(\sum_{|k-n\xi|\leqslant\sqrt{n}}|k-nx|^{2n\beta}p_{n,k}(x))^{\frac{1}{2n}}\leqslant Cn^{\frac{\beta-\alpha}{2}}\varphi^{\beta}(x).$$

4 Proof of Theorem 1

4.1 Proof of (2.2)

We first prove $x\in[0,\frac{1}{n}]$ (The same as $x\in[1-\frac{1}{n},1]$),now

$$\begin{aligned}
|\bar{w}(x)\bar{B}_{n,r}^{(2r)}(f,x)| &\leqslant \bar{w}(x)\sum_{i=0}^{r-1}\frac{n_i!}{(n_i-2r)!}\sum_{k=0}^{n_i-2r}|C_i(n)\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{k}{n_i})|p_{n_i-2r,k}(x)\\
&\leqslant C\bar{w}(x)\sum_{i=0}^{r-1}n_i^{2r}\sum_{k=0}^{n_i-2r}|C_i(n)\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{k}{n_i})|p_{n_i-2r,k}(x)\\
&\leqslant C\bar{w}(x)\sum_{i=0}^{r-1}n_i^{2r}\sum_{k=0}^{n_i-2r}\sum_{j=0}^{2r}C_{2r}^{j}|C_i(n)\bar{F}_n(\frac{k+2r-j}{n_i})|p_{n_i-2r,k}(x)\\
&\leqslant C\bar{w}(x)\sum_{i=0}^{r-1}n_i^{2r}\sum_{j=0}^{2r}C_{2r}^{j}|C_i(n)\bar{F}_n(\frac{2r-j}{n_i})|p_{n_i-2r,0}(x)\\
&\quad +C\bar{w}(x)\sum_{i=0}^{r-1}n_i^{2r}\sum_{j=0}^{2r}C_{2r}^{j}|C_i(n)\bar{F}_n(\frac{n_i-j}{n_i})|p_{n_i-2r,n_i-2r}(x)\\
&\quad +C\bar{w}(x)\sum_{i=0}^{r-1}n_i^{2r}\sum_{k=1}^{n_i-2r-1}\sum_{j=0}^{2r}C_{2r}^{j}|C_i(n)\bar{F}_n(\frac{k+2r-j}{n_i})|p_{n_i-2r,k}(x)\\
&:= H_1+H_2+H_3.
\end{aligned} \tag{4.1}$$

We have

$$\begin{aligned} H_1 &\leqslant C\bar{w}(x)\sum_{i=0}^{r-1} n_i^{2r}(\sum_{j=0}^{2r-1}|C_i(n)\bar{F}_n(\frac{2r-j}{n_i})|+|\bar{F}_n(0)|)p_{n_i-2r,0}(x) \\ &\leqslant Cn^{2r}\|\bar{w}f\|\sum_{i=0}^{r-1}\sum_{j=0}^{2r-1}(\frac{n_i|x-\xi|}{2r-j-n_i\xi})^{\alpha}(1-x)^{n_i-2r} \\ &\leqslant Cn^{2r}\|\bar{w}f\|\sum_{i=0}^{r-1}(n_i|x-\xi|)^{\alpha}(1-x)^{n_i-2r} \\ &\leqslant Cn^{2r}\|\bar{w}f\|. \end{aligned}$$

Similarly, we can get $H_2 \leqslant Cn^{2r}\|\bar{w}f\|$and $H_3 \leqslant Cn^{2r}\|\bar{w}f\|$.

When $x \in [\frac{1}{n}, 1-\frac{1}{n}]$, according to [5], we have

$$\begin{aligned} & |\bar{w}(x)\bar{B}_{n,r}^{(2r)}(f,x)| \\ = & |\bar{w}(x)B_{n,r}^{(2r)}(\bar{F}_n,x)| \\ = & \bar{w}(x)(\varphi^2(x))^{-2r}\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{k/n_i\in A}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ & +\bar{w}(x)(\varphi^2(x))^{-2r}\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{x_2'\leqslant k/n_i\leqslant x_3'}|(x-\frac{k}{n_i})^jH(\frac{k}{n_i})|p_{n_i,k}(x) \\ := & \sigma_1+\sigma_2. \end{aligned} \tag{4.2}$$

Where $A := [0, x_2'] \cup [x_3', 1]$, H is a linear function. If $\frac{k}{n_i} \in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n_i})} \leqslant C(1+n_i^{-\frac{\alpha}{2}}|k-n_ix|^{\alpha})$, we have $|k-n_i\xi| \geqslant \frac{\sqrt{n_i}}{2}$, then

$Q_j(x,n_i) = (n_ix(1-x))^{[(2r-j)/2]}$, and $(\varphi^2(x))^{-2r}Q_j(x,n_i)n_i^j \leqslant C(n_i/\varphi^2(x))^{r+j/2}$. By (3.8), then

$$\begin{aligned} \sigma_1 &\leqslant C\bar{w}(x)\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ &\leqslant C\|\bar{w}f\|\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k=0}^{n_i}[1+n_i^{-\frac{\alpha}{2}}|k-n_ix|^{\alpha}]|x-\frac{k}{n_i}|^jp_{n_i,k}(x) \\ &:= I_1+I_2. \end{aligned}$$

By a simple calculation, we have $I_1 \leqslant Cn^{2r}\|\bar{w}f\|$. By (3.1), then

$$I_2 \leqslant C\|\bar{w}f\|\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|C_i(n)|n_i^{-(\frac{\alpha}{2}+j)}(\frac{n_i}{\varphi^2(x)})^{j/2}\sum_{k=0}^{n_i}|k-n_ix|^{\alpha+j}p_{n_i,k}(x) \leqslant Cn^{2r}\|\bar{w}f\|.$$

We note $|H(\frac{k}{n_i})| \leqslant max(|H(x_1')|, |H(x_4')|) := H(a)$.

If $x \in [x_1', x_4']$, we have $\bar{w}(x) \leqslant \bar{w}(a)$.

$\sigma_2 \leqslant C\bar{w}(a)H(a)n^r\varphi^{-2r}(x) \leqslant Cn^{2r}\|\bar{w}f\|$.

If $x \notin [x_1', x_4']$, then $\bar{w}(a) > n_i^{-\frac{\alpha}{2}}$ we have

$$\sigma_2 \leqslant C\bar{w}(a)H(a)\varphi^{-2r}(x)\bar{w}(x)\sum_{i=0}^{r-1}C_i(n)n_i^{r+\frac{\alpha}{2}}\sum_{x_2' \leqslant k/n_i \leqslant x_3'}p_{n_i,k}(x) \leqslant Cn^{2r}\|\bar{w}f\|.$$

It follows from combining the above inequalities (4.1) and (4.2) that the theorem is proved.

4.2 Proof of (2.3)

(1) When $f \in C_{\bar{w}}$, we discuss it as follows:

Case 1. If $0 \leqslant \varphi(x) \leqslant \frac{1}{\sqrt{n}}$, by (2.2), we have

$$|\bar{w}(x)\varphi^{2r\lambda}(x)\bar{B}_{n,r}^{(2r)}(f,x)| \leqslant Cn^{-r\lambda}|\bar{w}(x)\bar{B}_{n,r}^{(2r)}(f,x)| \leqslant Cn^{r(2-\lambda)}\|\bar{w}f\|. \tag{4.3}$$

Case2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned} &|\bar{B}_{n,r}^{(2r)}(f,x)| = |B_{n,r}^{(2r)}(\bar{F}_n,x)| \\ \leqslant\ & (\varphi^2(x))^{-2r}\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|Q_j(x,n_i)C_i(n)|n_i^j\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x), \end{aligned}$$

$Q_j(x,n_i)=(n_ix(1-x))^{[(2r-j)/2]}$, and $(\varphi^2(x))^{-2r}Q_j(x,n_i)n_i^j\leqslant C(n_i/\varphi^2(x))^{r+j/2}$. So

$$\begin{aligned}
&|\bar{w}(x)\varphi^{2r\lambda}(x)\bar{B}_{n,r}^{(2r)}(f,x)| \\
\leqslant\ & C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k=0}^{n_i}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\
=\ & C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{k/n_i\in A}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\
&+C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}\sum_{j=0}^{2r}|C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2}\sum_{x_2'\leqslant k/n_i\leqslant x_3'}|(x-\frac{k}{n_i})^j\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\
:=\ & \sigma_1+\sigma_2. \qquad (4.4)
\end{aligned}$$

Where $A:=[0,x_2']\cup[x_3',1]$. We can easily get $\sigma_1\leqslant n^r\varphi^{2r(\lambda-1)}(x)\|\bar{w}f\|$, $\sigma_2\leqslant n^r\varphi^{2r(\lambda-1)}(x)\|\bar{w}f\|$. By bringing these facts (4.3) and (4.4) together, the theorem is proved.

2) When $f\in W_{\bar{w}}^{2r}$, we have

$$B_{n,r}^{(2r)}(\bar{F}_n,x)=\sum_{i=0}^{r-1}C_i(n)n_i^{2r}\sum_{k=0}^{n_i-2r}\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{k}{n_i})p_{n_i-2r,k}(x). \qquad (4.5)$$

If $0<k<n_i-2r$, we have

$$|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{k}{n_i})|\leqslant Cn_i^{-2r+1}\int_0^{\frac{2r}{n_i}}|\bar{F}_n^{(2r)}(\frac{k}{n_i}+u)|du, \qquad (4.6)$$

If $k=0$, we have

$$|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(0)|\leqslant Cn_i^{-r+1}\int_0^{\frac{2r}{n_i}}u^{2r-1}|\bar{F}_n^{(2r)}(u)|du, \qquad (4.7)$$

Similarly

$$|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{n_i-2r}{n_i})| \ \leqslant\ Cn_i^{-2r+1}\int_{1-\frac{2r}{n_i}}^{1}(1-u)^{2r-1}|\bar{F}_n^{(2r)}(u)|du.$$

By (4.5) and (4.6), we have

$$\begin{aligned}
|\bar{w}(x)\varphi^{2r\lambda}(x)\bar{B}_{n,r}^{(2r)}(f,x)| &\leqslant C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i^{2r}\sum_{k=0}^{n_i-2r}|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{k}{n_i})|p_{n_i-2r,k}(x)\\
&= C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i^{2r}\sum_{k=1}^{n_i-2r-1}|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{k}{n_i})|p_{n_i-2r,k}(x)\\
&\quad +C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i^{2r}|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(0)|p_{n_i-2r,0}(x)\\
&\quad +C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i^{2r}|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(\frac{n_i-2r}{n_i})|p_{n_i-2r,n_i-2r}(x)\\
&:= I_1+I_2+I_3.
\end{aligned}$$

By (4.6), we have

$$\begin{aligned}
I_1 &\leqslant C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i\sum_{k=1}^{n_i-2r-1}\int_0^{\frac{2r}{n_i}}|\bar{F}_n^{(2r)}(\frac{k}{n_i}+u)|du\,p_{n_i-2r,k}(x)\\
&= C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i\sum_{k/n_i\in A}\int_0^{\frac{2r}{n_i}}|\bar{F}_n^{(2r)}(\frac{k}{n_i}+u)|du\,p_{n_i-2r,k}(x)\\
&\quad +C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i\sum_{x_2'\leqslant k/n_i\leqslant x_3'}\int_0^{\frac{2r}{n_i}}|H_n^{(2r)}(\frac{k}{n_i}+u)|du\,p_{n_i-2r,k}(x)\\
&:= T_1+T_2.
\end{aligned}$$

Where $A:=[0,x_2']\cup[x_3',1],\ H$ is a linear function. If $\frac{k}{n_i}\in A,$ when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n_i})}\leqslant C(1+n_i^{-\frac{\alpha}{2}}|k-n_ix|^{\alpha}),$ we have $|k-n_i\xi|\geqslant\frac{\sqrt{n_i}}{2},$ by (3.1) and (3.7), then

$$\begin{aligned}
T_1 &\leqslant C\|\bar{w}\varphi^{2r\lambda}F^{(2r)}\|\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i\sum_{k/n_i\in A}\int_0^{\frac{2r}{n_i}}\bar{w}^{-1}(\frac{k}{n_i}+u)\varphi^{-2r\lambda}(\frac{k}{n_i}+u)du\,p_{n_i-2r,k}(x)\\
&\leqslant C\|\bar{w}\varphi^{2r\lambda}F^{(2r)}\|\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i\sum_{k=0}^{n_i}\int_0^{\frac{2r}{n_i}}[1+n_i^{-\frac{\alpha}{2}}|k-n_ix|^{\alpha}]\varphi^{-2r\lambda}(\frac{k}{n_i})du\,p_{n_i-2r,k}(x)\\
&\leqslant C\|\bar{w}\varphi^{2r\lambda}\bar{F}_n^{(2r)}\|\leqslant C(\|\bar{w}f\|+\|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|).
\end{aligned}$$

Similarly, we can get $T_2\leqslant C(\|\bar{w}f\|+\|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|).$ So $I_1\leqslant C(\|\bar{w}f\|+\|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|)$ and by (4.7), we have

$$\begin{aligned} I_2 &\leqslant C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i^{2r}|\overrightarrow{\Delta}_{\frac{1}{n_i}}^{2r}\bar{F}_n(0)|p_{n_i-2r,0}(x) \\ &\leqslant C\bar{w}(x)\varphi^{2r\lambda}(x)\sum_{i=0}^{r-1}|C_i(n)|n_i^{r+1}\int_0^{\frac{2r}{n_i}}u^{2r-1}|\bar{F}_n^{(2r)}(u)|du p_{n_i-2r,0}(x) \\ &\leqslant C\|\bar{w}\varphi^{2r\lambda}\bar{F}_n^{(2r)}\|\sum_{i=0}^{r-1}(n_ix)^{r(1+\lambda)}(1-x)^{r\lambda} \leqslant C\|\bar{w}\varphi^{2r\lambda}\bar{F}_n^{(2r)}\| \\ &\leqslant C(\|\bar{w}f\|+\|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|). \end{aligned}$$

Analogously, $I_3 \leqslant C(\|\bar{w}f\|+\|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|)$, then the theorem is proved.

Corollary 1. If $\alpha > 0$ and $\lambda = 0$, we have

$$|\bar{w}(x)\bar{B}_{n,r}^{(2r)}(f,x)| \leqslant \begin{cases} Cn^{2r}\|\bar{w}f\|, & f\in C_{\bar{w}}, \\ C(\|\bar{w}f\|+\|\bar{w}f^{(2r)}\|), & f\in W_{\bar{w}}^{2r}. \end{cases}$$

Corollary 2. If $\alpha > 0$ and $\lambda = 1$, we have

$$|\bar{w}(x)\varphi^{2r}(x)\bar{B}_{n,r}^{(2r)}(f,x)| \leqslant \begin{cases} Cn^{r}\|\bar{w}f\|, & f\in C_{\bar{w}}, \\ C(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|), & f\in W_{\bar{w}}^{2r}. \end{cases}$$

4.3 Proof of (2.4)

$$\begin{aligned} |\bar{w}(x)\bar{B}_{n,r}(f,x)| &= |\bar{w}(x)B_{n,r}(\bar{F}_n,x)| \leqslant \bar{w}(x)\sum_{i=0}^{r-1}\sum_{k=1}^{n_i-1}|C_i(n)\bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\ &\quad +\bar{w}(x)\sum_{i=0}^{r-1}|C_i(n)\bar{F}_n(0)|p_{n_i,0}(x)+\bar{w}(x)\sum_{i=0}^{r-1}|C_i(n)\bar{F}_n(1)|p_{n_i,n_i}(x) \\ &:= I_1+I_2+I_3. \end{aligned}$$

Analogously, the theorem can be proved easily.

4.4. Proof of (2.5)

We assume $f\in W_{\bar{w}}^{2r}$, then $\|\bar{w}(\bar{B}_{n,r}(f)-\bar{F}_n)\| \leqslant \frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|)$.

Recall that [?], then

$$B_{n,r}((t-x)^j,x)=0,\ j=1,2,\cdots,r, \tag{4.8}$$

$$B_{n,r}((t-x)^{2r-j},x)=O(n^{-r}\varphi^{2r-2j}(x)),\ x\in[\frac{1}{n},1-\frac{1}{n}],\ j=0,1,2,\cdots,r. \tag{4.9}$$

Case 1. $x \in [\frac{1}{n}, 1-\frac{1}{n}]$. By using Taylor expansion, we have

$$\begin{aligned}
& \bar{w}(x)(\bar{F}_n(x) - B_{n,r}(\bar{F}_n, x)) \\
= \ & \bar{w}(x)\sum_{j=1}^{2r-1}\frac{1}{(2r-j)!}B_{n,r}((t-x)^{2r-j}, x)\bar{F}_n^{(2r-j)}(x) \\
& +\bar{w}(x)B_{n,r}(\frac{1}{(2r-j)!}\int_x^t (t-u)^{2r-1}\bar{F}_n^{(2r)}(u)du, x) \\
:= \ & I_1 + I_2.
\end{aligned}$$

By (3.2), (3.7) and (4.9), we have f$1 \leqslant j \leqslant r,$ then

$$\frac{\bar{w}(x)\varphi^{2r-2j}(x)}{n^r}\bar{F}_n^{(2r-j)}(x) \leqslant \frac{C}{n^r}(\|\bar{w}\bar{F}_n\| + \|\bar{w}\varphi^{2r}\bar{F}_n^{(2r)}\|) \leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), \quad (4.10)$$

By (4.8) and (4.10) , we have

$$I_1 \leqslant \bar{w}(x)\sum_{j=1}^{r-1}\frac{1}{(2r-j)!}|B_{n,r}((t-x)^{2r-j}, x)\bar{F}_n^{(2r-j)}(x)| \leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

If u is between t and x we have $\frac{|u-x|^{2r-1}}{\varphi^{2r}(u)} \leqslant \frac{|t-x|^{2r-1}}{\varphi^{2r}(t)}$. Then

$$\begin{aligned}
& |\bar{w}(x)B_{n,r}(\frac{1}{(2r-j)!}\int_x^t (t-u)^{2r-1}\bar{F}_n^{(2r)}(u)du, x)| \\
\leqslant \ & C\bar{w}(x)\sum_{i=0}^{r-1}\sum_{k=0}^{n_i}|C_i(n)|\int_x^{\frac{k}{n_i}}|(\frac{k}{n_i}-u)^{2r-1}\bar{F}_n^{(2r)}(u)|du p_{n_i,k}(x) \\
= \ & C\bar{w}(x)\sum_{i=0}^{r-1}\sum_{k=1}^{n_i-1}|C_i(n)|\int_x^{\frac{k}{n_i}}|(\frac{k}{n_i}-u)^{2r-1}\bar{F}_n^{(2r)}(u)|du p_{n_i,k}(x) \\
& +C\bar{w}(x)\sum_{i=0}^{r-1}|C_i(n)|(1-x)^{n_i}\int_0^x u^{2r-1}|\bar{F}_n^{(2r)}(u)|du \\
& +C\bar{w}(x)\sum_{i=0}^{r-1}|C_i(n)|x^{n_i}\int_x^1 (1-u)^{2r-1}|\bar{F}_n^{(2r)}(u)|du \\
:= \ & J_1 + J_2 + J_3.
\end{aligned}$$

We have

$$\begin{aligned} J_1 &\leqslant C\bar{w}(x)\varphi^{-2r}(x)\sum_{i=0}^{r-1}\sum_{k/n_i\in A}|C_i(n)(\frac{k}{n_i}-x)^{2r-1}|\int_x^{\frac{k}{n_i}}\varphi^{2r}(v)|\bar{F}_n^{(2r)}(v)|dvp_{n_i,k}(x)\\ &+C\bar{w}(x)\varphi^{-2r}(x)\sum_{i=0}^{r-1}\sum_{x_2'\leqslant k/n_i\leqslant x_3'}|C_i(n)(\frac{k}{n_i}-x)^{2r-1}|\int_x^{\frac{k}{n_i}}\varphi^{2r}(v)|H^{(2r)}(v)|dvp_{n_i,k}(x)\\ &:= \sigma_1+\sigma_2. \end{aligned}$$

Analogously, we can get $\sigma_1 \leqslant \frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|)$.We note that $|\varphi^{2r}(v)H^{(2r)}(v)| \leqslant$ max $(|\varphi^{2r}(x_1')H^{(2r)}(x_1')|,\ |\varphi^{2r}(x_4')H^{(2r)}(x_4')|) := |\varphi^{2r}(a)H^{(2r)}(a)|,\ H^{(2r)}(x)$ is a linear function.

If $x\in[x_1',x_4']$, then $\bar{w}(x)\leqslant\bar{w}(a)$. So we have

$$\begin{aligned} \sigma_2 &\leqslant C\bar{w}(a)\varphi^{2r}(a)|H^{(2r)}(a)|\varphi^{-2r}(x)\sum_{i=0}^{r-1}\sum_{k=1}^{n_i-1}|C_i(n)|(\frac{k}{n_i}-x)^{2r}p_{n_i,k}(x)\\ &\leqslant \frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|), \end{aligned}$$

If $x\notin[x_1',x_4']$,by $\bar{w}(a)>n_i^{-\frac{\alpha}{2}}$,we have

$$\begin{aligned} \sigma_2 &\leqslant C\bar{w}(a)\varphi^{-2r}(a)|H^{(2r)}(a)|\sum_{i=0}^{r-1}\sum_{x_2'\leqslant k/n_i\leqslant x_3'}n_i^{\frac{\alpha}{2}}|C_i(n)|(\frac{k}{n_i}-x)^{2r}p_{n_i,k}(x)\\ &\leqslant \frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|). \end{aligned}$$

For J_2, we have

$$\begin{aligned} J_2 &\leqslant C\|\bar{w}\varphi^{2r}\bar{F}_n^{(2r)}\|\bar{w}(x)\sum_{i=0}^{r-1}|C_i(n)|(1-x)^{n_i}\int_0^x u^{2r-1}\bar{w}^{-1}(u)\varphi^{-2r}(u)du\\ &\leqslant \frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|). \end{aligned}$$

Similarly, we have

$$J_3\leqslant\frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

By bringing these facts together, we have

$$\|\bar{w}(\bar{B}_{n,r}(f)-\bar{F}_n)\|\leqslant\frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

Case 2. $x \in [0, \frac{1}{n}]$ (Similarly as $x \in [1-\frac{1}{n}, 1]$). By using Taylor expansion, we have

$$\begin{aligned} \bar{w}(x)|B_{n,r}(\bar{F}_n, x) - \bar{F}_n(x)| &\leqslant \frac{\bar{w}(x)}{r!}\sum_{i=0}^{r-1}|C_i(n)|B_{n_i}(\int_x^t |(t-u)^r \bar{F}_n^{(r+1)}(u)|du, x) \\ &+\frac{\bar{w}(x)}{r!}\sum_{i=0}^{r-1}|C_i(n)|(1-x)^{n_i}\int_0^x u^{2r-1}|\bar{F}_n^{(r+1)}(u)|du \\ &:= J_1 + J_2. \end{aligned}$$

$$\begin{aligned} J_1 &\leqslant C\bar{w}(x)\sum_{i=0}^{r-1}\sum_{k=0}^{n_i}\int_x^{\frac{k}{n_i}}|C_i(n)(\frac{k}{n_i}-u)^r \bar{F}_n^{(r+1)}(u)|du p_{n_i,k}(x) \\ &:= C\bar{w}(x)\sum_{i=0}^{r-1}\sum_{k=1}^{n_i-1}\int_x^{\frac{k}{n_i}}|C_i(n)(\frac{k}{n_i}-u)^r \bar{F}_n^{(r+1)}(u)|du p_{n_i,k}(x) \\ &+C\bar{w}(x)\sum_{i=0}^{r-1}|C_i(n)|x^{n_i}\int_x^1 (1-u)^r|\bar{F}_n^{(r+1)}(u)|du \\ &+C\bar{w}(x)\sum_{i=0}^{r-1}|C_i(n)|(1-x)^{n_i}\int_0^x u^r|\bar{F}_n^{(r+1)}(u)|du \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Analogously, we can get

$$\begin{aligned} I_1 &\leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), \\ I_2 &\leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), \\ I_3 &\leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|). \end{aligned}$$

$$\begin{aligned} J_1 &\leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), \\ J_2 &\leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|). \end{aligned} \qquad (4.11)$$

So, we have

$$\|\bar{w}(\bar{B}_{n,r}(f) - \bar{F}_n)\| \leqslant \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

Then

$$\begin{aligned}\|\bar{w}(\bar{B}_{n,r}(f)-f)\| &\leqslant \|\bar{w}(f-\bar{F}_n(f))\|+\|\bar{w}(\bar{F}_n(f)-\bar{B}_{n,r}(f))\|\\ &\leqslant \frac{C}{n^r}(\|\bar{w}f\|+\|\bar{w}\varphi^{2r}f^{(2r)}\|).\end{aligned}$$

If $f\in C_{\bar{w}}$, there exists $g\in W_{\bar{w}}^{2r}$, by (2.4) and the first inequality of (2.5), then

$$\begin{aligned}\|\bar{w}(\bar{B}_{n,r}(f)-f)\| &\leqslant \|\bar{w}(f-g)\|+\|\bar{w}\bar{B}_{n,r}(f-g)\|+\|\bar{w}(g-\bar{B}_{n,r}(g))\|\\ &\leqslant C(\|\bar{w}(f-g)\|+\frac{1}{n^r}(\|\bar{w}g\|+\|\bar{w}\varphi^{2r}g^{(2r)}\|))\\ &\leqslant C(\omega_\varphi^{2r}(f,n^{-1/2})_{\bar{w}}+n^{-r}\|\bar{w}f\|).\end{aligned}$$

4.5. Proof of (2.6)

From the proof of (2.5), we actually have

$$\|\bar{w}(\bar{B}_{n,r}(f)-f)\|\leqslant CK_{2r,\varphi}(f,t^r)_{\bar{w}}.$$

Therefore, $K_{2r,\varphi}(f,n^{-r})_{\bar{w}}=O(t^\alpha)$ implies

$$\|\bar{w}(\bar{B}_{n,r}(f)-f)\|\leqslant (n^{-\alpha/2}).$$

By (2.3) and (2.4), we may choose g properly such tha $\|\bar{w}\varphi^{2r}g^{(2r)}\|<\infty$ and

$$\begin{aligned}\omega_\varphi^{2r}(f,n^{-1/2})_{\bar{w}}+\frac{\|\bar{w}f\|}{n^r} &\leqslant \|\bar{w}(\bar{B}_{n,r}(f)-f)\|+\frac{1}{n^r}(\|\bar{w}\varphi^{2r}\bar{B}_{n,r}^{(2r)}(f-g)\|\\ &\quad+\|\bar{w}\varphi^{2r}\bar{B}_{n,r}^{(2r)}(g)\|)+\frac{\|\bar{w}f\|}{n^r}\\ &\leqslant \|\bar{w}(f-\bar{B}_{n,r}(f))\|+\frac{\|\bar{w}f\|}{n^r}+C(\frac{k}{n})^r(\|\bar{w}(f-g)\|\\ &\quad+k^{-r}\|\bar{w}\varphi^{2r}g^{(2r)}\|+k^{-r}\|\bar{w}f\|)\\ &\leqslant \|\bar{w}(f-\bar{B}_{n,r}(f))\|+C(\frac{k}{n})^r(\omega_\varphi^{2r}(f,k^{-1/2})_{\bar{w}}\\ &\quad+k^{-r}\|\bar{w}f\|).\end{aligned}$$

Hence, by [5], we obtain the converse inequality.

References

[1] P.L. Butzer, Linear combinations of Bernstein polynomials, Canad. J. Math. 5 (1953), pp. 559-567.

[2] H. Berens and G. Lorentz, Inverse theorems for Bernstein polynomials, Indiana Univ. Math. J. 21 (1972), pp. 693-708.

[3] D. Della Vechhia, G. Mastroianni and J. Szabados, Weighted approximation of functions with endpoint and inner singularities by Bernstein operators, Acta Math. Hungar. 103 (2004), pp. 19-41.

[4] Z. Ditzian, A global inverse theorem for combinations of Bernstein polynomials, J. Approx. Theory 26 (1979), pp. 277-292.

[5] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer-Verlag, Berlin, New York (1987).

[6] S.S. Guo, C.X. Li and X.W. Liu, Pointwise approximation for linear combinations of Bernstein operators, J. Approx. Theory 107 (2000), pp. 109-120.

[7] S.S. Guo, H. Tong and G. Zhang, Pointwise weighted approximation by Bernstein operators, Acta Math. Hungar. 101 (2003), pp. 293-311.

[8] G.G. Lorentz, Bernstein Polynomial, University of Toronto Press, Toronto (1953).

[9] L.S. Xie, Pointwise simultaneous approximation by combinations of Bernstein operators, J. Approx. Theory 137 (2005), pp. 1-21.

[10] L.S. Xie, The saturation class for linear combinations of Bernstein operators, Arch. Math. 91 (2008), pp. 86-96.

[11] D.S. Yu, Weighted approximation of functions with singularities by combinations of Bernstein operators, J.Applied Mathematics and Computation. 206(2008),pp.906-918.

[12] D.S. Yu and D.J. Zhao, Approximation of functions with singularities by truncated Bernstein operators, Southeast Bull. Math. 30 (2006), pp. 1178-1189.

[13] D.X. Zhou, Rate of convergence for Bernstein operators with Jacobi weights, Acta Math. Sinica 35 (1992), pp. 331-338.

[14] D.X. Zhou, On smoothness characterized by Bernstein type operators, J. Approx. Theory 81 (1994), pp. 303-315.