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The Use of Special Bilinear Functions in Computing Some Special Funtions

Ali M. Awin
Department of Mathematics, University of Tripoli, P.O. Box 13217, Tripoli, Libya.

Abstract

In this paper we give, at the beginning, a very quick review of the subject of bilinear functions in connection with the convolution of two n-tuples vectors(originally named quacroms); then we summarize some of its important applications such as the calculation of the product of two polynomials and hence two integers and their use in representing certain quantities. Then we divert our attention towards the main topic of this paper and use these special bilinear functions in computing special functions as in the case of Bernoulli, Hermite, and Legendre polynomials through simple recurrence relations using related linear equations; a related algorithm will then be discussed. Moreover, the first few polynomials of each fore-mentioned ones are computed.

Keywords:

Vectors; convolution; n-tuples; Bernoulli; Hermite; Legendre; polynomials; quacroms; $Q_{2\times n}(\vec{a},\vec{b})$; algorithm.

1. Introduction

Special bilinear functions(SBF) in connection with the convolution of two vectors $\vec{a}$ and $\vec{b}$ in $\Re$ were introduced and studied $[1]$, the name " quacroms of dimension $2\times n$ " was given to them then. The original applications for them were taking the product of two polynomials or of two integers and where the operation was shown to be more efficient and neater than the traditional way of doing that. More applications were found for them, applications such as using them in representing certain quantities and their applications in solving linear equations which showed to be very useful as we will discuss in this paper(linear quacrom equations LQE)$[2]$. Quacroms of the dimension $3\times n$ were then introduced and discussed $[3]$. In the next section we give some quick details regarding these special bilinear functions(SBF); then we advance to show some important applications .In the section to follow, we describe an algorithm to compute some special functions using SBF followed by practical calculations. Finally we conclude with a short discussion.

2. More Details

Definition 1

Consider a real-valued function f of a pair of n-vectors $\vec{a}$ and $\vec{b}$; where $\vec{a}=(a_1,a_2,.....,a_n)$ and $\vec{b}=(b_1,b_2,....,b_n)$,or of a real-valued matrix $A=\begin{pmatrix} a_1\, a_2 \ldots a_n \\ b_1\, b_2 \ldots b_n \end{pmatrix}$ which has the following properties

(i) $f(k_1\vec{a},k_2\vec{b})=k_1k_2 f(\vec{a},\vec{b});k_i\ ,i=1,2$ are scalars.

(ii) $f(\vec{a},\vec{b}+\vec{c})=f(\vec{a},\vec{b})+f(\vec{a},\vec{c})$ and $f(\vec{a}+\vec{b},\vec{c})=f(\vec{a},\vec{c})+f(\vec{b},\vec{c})$.

(iii) $f(\vec{e}_i,\vec{e}_j)=\delta_{i+j,n+1}$ where $\vec{e}_i$ is the i^{th} unit vector.

Then f is the SBF (or quacrom) of A written as

$$f(\vec{a},\vec{b})=Q_{2\times n}(A)=\left]\begin{matrix}a_1\, a_2 \ldots a_n\\ b_1\, b_2 \ldots b_n\end{matrix}\right[\qquad (1)$$

And by definition $Q_{2\times n}(A)$ is of second degree and n^{th} order, or of dimension $2\times n$.

2.1 Some Properties and Remarks

In this subsection, we present some properties which can be verified using definition 1 [1];

a- $\vec{a}=0 \; or \; \vec{b}=0 \Rightarrow f(\vec{a},\vec{b})=0$

b- $f(\vec{a},\vec{b})=\sum_{i=1}^{n} a_i b_{n-i+1}$. This property shows that $Q_{2\times n}(A)$ equals the convolution of the two vectors $\vec{a}$ and $\vec{b}$ .Moreover this property ascertain that f is well-defined since any function which has the value $\sum_{i=1}^{n} a_i b_{n-i+1}$ satisfies (i)-(iii) of definition 1.

c- $\left]\begin{matrix}a_1\, a_2 \ldots a_n\\ b_1\, b_2 \ldots b_n\end{matrix}\right[= \left]\begin{matrix}b_1\, b_2 \ldots b_n\\ a_1\, a_2 \ldots a_n\end{matrix}\right[$,i.e. rows can be interchanged.

d- If $\vec{a}=k\vec{c}$ then $Q_{2\times n}(\vec{a},\vec{b})=kQ_{2\times n}(\vec{c},\vec{b})$,which means that the scalar can be taken out as a common factor for rows; this is not the case for columns.

e- For any vector $\vec{a}$,define $\bar{a}$ as $\bar{a}=(a_n,a_{n-1},\ldots,a_1)$,then $Q_{2\times n}(\vec{a},\vec{b})=\bar{a}.\vec{b}$; this another definition for $Q_{2\times n}(\vec{a},\vec{b})$.

f- The set $\{Q_{2\times n}\}$ with the binary operation "addition" does not form a group for a fixed n.

g- $\because \begin{vmatrix}a & b\\ c & d\end{vmatrix} = \left]\begin{matrix}a & -b\\ c & d\end{matrix}\right[= \left]\begin{matrix}a & b\\ -c & d\end{matrix}\right[$, $\therefore \; \{all\ determinants\ of\ order\ 2\} \subseteq \{Q_{2\times n}\}$ [1],[4].

h- If $\vec{a}=\vec{a}(t)$ and $\vec{b}=\vec{b}(t)$ then $D_t f(\vec{a},\vec{b})=f(D_t\vec{a},\vec{b})+f(\vec{a},D_t\vec{b})$ provided that $\vec{a}$ and $\vec{b}$ are differentiable with respect to t .

2.2 Sample Applications

1- If $f(x)=\sum_{i=1}^{n} a_i x^{n-i}$ and $g(x)=\sum_{i=1}^{n} b_i x^{n-i}$ are two polynomials, then it clear that their product is given by

$$f(x)g(x)=\left]\begin{matrix}a_1\\ b_1\end{matrix}\right[x^{2n-2} + \left]\begin{matrix}a_1 & a_2\\ b_1 & b_2\end{matrix}\right[x^{2n-3} + \ldots + \left]\begin{matrix}a_1 & \cdots & a_n\\ b_1 & \cdots & b_n\end{matrix}\right[x^{n-1} + \ldots \left]\begin{matrix}a_n\\ b_n\end{matrix}\right[\qquad (2)$$

2- In Equation(2) ,if we put x=10; then we get the product of two integers N_1 and N_2 as

$$N_1 N_2 = \left]\begin{matrix}a_1\\ b_1\end{matrix}\right[10^{2n-2} + \left]\begin{matrix}a_1 & a_2\\ b_1 & b_2\end{matrix}\right[10^{2n-3} + \ldots + \left]\begin{matrix}a_1 & \cdots & a_n\\ b_1 & \cdots & b_n\end{matrix}\right[10^{n-1} + \ldots \left]\begin{matrix}a_n\\ b_n\end{matrix}\right[$$

Where we have to note that an integer N of n digits can be written as $N=\sum_{i=1}^{n} a_i (10)^{n-i} (0 \le a_i \le 9)$

We note here that this method of taking the product is different from the traditional one .It is easier, faster ,and takes place in one line .Moreover it is applicable to all bases.

In practice to calculate any SBF(quacrom) –put in an array form- we imagine that a pair of scissors is opened with angle θ with its two ends joining the first and the n^{th} columns; we multiply crosswise and add, then we start closing the scissors repeating the process of crosswise multiplication whenever the ends meet with any digits until it is completely closed .This is where the word "quacrom" came from.

To clarify the above remarks we give the following example

Example 1.

To evaluate the product 23X34 using the SBF method we proceed as follows;-

Step 1

$$14\times 62=\begin{matrix}1\,4\\ 6\,2\end{matrix}=\left(\left]\begin{matrix}1\\6\end{matrix}\right[\right)\left(\left]\begin{matrix}1 & & 4\\6 & & 2\end{matrix}\right[\right)\left(\left]\begin{matrix}4\\2\end{matrix}\right[\right)$$

Step 2

Now we compute different quacroms of different dimension ,i.e.

$$\left]\begin{matrix}1\\6\end{matrix}\right[=1\times 6=6(\theta=0),\ \left]\begin{matrix}1 & & 4\\6 & & 2\end{matrix}\right[=1\times 2+4\times 6=26(\theta=\pi/2),\ ,\ \left]\begin{matrix}4\\2\end{matrix}\right[=4\times 2=8(\theta=0).$$

Step 3

14x62=(6)(26)(8)=868.

Therefore the product is 782 and we should note that performing the process in the manner we showed is very formal but in practice the process is very quick and the result is given in one line[1].

3- Discrete convolution is defined by the summation $y(kt)=\sum_{i=0}^{N-1} x(it)h[(k-i)t]$ where both $x(kt)$ and $h(kt)$ are periodic functions with a period of N .If we define $\vec{x}=(x(0t),x(t),......,x([N-1]t))$ and $\vec{h}=(h([k-N+1]t),........,h(kt))$,we can easily see that $y(kt)=Q_{2\times n}(\vec{x},\vec{h})$ and hence the $2\times n$ SBF represents the discrete convolution of the two functions $\vec{x}$ and $\vec{h}$ [2].

3. An Algorithm to Compute some Special Functions

Focusing on $Q_{2\times n}(\vec{a},\vec{b})$,we give the following definition

Definition 2.

A linear SBF equation(LQE) is an equation of the form

$$\left]\begin{matrix}a_1 & .. & a_i\,.. & a_j\,.. & a_n\\ b_1 & .. & x\,.. & b_j\,.. & b_n\end{matrix}\right[=c(i+j=n+1) \qquad (3)$$

Where the a's and b's and c are real.

The above equation is equivalent to $ax=b$;however this definition involving SBF will lead to a very important application and actually to an interesting algorithm which will enable us to compute any of Bernoulli ,Hermite ,or Legendre polynomials in a simple and straitforward manner. Note that the calculation for Bernoulli polynomials was, neatly, introduced very recently[4]; and to stress the importance of SBF in this concern,we compute , in this paper, other special functions, namely Hermite and Legendre polynomials.

Now ,equation(3) can be simplified by division by a_j and put in the form

$$\left]\begin{matrix} a_1 & .. & a_i\,.. & 1\,.. & a_n \\ b_1 & .. & x\,.. & b_j\,.. & b_n \end{matrix}\right[= c(i+j=n+1) \qquad (4)$$

Where the new a's and c are the old ones divided by a_j. The solution is clearly given by

$$x = \left]\begin{matrix} -a_1 & .. & -a_i\,.. & 1\,.. & -a_n \\ b_1 & .. & c\,.. & b_j\,.. & b_n \end{matrix}\right[\quad (i+j=n+1) \qquad (5)$$

Example 2.

To solve the LQE $\left]\begin{matrix} 2 & 2 \\ x & 5 \end{matrix}\right[= 2$,we see that $2\left]\begin{matrix} 1 & 1 \\ x & 5 \end{matrix}\right[= 2$ and hence $\left]\begin{matrix} 1 & 1 \\ x & 5 \end{matrix}\right[= 1$ .Therefore $x = \left]\begin{matrix} -1 & 1 \\ 1 & 5 \end{matrix}\right[= -1\times 5 + 1\times 1 = -4$.

3.1 Some Special Functions as Sample Applications

Bernoulli Polynomials

Bernoulli polynomials $B_n(x)$ are generated by [5]

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi) \qquad (6)$$

Equation(6) can be written as

$$\frac{1 + xt + x^2t^2/2! + ...}{1 + t/2! + t^2/3! + ...} = \sum_{i=0}^{k} B_i(x)\frac{t^i}{i!} + t^{k+1}\left(\frac{p(t)}{1 + t/2! + t^2/3! + ...}\right) \qquad (7)$$

Where p(t) contains the rest of the terms in the series. However, the numerator in the right hand side of equation(7) can be rewritten as

$$\left]\begin{matrix} 1 \\ B_0(x) \end{matrix}\right[+ \left]\begin{matrix} 1 & 1/2! \\ B_0(x) & B_1(x)/1! \end{matrix}\right[t + \left]\begin{matrix} 1 & 1/2! & 1/3! \\ B_0(x) & B_1(x)/1! & B_2(x)/2! \end{matrix}\right[t^2 + ...$$
$$+ \left]\begin{matrix} 1 & 1/2! & ...1/(k+1)! \\ B_0(x) & B_1(x)/1! & ...B_k(x)/k! \end{matrix}\right[t^k + .. \qquad (8)$$

Comparing the numerator given by Equation(8) with the numerator in the left hand side of Equation(7), we get

$$\left]\begin{matrix} 1 \\ B_0(x) \end{matrix}\right[= 1 \qquad (9)$$

,

$$\left]\begin{matrix} 1 & 1/2! \\ B_0(x) & B_1(x)/1! \end{matrix}\right[= x \qquad (10)$$

,and

$$\left]\begin{array}{ccc}1 & 1/2! & 1/3! \\ B_0(x) & B_1(x)/1! & B_2(x)/2!\end{array}\right[= x^2/2! \qquad (11)$$

In general for ,any i=n, we have

$$\left]\begin{array}{cccccc}1 & 1/2! & 1/3! & \ldots & 1/n! & 1/(n+1)! \\ B_0(x) & B_1(x)/1! & B_2(x)/2! & \ldots B_{n-1}(x)/(n-1)! & B_n(x)/n! & \end{array}\right[= \frac{x^n}{n!} \qquad (12)$$

We should note that Equation (12) is one of the main cores of our algorithm and the beauty of using LQE technique lies in the linearity of these recurrence relations in the $B_n(x)'s$ .Moreover, the LQE made the process easy to get the various Bernoulli polynomials since from these recurrence relations we can solve for all $B_n(x)\ (n = 0,1,2,3,\ldots.)$ directly without needing any further information .

Later on, we will illustrate this by solving for the first few Bernoulli polynomials and which can be compared with their well-known forms from the literature[5].

Hermite Polynomials

The generating function for Hermite polynomials is given by[6]

$$e^{-t^2+2tx} = \sum_{n=o}^{\infty} \frac{H_n(x)}{n!} t^n \qquad (13)$$

Writing the term e^{-t^2+2tx} as e^{2tx}/e^{t^2} ,then expanding both the numerator and denominator we get

$$\frac{1+2tx+4t^2x^2/2!+\ldots\ldots\ldots}{1+t^2+t^4/2!+\ldots\ldots\ldots\ldots} = \sum_{n=0}^{n=j}(H_n(x)/n!)t^n + t^{j+1}\left(\frac{s(t)}{1+t^2+t^4/2!+\ldots.}\right) \qquad (14)$$

Again, the numerator of the right hand side can be written as

$$\left]\begin{array}{c}1 \\ H_0(x)\end{array}\right[+ \left]\begin{array}{c}1 \\ H_1(x)\end{array}\right[t + \left]\begin{array}{cc}1 & 1 \\ H_0(x) & H_2(x)/2!\end{array}\right[t^2 + . \left]\begin{array}{cc}1 & 1 \\ H_1(x) & H_3(x)/3!\end{array}\right[t^3 + \left]\begin{array}{ccc}1 & 1 & 1/2! \\ H_0(x) & H_2(x)/2! & H_4(x)/4!\end{array}\right[+ \ldots. \qquad (15)$$

Comparing this with the numerator of the left hand side of equation(14), we get

$$\left]\begin{array}{c}1 \\ H_0(x)\end{array}\right[= 1 \qquad (16)$$

,

$$\left]\begin{array}{c}1 \\ H_1(x)\end{array}\right[= 2x \qquad (17)$$

,

$$\left]\begin{array}{cc}1 & 1 \\ H_0(x) & H_2(x)/2!\end{array}\right[= 2x^2 \quad (18)$$

,

$$\begin{vmatrix} 1 & 1 \\ H_1(x) & H_3(x)/3! \end{vmatrix} = 8x^3/3! \tag{19}$$

,

$$\begin{vmatrix} 1 & 1 & 1/2! \\ H_0(x) & H_2(x)/2! & H_4(x)/4! \end{vmatrix} = 16x^4/4! \tag{20}$$

And so on for the rest of the recurrence relations.

Legendre Polynomials

The generating function in this case is[6]

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_0^\infty p_n(t)t^n \, (|t|<1) \tag{21}$$

The calculation here is a little more involved than before where we note that

$$\frac{1}{\sqrt{1-2tx+t^2}} = \frac{\sqrt{1-2tx+t^2}}{1-2tx+t^2} = \sum_0^\infty p_n(t)t^n \, (|t|<1) \tag{22}$$

Hence, using binomial expansion for the term $\sqrt{1-2tx+t^2}$,and following the same technique adopted for the previous polynomials we get the following recurrence relations

$$\begin{vmatrix} 1 \\ p_0(x) \end{vmatrix} = 1 \tag{23}$$

,

$$\begin{vmatrix} 1 & -2x \\ p_0(x) & p_1(x) \end{vmatrix} = -x \tag{24}$$

,

$$\begin{vmatrix} 1 & -2x & 1 \\ p_0(x) & p_1(x) & p_2(x) \end{vmatrix} = \frac{1}{2} - \frac{1}{2}x^2 \tag{25}$$

and so on.

Now we proceed with computing the first few Bernoulli ,Hermite, and Legendre polynomials:

From Equations (5) and (9)-(11) we obtain for the first three Bernoulli polynomials

$$B_0(x) = \begin{vmatrix} 1 \\ 1 \end{vmatrix} = 1 \quad , B_1(x) = 1! \begin{vmatrix} 1 & -1/2! \\ B_0(x) & x \end{vmatrix} = \begin{vmatrix} 1 & -1/2 \\ 1 & x \end{vmatrix} = x - 1/2$$

$$, B_2(x) = 2! \begin{vmatrix} 1 & -1/2! & -1/3! \\ B_0(x) & B_1(x)/1! & x/2! \end{vmatrix} = 2! \begin{vmatrix} 1 & -1/2 & -1/6 \\ 1 & x-1/2 & x^2/2 \end{vmatrix} = x^2 - x + 1/6$$

As for Hermite polynomials we use Equations(16)-(20) to get

$$H_0(x)=\begin{vmatrix}1\\1\end{vmatrix}=1\,,H_1(x)=\begin{vmatrix}1\\2x\end{vmatrix}=2x\,,H_2(x)=2!\begin{vmatrix}1 & -1\\1 & 2x^2\end{vmatrix}=4x^2-2\,,$$

$$H_3(x)=3!\begin{vmatrix}1 & -1\\2x & 8x^3/3!\end{vmatrix}=8x^3-12x\,,H_4(x)=4!\begin{vmatrix}1 & -1 & -1/2!\\1 & (4x^2-2)/2! & 16x^4/4!\end{vmatrix}$$

$$=16x^4-48x^2+12.$$

For Legendre polynomials we use Equations(23)-(25) to get

$$p_0(x)=\begin{vmatrix}1\\1\end{vmatrix}=1\,,p_1(x)=\begin{vmatrix}1 & 2x\\1 & -x\end{vmatrix}=2x-x=x\,,p_2(x)=\begin{vmatrix}1 & 2x & -1\\1 & x & \frac{1}{2}-\frac{1}{2}x^2\end{vmatrix}$$

$$=\frac{1}{2}-\frac{1}{2}x^2-1+2x^2=\frac{3}{2}x^2-\frac{1}{2}\,,p_3(x)=\begin{vmatrix}1 & 2x & -1\\x & \frac{1}{2}(3x^2-1) & \frac{1}{2}x(1-x^2)\end{vmatrix}=\frac{1}{2}(5x^3-3x)$$

Note that one can compare the above results for the three polynomials withthe well-known forms of these polynomials from the literature[5]-[6]

In the following and implementing Equation (12) we proceed to describe our interesting algorithm which can be used to compute Bernoulli polynomials(described before in [4])

Step 1.

Define a function of two variable vectors(an SBF) as in equation (1).

Step 2.

Compute Bernoulli polynomials using the defined function and equations (5) and (12).The steps to get $B_0(x)-B_2(x)$ are to be taken as a guide.

Step 3.

Results are to be compared with the values given in Reference [5].

It is worthwhile to mention that we described the algorithm just for Bernoulli polynomials, but the same procedure can be used for the other two polynomials.

4. Conclusion

As we have seen SBF and LQE have many useful applications some of which are computing product of two polynomials and numbers,their use in expressing various quantities and finally their use in describing a method by which one can compute special functions such as Bernoulli ,Hermite, and Lrgendre polynomials.In fact one expect that such an algorithm can be used to calculate most of other polynomials. Chebyshev Polynomials are ones to be cited here.Moreover, detailed numerical calculations using the above-mentioned algorithm can be made and this work will constitute a part of a future research.

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