SCITECH
RESEARCH ORGANISATION|

Volume 2, Issue 2
March 2, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com

On Indexed Riesz Summability of an Infinite Series

B. P. Padhy[1], S.Nayak[2], U.K.Misra[4].

[1]Department of Mathematics, Roland Institute of Technology, Golanthara-761008, Odisha, India, Email: iraady@gmail.com

[2] Department of Mathematics, J.J.Mahavidyalaya,Kodala,Ganjam, Odisha,India
Email: santoshnayak5151@gmail.com

[3] Department of Mathematics,Binayak Acharya College,Berhampur, Odisha, India
Email: mahendramisra@2007.gmail.com

[4] Department of Mathematics, National Institute of Science and Technology, PallurHills-761008, Odisha, India, Email: umakanta_misra@yahoo.com

Abstract

Generalizing the results of Seyhan and Misra et al. a theorem on indexed-Reisz summability has been established.

Keywords: $\left|\bar{N}, p_n\right|_k$ Summability; $\varphi-\left|\bar{N}, p_n\right|_k$ Summability; $\varphi-\left|\bar{N}, p_n, \delta\right|_k, k \geq 1$ Summability.

2010-Mathematics Subject Classification: 40D25.

1. Introduction

Let $\{s_n\}$ denote the nth partial sum of an infinite series $\sum a_n$ and let $\{p_n\}$ be a sequence of positive real constants such that

(1.1) $$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \text{ for } n = 0,1,2,... \left(P_i = p_i = 0,\ i < 0\right).$$

Then the sequence-to-sequence transformation given by

(1.2) $$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu ,$$

defines the $\left(\bar{N}, p_n\right)$ mean of the sequence $\{s_n\}$.

The series $\sum a_n$ is said to be $\left|\bar{N}, p_n\right|_k, k \geq 1$ summable[1] if

(1.3) $$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_n - T_{n-1}|^k < \infty.,$$

Taking p_n =1 for all n, $\left|\bar{N}, p_n\right|_k$ summability reduces to $|C,1|_k$ summability method.

Further, for a sequence $\{\phi_n\}$ of positive real numbers the series $\sum a_n$ is said to be $\phi - \left|\bar{N}, p_n\right|_k, k \ge 1,$ summable if

(1.4) $$\sum_{n=1}^{\infty}\varphi_n^{k-1}|T_n - T_{n-1}|^k < \infty..$$

Taking $\varphi_n = \frac{P_n}{p_n}$, for all n, $\phi - \left|\bar{N}, p_n\right|_k$ − Summability method reduces to $\left|\bar{N}, p_n\right|_k$ − Summability method. The series $\sum a_n$ is said to be $\varphi - \left|\bar{N}, p_n.\delta\right|_k, k \ge 1, \delta \ge 0,$ summable if

(1.5) $$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|T_n - T_{n-1}|^k < \infty.$$

Taking $\delta = 0$, $\phi - \left|\bar{N}, p_n.\delta\right|_k$ − Summability method reduces to $\phi - \left|\bar{N}, p_n\right|_k$ − Summability method.

2. Known theorems:

Concerning with $|C,1|_k$ − Summability of infinite series $\sum a_n$, in 1957 Flett [2] has established the following result. He proved

Theorem-A:

Let σ_n and τ_n denote the $(C,1)$ mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively that is

(i) $$\sigma_n = \frac{1}{n+1}\sum_{\nu=0}^{n} s_\nu$$

and

(ii) $$\tau_n = \frac{1}{n+1}\sum_{\nu=0}^{n} \nu a_\nu .$$

Then the series $\sum a_n$ is summable $|C,1|_k, k \ge 1$, if and only if

(2.1) $$\sum_{n=1}^{\infty}\frac{1}{n}|\tau_n|^k < \infty.$$

Further in 1995, Seyhan [5] extended the result of Flett to $\phi - |C,1|_k$ − summability by establishing

Theorem-B:

Let σ_n and τ_n be as defined in theorem-A and let $\{\varphi_n\}$ be a sequence of positive real numbers. Then the series $\sum a_n$ is summable $\varphi - |C,1|_k, k \geq 1$, if and only if

(2.2) $$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |\tau_n|^k < \infty.$$

In 2010, Mishra et al. [3] have established a theorem similar to theorem-A, by $|\overline{N}, p_n|_k$ − summability method. They proved:

Theorem-C:

Let $\{t_n\}$ denote the $(\overline{N}, p_n)$-mean of the sequence $\{na_n\}$ and $\{T_n\}$ be the sequence as defined in (1.2), where $\{p_n\}$ be a sequence of positive real constants satisfying the following conditions:

(a) $$np_n = O(P_n)$$

(b) $$P_n = O(np_n)$$

and

(c) $$n|\Delta p_n| = O(p_n)$$

Then $\sum a_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$, if and only if

(2.3) $$\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty.$$

Recently Misra et al [4] established a similar theorem for $\varphi - |\overline{N}, p_n|_k, k \geq 1$, summability method. They prove the following:

Theorem-D:

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences of $(\overline{N}, p_n)$-mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Let $\{\varphi_n\}$, $\{p_n\}$ be the sequences of positive real constants satisfying the following conditions:

(2.4) $$np_n = O(P_n)$$

(2.5) $$P_n = O(np_n)$$

(2.6) $$n|\Delta p_n| = O(p_n)$$

and

(2.7) $$\frac{\phi_n}{n} = O(1).$$

Then the series $\sum a_n$ is summable $\varphi - \left|\overline{N}, p_n\right|_k, k \geq 1$, if and only if

(2.8) $$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left|t_n\right|^k < \infty.$$

In what follows, in this paper we prove a similar theorem on $\varphi - \left|\overline{N}, p_n, \delta\right|_k, k \geq 1, \delta \geq 0$ summabilty method. We prove:

3. Main theorem :

Let $\{\varphi_n\}$, $\{p_n\}$ be the sequences of positive real constants such that

(3.1) $$np_n = O(P_n)$$

(3.2) $$P_n = O(np_n)$$

(3.3) $$n\left|\Delta p_n\right| = O(p_n)$$

(3.4) $\{\phi_n^{\delta k+k-1}\}, k \geq 1, \delta \geq 0$, is monotonically decreasing .

Let $\{T_n\}$ and $\{t_n\}$ denote the sequences of $(\overline{N}, p_n)$-mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Then the series $\sum a_n$ is summable $\phi - \left|\overline{N}, p_n, \delta\right|_k, k \geq 1, \delta \geq 0$, if and only if

(3.5) $$\sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} \left|t_n\right|^k < \infty.$$

4.Required Lemma:

We require the following lemma to prove theorem-3.1.

Lemma-4.1[3]:

Let $\{p_n\}$ be a sequence of positive real constants satisfying (i) and (ii) of Theorem-C , then

(4.1) $$p_{n+1} = O(p_n)$$

and

(4.2) $$p_n = O(p_{n+1})$$

holds good.

5. Proof of the Main Theorem:

Sufficient Part $(\Leftarrow)$:

Since $\{t_n\}$ is the $(\overline{N},\ p_n)$-mean of the sequence $\{na_n\}$, we have

(5.1) $$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \nu a_\nu = \frac{1}{P_n}\sum_{\nu=1}^{n} p_\nu \nu a_\nu$$

Then $$P_n t_n - P_{n-1}t_{n-1} = np_n a_n$$

(5.2) $$\Rightarrow a_n = \frac{P_n t_n - P_{n-1}t_{n-1}}{np_n}$$

Now we have

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{\lambda=0}^{\nu} a_\lambda$$

$$= \frac{1}{P_n}\sum_{\lambda=0}^{n} a_\lambda \sum_{\nu=\lambda}^{n} p_\nu$$

$$= \frac{1}{P_n}\sum_{\lambda=0}^{n} a_\lambda \left(P_n - P_{\lambda-1}\right)$$

$$= \sum_{\lambda=0}^{n} a_\lambda - \frac{1}{P_n}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1}$$

Then

(5.3) $$\nabla T_n = T_n - T_{n-1}$$

$$= \sum_{\lambda=0}^{n} a_\lambda - \frac{1}{P_n}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1} - \sum_{\lambda=0}^{n-1} a_\lambda + \frac{1}{P_{n-1}}\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1}$$

$$= a_n + \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1} - \frac{P_{n-1}a_n}{P_n}$$

$$= a_n\left(1 - \frac{P_{n-1}}{P_n}\right) + \frac{p_n}{P_n P_{n-1}}\sum_{\lambda=1}^{n-1} a_\lambda P_{\lambda-1}$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{\lambda=1}^{n} a_\lambda P_{\lambda-1}$$

Using (5.2) we get

$$\nabla T_n = \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}\frac{P_\nu t_\nu - P_{\nu-1}t_{\nu-1}}{\nu p_\nu}$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \frac{P_{\nu-1} P_\nu t_\nu}{\nu p_\nu} - \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} \frac{P_{\nu-1}^2 t_{\nu-1}}{\nu p_\nu}$$

$$= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_{\nu-1} P_\nu t_\nu}{\nu p_\nu} - \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_\nu^2 t_\nu}{(\nu+1) p_{\nu+1}}$$

$$= \frac{t_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu t_\nu \left(\frac{P_{\nu-1}}{\nu p_\nu} - \frac{P_\nu}{(\nu+1) p_{\nu+1}} \right)$$

$$= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu t_\nu}{\nu} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu^2 t_\nu \left(\frac{1}{\nu p_\nu} - \frac{1}{(\nu+1) p_{\nu+1}} \right)$$

$$= \frac{t_n}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu t_\nu}{\nu} + \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu^2 t_\nu}{\nu(\nu+1) p_{\nu+1}}$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu^2 t_\nu}{\nu} \left(\frac{1}{p_\nu} - \frac{1}{p_{\nu+1}} \right)$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ (Say)}.$$

In order to complete the proof of the sufficient part, by using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \phi_n^{\delta k+k-1} |T_{n,i}|^k < \infty \text{ for } i = 1, 2, 3, 4.$$

Now we have

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |T_{n,1}|^k = \sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_n|^k < \infty. \text{ By (3.5).}$$

Next we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,2}|^k \leq \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \left(\sum_{\nu=1}^{n-1} \frac{P_\nu |t_\nu|}{\nu} \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} |t_\nu| p_\nu \right)^k \quad \text{, using (3.2)}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} p_\nu \right)^{k-1} \left(\sum_{\nu=1}^{n-1} p_\nu |t_\nu|^k \right)$$

Using Holder's inequality

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{p_n}{P_n P_{n-1}} \left(\sum_{\nu=1}^{n-1} |t_\nu|^k p_\nu \right)$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu |t_\nu|^k \sum_{n=\nu+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k}$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu |t_\nu|^k \left(\frac{\varphi_\nu p_\nu}{P_\nu}\right)^{\delta k+k-1} \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} \sum_{n=\nu+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) \text{, using (3.4)}$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu |t_\nu|^k \left(\frac{\varphi_\nu p_\nu}{P_\nu}\right)^{\delta k+k-1} \left(\frac{P_\nu}{p_\nu}\right)^{\delta k} \frac{1}{P_\nu}$$

$$= O(1)\sum_{\nu=1}^{m} \left(\frac{\varphi_\nu p_\nu}{P_\nu}\right)^{\delta k+k-1} \left(\frac{P_\nu}{p_\nu}\right)^{\delta k-1} |t_\nu|^k$$

$$= O(1)\sum_{\nu=1}^{m} \frac{\varphi_\nu^{\delta k+k-1}}{\nu^k} |t_\nu|^k \text{, using (3.2)}$$

$$= O(1) \text{, using (3.5)}$$

Further we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,3}|^k \le \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{\nu=1}^{n-1} \frac{P_\nu^2 |t_\nu|}{\nu(\nu+1) p_{\nu+1}}\right)^k$$

$$\le \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} \frac{P_\nu P_{\nu+1} |t_\nu|}{\nu(\nu+1) p_{\nu+1}}\right)^k$$

$$\le \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} |t_\nu| p_\nu\right)^k \text{, using (3.2)}$$

$$= O(1) \text{ as } m \to \infty \text{, proceeding as above.}$$

Next we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |T_{n,4}|^k \le \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{\nu=1}^{n-1} \frac{P_\nu^2 |t_\nu|}{\nu} \left|\frac{1}{p_\nu} - \frac{1}{p_{\nu+1}}\right|\right)^k$$

$$\le \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} \frac{P_\nu^2 |t_\nu|}{\nu} \left|\frac{p_{\nu+1} - p_\nu}{p_{\nu+1} p_\nu}\right|\right)^k,$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n P_{n-1}^k} \left(\sum_{\nu=1}^{n-1} \nu |\Delta p_\nu| |t_\nu|\right)^k \text{, Using (3.2)}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{p_n}{P_n P_{n-1}^k}\left(\sum_{\nu=1}^{n-1} p_\nu |t_\nu|\right)^k \text{, Using (3.3)}$$

$$= O(1) \ as\ m \to \infty \text{, proceeding as above}$$

This proves the sufficient part of the theorem.

Necessary Part $(\Rightarrow)$:

From (5.3) we have

$$\frac{P_{n-1}P_n}{p_n}\nabla T_n = \sum_{\nu=1}^{n} P_{\nu-1}a_\nu$$

$$a_n = \frac{1}{P_{n-1}}\left[\frac{P_{n-1}P_n}{p_n}\nabla T_n - \frac{P_{n-2}P_{n-1}}{p_{n-1}}\nabla T_{n-1}\right]$$

$$= \frac{P_n}{p_n}\nabla T_n - \frac{P_{n-2}}{p_{n-1}}\nabla T_{n-1}$$

Now

$$t_n = \frac{1}{P_n}\sum_{\nu=1}^{n} p_\nu \nu\, a_\nu$$

$$= \frac{1}{P_n}\sum_{\nu=1}^{n}\left(\nu P_\nu \nabla T_\nu - \frac{\nu\, p_\nu P_{\nu-2}}{p_{\nu-1}}\nabla T_{\nu-1}\right)$$

$$= \frac{1}{P_n}\sum_{\nu=1}^{n}\nu P_\nu \nabla T_\nu - \frac{1}{P_n}\sum_{\nu=0}^{n-1}\frac{(\nu+1)p_{\nu+1}P_{\nu-1}}{p_\nu}\nabla T_\nu$$

$$= n\nabla T_n + \frac{1}{P_n}\sum_{\nu=1}^{n}\nu P_\nu \nabla T_\nu - \frac{1}{P_n}\sum_{\nu=1}^{n-1}\frac{(\nu+1)p_{\nu+1}P_{\nu-1}}{p_\nu}\nabla T_\nu$$

$$= n\nabla T_n + \frac{1}{P_n}\sum_{\nu=1}^{n}\nu \nabla T_\nu (P_\nu - P_{\nu-1})$$

$$+\frac{1}{P_n}\sum_{\nu=1}^{n-1}\nu \nabla T_\nu P_{\nu-1}\left(1-\frac{p_{\nu+1}}{p_\nu}\right)+\frac{1}{P_n}\sum_{\nu=1}^{n-1}P_{\nu-1}\frac{p_{\nu+1}}{p_\nu}\nabla T_\nu$$

$$= t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4} \text{, say.}$$

To complete the necessary part, using Minokowski's inequality , we need to show only

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,i}|^k < \infty. \ for \ i = 1,2,3,4.$$

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,1}|^k = \sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1} n^k}{n^k} |\nabla T_n|^k$$

$$= \sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |\nabla T_n|^k$$

$$= O(1).$$

Further,

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,2}|^k \leq \sum_{n=2}^{m+1} \frac{\varphi_n^{\delta k+k-1}}{n^k} \frac{1}{P_n^k} (\sum_{v=1}^{n-1} v |\nabla T_v| p_v)^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n^k P_{n-1}} (\sum_{v=1}^{n-1} v |\nabla T_v| p_v)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n^k P_{n-1}} (\sum_{v=1}^{n-1} p_v)^{k-1} (\sum_{v=1}^{n-1} v^k |\nabla T_v|^k p_v),$$

Using Holder's inequality

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{p_n}{P_n P_{n-1}} (\sum_{v=1}^{n-1} v^k |\nabla T_v|^k p_v)$$

$$= O(1) \sum_{v=1}^{m} p_v v^k |\nabla T_v|^k \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) \left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1} \left(\frac{P_n}{p_n}\right)^{\delta k}$$

$$= O(1) \sum_{v=1}^{m} p_v v^k |\nabla T_v|^k \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) (\varphi_n)^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} p_v v^k |\nabla T_v|^k \left(\frac{1}{P_v} - \frac{1}{P_{m+1}}\right) (\varphi_v)^{\delta k+k-1} \left(\frac{p_v}{P_v}\right)^{k-1}$$

$$\leq O(1) \sum_{v=1}^{m} \frac{p_v}{P_v} v^k |\nabla T_v|^k (\varphi_v)^{\delta k+k-1} \left(\frac{p_v}{P_v}\right)^{k-1} \quad \text{using (3.4)}$$

$$= O(1)\sum_{\nu=1}^{m} \varphi_\nu^{\delta k+k-1} |\nabla T_\nu|^k$$

$$= O(1) \; as \; m \to \infty$$, using (3.5)

Again

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{\delta k+k-1}}{n^k} |t_{n,3}|^k \le \sum_{n=2}^{m+1} \frac{\varphi_n^{\delta k+k-1}}{n^k} \frac{1}{P_n^k} \left(\sum_{\nu=1}^{n-1} \nu P_{\nu-1} |\nabla T_\nu| \frac{|p_\nu - p_{\nu-1}|}{p_\nu} \right)^k$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{nP_n^k} \left(\sum_{\nu=1}^{n-1} \nu P_{\nu-1} |\nabla T_\nu| \frac{|\Delta p_\nu|}{p_\nu} \right)^k,$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{nP_n^k} \left(\sum_{\nu=1}^{n} P_{\nu-1} |\nabla T_\nu| \right)^k$$, using (3.3)

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{1}{nP_n^k} \left(\sum_{\nu=1}^{n} P_\nu |\nabla T_\nu| \right)^k$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{n^{k-1}}{P_n^k} \left(\sum_{\nu=1}^{n} p_\nu |\nabla T_\nu| \right)^k$$, using (3.2)

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{n^{k-1}}{P_n^k} \left(\sum_{\nu=1}^{n} p_\nu \right)^{k-1} \left(\sum_{\nu=1}^{n} p_\nu |\nabla T_\nu|^k \right)$$

$$= O(1)\sum_{n=2}^{m+1} \left(\frac{\phi_n p_n}{P_n} \right)^{\delta k+k-1} \left(\frac{P_n}{p_n} \right)^{\delta k} \frac{n^{k-1}}{P_n} \left(\sum_{\nu=1}^{n} p_\nu |\nabla T_\nu|^k \right)$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu |\nabla T_\nu|^k \sum_{n=\nu+1}^{m+1} (\phi_n)^{\delta k+k-1} \left(\frac{p_n}{P_n} \right)^{k-1} \frac{n^{k-1}}{P_n}$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu |\nabla T_\nu|^k (\phi_\nu)^{\delta k+k-1} \frac{\nu}{P_\nu}$$ using (3.4)

$$= O(1)\sum_{\nu=1}^{m} \varphi_\nu^{\delta k+k-1} |\nabla T_\nu|^k$$

$$= O(1) \; as \; m \to \infty$$, using (3.5).

Finally,

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{\delta k+k-1}}{n^k}\left|t_{n,4}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}P_{\nu-1}\frac{p_{\nu+1}}{p_\nu}\left|\nabla T_\nu\right|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}P_{\nu-1}\left|\nabla T_\nu\right|\right)^k \text{, by lemma}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{\delta k+k-1}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{1}{nP_n^k}\left(\sum_{\nu=1}^{n-1}P_\nu\left|\nabla T_\nu\right|\right)^k$$

$$= O(1) \text{ as } m \to \infty \text{, proceeding as above.}$$

This completes the proof of the theorem.

References

[1] Bor,H. (1975), A Note on the Summability methods, Math.Proc. of Cambridge Phil. Doc.,Vol 97 (1975),pp:147-149.

[2] Flett,T.M. (1957), On an extension of absolute Summability and some theorem of Littlewood and Pabey, Proc. Lond.Math. Soc. , 7, pp: 113-141.

[3] Misra, U.K., Panda, S.P., and Panda, S.P. (2010), A theorem on $\left|\overline{N}, p_n\right|_k$ summabilty of infinite series, Journal. Of Comp. and Math. Soc., Vol. 1(2), pp: 103-110.

[4] Misra, U.K., Padhy, B.P., and Bisoyi, D. (2014), A theorem on $\varphi - \left|\overline{N}, p_n\right|_k$ summabilty of infinite series, International Journal of Mathematical Sciences and Humanities(International eJournal),c., 112 ,pp: 1209-1220.

[5] Seyhan, H. (1995), The absolute Summability methods, Ph.D Thesis,Kayseri, pp:01-57.