Journal of Progressive Research in Mathematics(JPRM)
ISSN: 2395-0218

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RESEARCH ORGANISATION|

Volume 4, Issue 3
Published online: July 23, 2015|

Journal of Progressive Research in Mathematics
www.scitecresearch.com/journals

Numerical Quenching solutions of Localized Semilinear Parabolic Equation with a Variable Reaction

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Abstract

In this paper, we study the semidiscrete approximation for the following initial-boundary value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + (1-u(0,t))^{-p(x)}, & -l < x < l, \quad t > 0, \\ u(-l,t) = 0, \quad u(l,t) = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0, & -l \leq x \leq l, \end{cases}$$

where $p(x) \in C^0([-l,l])$, symmetric and non decreasing on the interval $(-l,0)$, $\inf_{x\in(-l,0)} p(x) > 1$ and $l = \frac{1}{2}$. We prove, under suitable conditions on $p(x)$ and initial datum, that the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

Keywords: Semidiscretizations; localized semilinear parabolic equation; semidiscrete quenching time; convergence.

1. Introduction

We consider the following initial-boundary value problem

$$u_t(x,t) = u_{xx}(x,t) + (1-u(0,t))^{-p(x)}, \quad -l < x < l, \quad t > 0, \tag{1}$$

$$u(-l,t) = 0, \quad u(l,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) \geq 0, \quad -l \leq x \leq l, \tag{3}$$

where $p(x) \in C^0([-l,l])$, symmetric and nondecreasing on the interval $(-l,0)$, $\inf_{x\in(-l,0)} p(x) > 1$, $l = \frac{1}{2}$ and $u_0(x)$ is a function which is bounded and symmetric. In addition, $u_0(x)$ is nondecreasing on the interval $(-l,0)$ and $u_0''(x) + (1-u_0(0))^{-p(x)} \geq 0$ on $(-l,l)$.

1.1 Definition

We say that the classical solution u of (1)-(3) quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_q)$ but

$$\lim_{t \to T_q} \|u(.,t)\|_\infty = 1,$$

Where $\|u(.,t)\|_\infty = \max_{-l \le x \le l} |u(x,t)|$. The time T_q is called the quenching time of the solution u.

The present problem is connected with the diffusion equation generated by a polarization phenomenon in ionic conductors, local structures in chemical reactions with heterogeneous catalysis or localization phenomenon that arise from more sophisticated modelling of biological systems and chemical reaction diffusion processes in which the reaction takes place only at some local sites (see [4], [14] and the reference therein).

The theoretical analysis of quenching solutions for semilinear parabolic equations has been investigated by many authors (see [3], [6], [7], [8] and the references cited therein). Local in time existence and the uniqueness of a classical solution have been proved. But the problem presented in this article has not been treated. In [8], the authors have considered a semilinear parabolic equation with variable reaction term for the study of the blow-up phenomenon (we say that a solution blows up in a finite time if it attains the value infinity in a finite time). In our case, we remark that, if $p_- > 0$, then the reaction term $f(x, u(x,t)) = (1 - u(0,t))^{-p(x)}$ is continuous in both variables and locally Lipschitz in the second one. We can deduce, the local in time existence and uniqueness of a classical solution for any bounded initial datum (see [9]).

This paper concerns the numerical study of the phenomenon of quenching, using a semidiscrete form of the problem (1)-(3). We obtain some conditions, under which, the solution of a semidiscrete form of (1)-(3) quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time to the theoretical one when the mesh size tends to zero. One may find in [12] and [13], some results concerning the numerical approximations of quenching solutions. A similar study has been undertaken in [1] for the phenomenon of blow-up where the authors have considered the problem (1)-(3) in the case where the reaction term $(1 - u(0,t))^{-p}$ is replaced by $(u(x,t))^p$ with $p > 1$. In the same way in [2] the numerical extinction has been studied using some discrete and semidiscrete schemes (we say that a solution u extincts in a finite time if it reaches the value zero in a finite time).

Our paper is structured as follows. In the next section, we give some lemmas which will be used throughout the paper. In the third section, under some hypotheses, we show that the solution of the

semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we give a result about the convergence of the semidiscrete quenching time to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used throughout the paper. Let us begin with the construction of a semidiscrete scheme. Let I be a positive integer, and consider the grid $x_i = -l + ih$, $0 \le i \le I$, where $h = \frac{2l}{I}$. We approximate the solution u of (1)-(3) by the solution $U_h(t) = (U_0(t), U_1(t), \cdots U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) + (1 - U_k(t))^{-b_i}, \quad 1 \le i \le I-1, \quad t \in (0, T_q^h), \tag{4}$$

$$U_0(t) = 0, \quad U_I(t) = 0, \quad t \in (0, T_q^h), \tag{5}$$

$$U_i(0) = \varphi_i \ge 0, \quad 0 \le i \le I, \tag{6}$$

where b_i is an approximation of $p(x_i)$, $0 \le i \le I$, $b_0 = 0$, $b_I = 0$, $b_i > 0$, $1 \le i \le I-1$ and

$b_{I-i} = b_i$, $1 \le i \le I-1$, $b_{i+1} \ge b_i$, $1 \le i \le k-1$,

k is the integer part of the number $\frac{I}{2}$,

$$\delta^2 U_i(t)=\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{h^2}, \qquad 1\le i\le I-1,$$

$$\varphi_0=0,\ \varphi_I=0,\ \varphi_{I-i}=\varphi_i,\ 0\le i\le I,\ \delta^+\varphi_i>0,\ 0\le i\le k-1,\ \delta^+\varphi_i=\frac{\varphi_{i+1}-\varphi_i}{h}.$$

Here $\left(0,T_q^h\right)$ is the maximal time interval on which $\|U_h(t)\|_\infty<1$ where $\|U_h(t)\|_\infty=\max_{0\le i\le I}|U_i(t)|$. When the time T_q^h is finite, then we say that the solution $U_h(t)$ of (4)-(6) quenches in a finite time, and the time T_q^h is called the quenching time of the solution $U_h(t)$.

The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1 Let $a_h(t)\in C^0\left([0,T),\Re^{I+1}\right)$ and let $V_h(t)\in C^1\left([0,T),\Re^{I+1}\right)$ be such that

$$\frac{dV_i(t)}{dt}-\delta^2V_i(t)+a_k(t)V_k(t)\ge 0, \quad 1\le i\le I-1, \quad t\in(0,T), \tag{7}$$

$$V_0(t)\ge 0, \quad V_I(t)\ge 0, \quad t\in(0,T), \tag{8}$$

$$V_i(0)\ge 0, \quad 0\le i\le I. \tag{9}$$

Then $V_i(t)\ge 0$, $0\le i\le I$, $t\in(0,T)$.

Proof. Let T_0 be any quantity satisfying the following inequality $T_0<T$, and let $m=\min_{0\le i\le I, 0\le t\le T_0}V_i(t)$. Since for $i\in\{0,\cdots,I\}$, $V_i(t)$ is a continuous function on the compact $[0,T_0]$, there exists $t_0\in[0,T_0]$ such that $m=V_{i_0}(t_0)$ for a certain $i_0\in\{0,\cdots,I\}$. We argue by contradiction. Assume that $m<0$. If $i_0=0$ or $i_0=I$, then we have a contradiction because of (8). For $i_0\in\{1,\cdots,I-1\}$, it is not hard to see that

$$\frac{dV_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{V_{i_0}(t_0)-V_{i_0}(t_0-k)}{k}\le 0, \tag{10}$$

$$\delta^2V_{i_0}(t_0)=\frac{V_{i_0+1}(t_0)-2V_{i_0}(t_0)+V_{i_0-1}(t_0)}{h^2}\ge 0. \tag{11}$$

Define the vector $Z_h(t)=e^{\lambda t}V_h(t)$ where λ is large enough that $a_k(t_0)V_k(t_0)-\lambda m<0$. Use (10) and (11) to obtain $\frac{dZ_{i_0}(t_0)}{dt}\le 0$ and $\delta^2Z_{i_0}(t_0)\ge 0$, which implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2Z_{i_0}(t_0)+e^{\lambda t_0}\left(a_k(t_0)V_k(t_0)-\lambda m\right)<0. \tag{12}$$

From (7), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2Z_{i_0}(t_0)+e^{\lambda t_0}\left(a_k(t_0)V_k(t_0)-\lambda m\right)\ge 0.$$

Therefore, we have a contradiction because of (12). This ends the proof.

The next lemma is another form of the maximum principle for semidiscrete equations.

Lemma 2.2 Let $f\in C^1(\Re\times\Re,\Re)$. If $U_h(t),V_h(t)\in C^1\left([0,T),\Re^{I+1}\right)$ are such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_k(t),t) \geq \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_k(t),t),\ 1 \leq i \leq I-1,\ t \in (0,T), \quad (13)$$

$$V_0(t) \geq U_0(t), \quad V_I(t) \geq U_I(t), \quad t \in (0,T), \quad (14)$$

$$V_i(0) \geq U_i(0), \quad 0 \leq i \leq I. \quad (15)$$

Then $V_i(t) \geq U_i(t)$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Consider the vector $Z_h(t) = V_h(t) - U_h(t)$. A direct calculation yields

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + f'(\theta_k(t),t) Z_k(t) \geq 0, \quad 1 \leq i \leq I-1, \quad t \in (0,T),$$

$$Z_0(t) \geq 0, \quad Z_I(t) \geq 0, \quad t \in (0,T),$$

$$Z_i(0) \geq 0, \quad 0 \leq i \leq I,$$

where θ_k is an intermediate value between U_k and V_k and f' is the partial derivative of f with respect to the second variable. By hypothesis $f \in C^1$ then $f'(\theta_k(t),t)$ is bounded on $(0,T)$. Apply Lemma 2.1 to complete the rest of the proof.

The next lemma shows that when *i* is between 1 and *I*-1, then $U_i(t)$ is positive where $U_h(t)$ is the solution of the semidiscrete problem.

Lemma 2.3 Let $U_h(t)$ be the solution of (4)-(6). Then, we have

$$U_i(t) > 0,\ 1 \leq i \leq I-1.$$

Proof. Let $\gamma = \min_{1 \leq i \leq I-1} \varphi_i$ and introduce the vector V_h defined by $V_i = \gamma e^{-\lambda_h t} \sin(i\pi h)$, $0 \leq i \leq I$, where $\lambda_h = \frac{2 - 2\cos(h\pi)}{h^2}$. It is not hard to see that

$$\frac{dU_i(t)}{dt} - \delta^2 U_i(t) \geq \frac{dV_i(t)}{dt} - \delta^2 V_i(t) = 0,\ 1 \leq i \leq I-1, \quad t \in (0,T)$$

$$U_0(t) = V_0(t) = 0, \quad U_I(t) = V_I(t) = 0, \quad t \in (0,T),$$

$$U_i(0) \geq V_i(0), \quad 1 \leq i \leq I-1.$$

We deduce from Lemma 2.2 that $U_i(t) \geq \gamma e^{-\lambda_h t} \sin(i\pi h)$, $0 \leq i \leq I$. This implies that $U_i(t) > 0$, $1 \leq i \leq I-1$, and the proof is complete.

The following lemma reveals that the solution $U_h(t)$ of the semidiscrete problem is symmetric and $\delta^+ U_i(t)$ is positive when *i* is between 1 and *k*-1.

Lemma 2.4 Let U_h be the solution of (4)-(6). Then, we have for $t \in (0, T_q^h)$

$$U_{I-i}(t) = U_i(t),\ 0 \leq i \leq I,\ \ \delta^+ U_i(t) > 0,\ 0 \leq i \leq k-1. \quad (16)$$

Proof. Consider the vector V_h defined as follows $V_i(t) = U_{I-i}(t)$ for $0 \leq i \leq I$. For *i*=0, then we have $V_0(t) = U_{I-0}(t) = U_I(t) = 0$, and *i*=I, then we also have $V_I(t) = U_{I-I}(t) = U_0(t) = 0$. For $i \in \{1, \cdots, I-1\}$, it follows that

$$\frac{dU_{I-i}(t)}{dt}=\delta^2U_{I-i}(t)+(1-U_k(t))^{-b_{I-i}},\quad 1\le i\le I-1,\quad t\in\left(0,T_q^h\right).$$

If we replace $U_{I-i}(t)$ by $V_i(t)$ and use the fact that $b_{I-i}=b_i$, we obtain

$$\frac{dV_i(t)}{dt}-\delta^2V_i(t)=(1-U_k(t))^{-b_i},\quad 1\le i\le I-1,\quad t\in\left(0,T_q^h\right),$$

which implies that $V_h(t)$ is a solution of (4)-(6).

Define the vector $W_h(t)$ such that $W_h(t)=U_h(t)-V_h(t)$. We observe that

$$\frac{dW_i(t)}{dt}-\delta^2W_i(t)=0,\ 1\le i\le I-1,\quad t\in\left(0,T_q^h\right),$$

$$W_0(t)=0,\quad W_I(t)=0,\quad t\in\left(0,T_q^h\right),$$

$$W_i(0)=0,\quad 0\le i\le I.$$

From Lemma 2.1, it follows that

$W_i(t)=0$ for $0\le i\le I$, $t\in\left(0,T_q^h\right)$,

which implies that $V_h(t)=U_h(t)$.

Now, define the vector $Z_h(t)$ such that

$$Z_i(t)=U_{i+1}(t)-U_i(t),\ 0\le i\le k-1,$$

and let t_0 be the first $t>0$ such that $Z_i(t)>0$ for $t\in\left[0,t_0\right)$ but $Z_{i_0}(t_0)=0$. Without loss of the generality, we assume that i_0 is the smallest integer such that $Z_{i_0}(t_0)=0$. If $i_0=0$ then we have $U_1(t_0)=U_0(t_0)=0$, which is a contradiction because from Lemma 2.3. $U_1(t_0)>0$. It is easy to check that

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\le 0,\ \text{ if } 1\le i_0\le k-1. \tag{17}$$

We observe that

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2Z_{i_0}(t_0)=(1-U_k(t_0))^{-b_{i_0+1}}-(1-U_k(t_0))^{-b_{i_0}}.$$

Use the mean value theorem to obtain

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2Z_{i_0}(t_0)=(b_{i_0}-b_{i_0+1})(1-U_k(t_0))^{-l_{i_0}}\ln(1-U_k(t_0)),$$

where l_{i_0} is an intermediate value between the exponents b_{i_0+1} and b_{i_0}. Since $b_{i_0}\le b_{i_0+1}$ and $\ln(1-U_k(t_0))<0$, which implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2Z_{i_0}(t_0)\ge 0.$$

A direct calculation yields

$$\frac{dZ_{i_0}(t_0)}{dt}\ge\frac{U_{i_0+2}(t_0)-U_{i_0-1}(t_0)}{h^2}\ge\frac{U_{i_0+2}(t_0)-U_{i_0+1}(t_0)}{h^2}.$$

For $i_0 \in \{1,\cdots,k-2\}$, we obtain

$$\frac{dZ_{i_0}(t_0)}{dt} \geq \frac{Z_{i_0+1}(t_0)}{h^2} > 0,$$

which is a contradiction because of (17).

If $i_0 = k-1$, use the fact that the semidiscrete solution U_h is symmetric and since k is the integer part of the Number $\frac{I}{2}$, we have either $U_{k+1}(t) = U_{k-1}(t)$ or $U_{k+1}(t) = U_k(t)$. In both cases, we find that

$$\frac{dZ_{k-1}(t_0)}{dt} \geq \frac{U_{k+1}(t_0) - U_k(t_0)}{h^2} = 0.$$

We have a contradiction because of (17) and the proof is complete.

The following lemma is the discrete version of the Green's formula.

Lemma 2.5 Let $U_h, V_h \in \mathfrak{R}^{I+1}$ be two vectors such that $U_0 = 0$, $U_I = 0$, $V_0 = 0$, $V_I = 0$. Then, we have

$$\sum_{i=1}^{I-1} hU_i\delta^2 V_i = \sum_{i=1}^{I-1} hV_i\delta^2 U_i. \tag{18}$$

Proof. A routine calculation yields

$$\sum_{i=1}^{I-1} hU_i\delta^2 V_i = \sum_{i=1}^{I-1} hV_i\delta^2 U_i + \frac{V_I U_{I-1} - U_I V_{I-1} + V_0 U_1 - U_0 V_1}{h},$$

and using the assumptions of the lemma, we obtain the desired result.

Now, let us state a result on the operator δ^2.

Lemma 2.6 Let $U_h \in \mathfrak{R}^{I+1}$ be such that $\|U_h\|_\infty < 1$ and let β be a positive constant. Then, we have

$$\delta^2(1-U_i)^{-\beta} \geq \beta(1-U_i)^{-\beta-1}\delta^2 U_i \text{ for } 1 \leq i \leq I-1.$$

Proof. Using Taylor's expansion, we get

$$\delta^2(1-U_i)^{-\beta} = \beta(1-U_i)^{-\beta-1}\delta^2 U_i + (U_{i+1}-U_i)^2\frac{\beta(\beta+1)}{2h^2}(1-\theta_i)^{-\beta-2} + (U_{i-1}-U_i)^2\frac{\beta(\beta+1)}{2h^2}(1-\eta_i)^{-\beta-2}$$

if $1 \leq i \leq I-1$, where θ_i is an intermediate value between U_i and U_{i+1}, η_i the one between U_i and U_{i-1}. The result follows taking into account the fact that $\|U_h\|_\infty < 1$.

To end this section, let us give another property of the operator δ^2.

Lemma 2.7 Let $U_h, V_h \in \mathfrak{R}^{I=1}$. If

$$\delta^+(U_i)\delta^+(V_i) \geq 0 \text{ and } \delta^-(U_i)\delta^-(V_i) \geq 0,\ 1 \leq i \leq I-1, \tag{19}$$

Then $\delta^2(U_iV_i) \geq U_i\delta^2(V_i) + V_i\delta^2(U_i)$, $1 \leq i \leq I-1$,

where $\delta^+(U_i) = \frac{U_{i+1}-U_i}{h}$ and $\delta^-(U_i) = \frac{U_{i-1}-U_i}{h}$.

Proof. A straightforward computation yields

$$h^2\delta^2(U_iV_i) = U_{i+1}V_{i+1} - 2U_iV_i + U_{i-1}V_{i-1}$$

$$h^2\delta^2(U_iV_i)=(U_{i+1}-U_i)(V_{i+1}-V_i)+V_i(U_{i+1}-U_i)+U_i(V_{i+1}-V_i)+U_iV_i-2U_iV_i$$

$$+(U_{i-1}-U_i)(V_{i-1}-V_i)+V_i(U_{i-1}-U_i)+U_i(V_{i-1}-V_i)+U_iV_i,\ 1\le i\le I-1,$$

which implies that

$$\delta^2(U_iV_i)=U_i\delta^2(V_i)+V_i\delta^2(U_i)+\delta^+(U_i)\delta^+(V_i)+\delta^-(U_i)\delta^-(V_i),\ 1\le i\le I-1.$$

Using (19), we obtain the desired result.

3. Quenching solutions

In this section, we show that under some assumptions, the solution U_h of (4)-(6) quenches in a finite time and estimate its semidiscrete quenching time.

Theorem 3.1 Let U_h be the solution of (4)-(6) and assume that there exists a constant $A\in(0,1]$ such that the initial datum at (6) satisfies

$$\delta^2\varphi_i+(1-\varphi_i)^{-b_i}\ge A\sin(\pi ih)(1-\varphi_i)^{-b_i},\ \ 0\le i\le I, \tag{20}$$

and

$$1-\frac{2\pi^2}{A(b_k+1)}(1-\|\varphi_h\|_\infty)^{b_k+1}>0. \tag{21}$$

Then, the solution U_h quenches in a finite time T_q^h and we have the following estimate

$$T_q^h\le-\frac{1}{\pi^2}\ln\left(1-\frac{2\pi^2}{A(b_k+1)}(1-\|\varphi_h\|_\infty)^{b_k+1}\right).$$

Proof. Let $(0,T_q^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty<1$. To prove the finite time quenching, we consider the function $J_h(t)$ defined as follows

$$J_i(t)=\frac{dU_i(t)}{dt}-C_i(t)(1-U_i(t))^{-b_i},\quad 0\le i\le I,$$

where $C_i(t)=Ae^{-\lambda_h t}\sin(ih\pi)$, with $\lambda_h=\dfrac{2-2\cos(h\pi)}{h^2}$. A straightforward computation reveals that

$$\frac{dJ_i}{dt}-\delta^2J_i=\frac{d}{dt}\left(\frac{dU_i}{dt}-\delta^2U_i\right)-(1-U_i)^{-b_i}\frac{dC_i}{dt}-b_iC_i(1-U_i)^{-b_i-1}\frac{dU_i}{dt}+\delta^2(C_i(1-U_i)^{-b_i}),\ 1\le i\le I-1.$$

From Lemmas 2.6 and 2.7, the last term on the right hand side of the equality $\delta^2(C_i(1-U_i)^{-b_i})$ is bounded from below by $(1-U_i)^{-b_i}\delta^2C_i+b_i(1-U_i)^{-b_i-1}C_i\delta^2U_i$ due to the fact $\delta^+(1-U_i)^{-b_i}\delta^+C_i$ and $\delta^-(1-U_i)^{-b_i}\delta^-C_i$ are nonnegative because the results of Lemma 2.4 hold for $U_h(t)$ and $C_h(t)$. We deduce that

$$\frac{dJ_i}{dt}-\delta^2J_i\ge\frac{d}{dt}\left(\frac{dU_i}{dt}-\delta^2U_i\right)-(1-U_i)^{-b_i}\left(\frac{dC_i}{dt}-\delta^2C_i\right)-b_iC_i(1-U_i)^{-b_i-1}\left(\frac{dU_i}{dt}-\delta^2U_i\right),\ 1\le i\le I-1.$$

Using (4) and the fact that $\dfrac{dC_i}{dt}-\delta^2C_i=0$, we find that

$$\frac{dJ_i}{dt}-\delta^2J_i\ge b_i(1-U_k)^{-b_i-1}\frac{dU_k}{dt}-b_i(1-U_k)^{-b_i-1}C_i(1-U_k)^{-b_i},\quad 1\le i\le I-1,$$

$$\frac{dJ_i}{dt}-\delta^2 J_i \geq b_i(1-U_k)^{-b_i-1}(J_k+C_k(1-U_k)^{-b_k})-b_i(1-U_i)^{-b_i-1}C_i(1-U_k)^{-b_i}, \quad 1\leq i\leq I-1,$$

$$\frac{dJ_i}{dt}-\delta^2 J_i \geq b_i(1-U_k)^{-b_i-1}J_k+b_i(1-U_k)^{-b_i-1}(C_k(1-U_k)^{-b_k}-C_i(1-U_k)^{-b_i}), \quad 1\leq i\leq I-1,$$

From Lemma 2.4, $U_k \geq U_i$. We remark that $C_k \geq C_i$. Since $b_k \geq b_i$ for $1\leq i\leq k-1$. We deduce that

$$\frac{dJ_i}{dt}-\delta^2 J_i \geq b_i(1-U_k)^{-b_i-1}J_k,\ 1\leq i\leq I-1,\ \left(0,T_q^h\right).$$

It is not hard to see that $J_0(t)=0$, $J_I(t)=0$ and the relation (20) implies that $J_h(0)\geq 0$. It follows from Lemma 2.1 that $J_h(t)\geq 0$, which implies that

$$\frac{dU_i(t)}{dt}\geq C_i(t)(1-U_i(t))^{-b_i}, \quad 0\leq i\leq I, \quad t\in(0,T_q^h).$$

Using Taylor's expansion, we find that $\cos(h\pi)\geq 1-\pi^2\frac{h^2}{2}$, which implies that $\lambda_h \leq \pi^2$. Obviously $\sin(kh\pi)\geq\frac{1}{2}$. We deduce that

$$\frac{dU_k}{dt}\geq\frac{A}{2}e^{-\pi^2 t}(1-U_k)^{-b_k}, \quad t\in(0,T_q^h).$$

This inequality can be rewritten as

$$(1-U_k)^{b_k}\,dU_k \geq \frac{A}{2}e^{-\pi^2 t}dt, \quad t\in(0,T_q^h). \tag{22}$$

A simple integration of the inequality (22) over $(0,T_q^h)$ yields

$$\frac{A(1-e^{-\pi^2 T_q^h})}{2\pi^2}\leq\frac{(1-U_k(0))^{b_k+1}}{b_k+1},$$

which implies that

$$e^{-\pi^2 T_q^h}\geq 1-\frac{2\pi^2}{A(b_k+1)}(1-U_k(0))^{b_k+1}.$$

By using the inequality (21), we obtain

$$T_q^h\leq-\frac{1}{\pi^2}\ln(1-\frac{2\pi^2}{A(b_k+1)}(1-\|\varphi_h\|_\infty)^{b_k+1}).$$

We have the desired result.

Remark Therefore by integrating the inequality (22) over interval (t_0,T_q^h), we have

$$T_q^h-t_0\leq-\frac{1}{\pi^2}\ln(1-\frac{2\pi^2}{A(b_k+1)}e^{\pi^2 t_0}(1-\|U_h(t_0)\|_\infty)^{b_k+1}).$$

The Remark 3.1 is crucial to prove the convergence of the semidiscrete quenching time.

4. Convergence of semidiscrete quenching times

In this section, under adequate hypotheses, we show the convergence of the semidiscrete quenching time to the theoretical one when the mesh size goes to zero. We denote by

$u_h(t)=(u(x_0,t),\cdots,u(x_I,t))^T$, $p_h=(p(x_0),\cdots,p(x_I))^T$ and $b_h=(b_0,\cdots,b_I)^T$.

In order to prove this result, firstly, we need the following theorem.

Theorem 4.1 Assume that (1)-(3) has a solution $u\in C^{3,1}([-l,l]\times[0,T-\tau])$ such that $\sup_{t\in[0,T-\tau]}\|u(.,t)\|_\infty=\alpha<1$ with $\tau\in(0,T)$. Suppose that the initial datum at (6) and the exponent at (4) satisfy respectively

$$\|\varphi_h-u_h(0)\|_\infty=o(1) \text{ and } \|b_h-p_h\|_\infty=o(1) \text{ as } h\to 0. \tag{23}$$

Then, for h sufficiently small, the problem (4)-(6) has a unique solution $U_h\in C^1([0,T_q^h],\Re^{I+1})$ such that

$$\max_{0\le t\le T-\tau}\|U_h(t)-u_h(t)\|_\infty=O(\|\varphi_h-u_h(0)\|_\infty+\|b_h-p_h\|_\infty+h) \text{ as } h\to 0.$$

Proof. Let K, L and M be positive constants such that

$$\frac{\|u_{xxx}\|_\infty}{3}\le K,\ b_h(1-\frac{\alpha}{2})^{-b_h-1}\le M,\ |(1-u_h(t))^{-l_h}\ln(1-u_h(t))|\le L. \tag{24}$$

The problem (4)-(6) has for each h, a unique solution $U_h\in C^1([0,T_q^h],\Re^{I+1})$. Let $t(h)\le\min\{T-\tau,T_q^h\}$ be the greatest value of $t>0$. There exists a positive real β (with $\alpha<\beta<1$) such that

$$\|U_h(t)-u_h(t)\|_\infty\le\frac{\beta-\alpha}{2} \text{ for } t\in(0,t(h)). \tag{25}$$

From (23), we deduce that $t(h)>0$ for h sufficiently small. By the triangle inequality, we obtain

$$\|U_h(t)\|_\infty\le\|u(.,t)\|_\infty+\|U_h(t)-u_h(t)\|_\infty \text{ for } t\in(0,t(h)),$$

which implies that

$$\|U_h(t)\|_\infty\le\alpha+\frac{\beta-\alpha}{2}=\frac{\beta+\alpha}{2}<1 \text{ for } t\in(0,t(h)). \tag{26}$$

Let $e_h(t)=U_h(t)-u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t\in(0,t(h))$,

$$\frac{de_i(t)}{dt}-\delta^2e_i(t)=\frac{h}{6}u_{xxx}(\tilde{x}_i,t)-\frac{h}{6}u_{xxx}(\tilde{\tilde{x}}_i,t)+b_i(1-\xi_k)^{-b_i-1}-(b_i-p(x_i))(1-u(x_k,t))^{-l_i}\ln(1-u(x_k,t)),\quad 1\le i\le I-1,$$

where ξ_k is an intermediate value between $U_k(t)$ and $u(x_k,t)$ and l_i the one between the exponents b_i and $p(x_i)$. Using (24) and (26), we arrive at

$$\frac{de_i(t)}{dt}-\delta^2e_i(t)\le M|e_k(t)|+L\|b_h-p_h\|_\infty+Kh,\ 1\le i\le I-1. \tag{27}$$

Let $z_h(t)$ the vector defined by

$$z_i(t)=e^{(M+1)t}(\|\varphi_h-u_h(0)\|_\infty+L\|b_h-p_h\|_\infty+Kh),\ 0\le i\le I.$$

A direct calculation yields

$$\frac{dz_i(t)}{dt} - \delta^2 z_i(t) > M|z_k(t)| + L\|b_h - p_h\|_\infty + Kh,\ 1 \le i \le I-1,\ t \in (0, t(h)),$$

$$z_0(t) > e_0(t),\ z_I(t) > e_I(t),\ t \in (0, t(h)),$$

$$z_i(0) > e_i(0),\ 0 \le i \le I.$$

It follows from Lemma 2.2 that $z_i(t) > e_i(t)$ for $t \in (0, t(h)),\ 0 \le i \le I$. By the same reasoning, we also prove that $z_i(t) > -e_i(t)$ for $t \in (0, t(h)),\ 0 \le i \le I$, which implies that

$$z_i(t) > |e_i(t)|,\ 0 \le i \le I,\ t \in (0, t(h)).$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + L\|b_h - p_h\|_\infty + Kh),\ t \in (0, t(h)).$$

In order to show that $t(h) = \min\{T - \tau, T_q^h\}$, we argue by contradiction. Suppose that $t(h) < \min\{T - \tau, T_q^h\}$. From (25), we obtain

$$\frac{\beta - \alpha}{2} \le \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + L\|b_h - p_h\|_\infty + Kh),\ t \in (0, t(h)). \quad (28)$$

We remark that when h tends to zero, $\frac{\beta - \alpha}{2} \le 0$, which is impossible. Consequently $t(h) = \min\{T - \tau, T_q^h\}$.
Let us show that $t(h) = T - \tau$. Suppose that $t(h) = T_q^h < T - \tau$. Arguing as above, we obtain a contradiction, which leads us to the desired result.

Now, we prove the main result of this section, the convergence of the quenching time.

Theorem 4.2 Suppose that the problem (1)-(3) has a solution u which quenches in a finite time T_q such that $u \in C^{3,1}([-l, l] \times [0, T_q])$ and the initial datum at (6) and the exponent at (4) satisfy the hypothesis (23). Under the assumptions of Theorem 3.1, the problem (4)-(6) has a solution U_h which quenches in a finite time T_q^h and $\lim_{h \to 0} T_q^h = T_q$.

. *Proof*. Let $0 < \varepsilon < \frac{T_q}{2}$. There exists $\gamma = \beta - \alpha$ (with $0 < \alpha < \beta < 1$) such that

$$-\frac{1}{\pi^2}\ln\left(1 - \frac{2\pi^2}{A(b_k + 1)} e^{\pi^2 T_q}(1 - y)^{b_k + 1}\right) \le \frac{\varepsilon}{2} \quad \text{for } y \in [1 - \gamma, 1). \quad (29)$$

Since $\lim_{t \to T_q} \|u(.,t)\|_\infty = 1$, there exist $T_1 < T_q$ and $|T_q - T_1| < \frac{\varepsilon}{2}$ such that $1 > \|u(.,t)\|_\infty \ge 1 - \frac{\gamma}{2}$ for $t \in [T_1, T_q)$. From Theorem 4.1, the problem (4)-(6) has for h sufficiently small, the unique solution $U_h(t)$ such that $\|U_h(t) - u_h(t)\|_\infty < \frac{\gamma}{2}$ for $t \in [0, T_2]$ where $T_2 = \frac{T_1 + T_q}{2}$. Using the triangle inequality, we get

$$\|U_h(t)\|_\infty \ge \|u(.,t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty \ge 1 - \frac{\gamma}{2} - \frac{\gamma}{2} \quad \text{for } t \in [T_1, T_2],$$

which implies that

$$\|U_h(t)\|_\infty \ge 1 - \gamma \quad \text{for } t \in [T_1, T_2].$$

From Theorem 3.1, $U_h(t)$ quenches at time T_q^h. Using inequality (29) and the Remark 3.1, we arrive at

$$\left|T_q^h - T_1\right| \le -\frac{1}{\pi^2}\ln(1-\frac{2\pi^2}{A(b_k+1)}e^{\pi^2 T_1}(1-\|U_h(T_1)\|_\infty)^{b_k+1}) \le \frac{\varepsilon}{2},$$

it follows that

$$\left|T_q^h - T_q\right| \le \left|T_q^h - T_1\right| + \left|T_1 - T_q\right| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This complete the proof.

5. Numerical results

In this section, we present some numerical approximations to the quenching time of the problem (1)-(3) in the case where $u_0(x) = 0$ and

$$p(x) = \begin{cases} 0 \quad if \quad x \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}; \\ 2 + \frac{1}{1+|x|} \quad if \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

Firstly, we consider the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + (1 - U_k^{(n)})^{-b_i}, \ 1 \le i \le I-1,$$

$$U_0^{(n)} = 0, \ U_I^{(n)} = 0,$$

$$U_i^{(0)} = 0, \ 0 \le i \le I,$$

and secondly, we use the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + (1 - U_k^{(n)})^{-b_i}, \ 1 \le i \le I-1,$$

$$U_0^{(n+1)} = 0, \ U_I^{(n+1)} = 0,$$

$$U_i^{(0)} = 0, \ 0 \le i \le I,$$

where $n \ge 0$, $k = \frac{I}{2}$, $b_i = 2 + \frac{1}{1+|x_i|}$ for $1 \le i \le I-1$, $\Delta t_n = h^2(1 - \|U_h^n\|_\infty)^{b_k+1}$, $\Delta t_n^e = \min\left\{\frac{h^2}{2}, \Delta t_n\right\}$

and $T^n = \sum_{j=0}^{n-1} \Delta t_j$. In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. The numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = \left|T^{n+1} - T^n\right| \le 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}$$

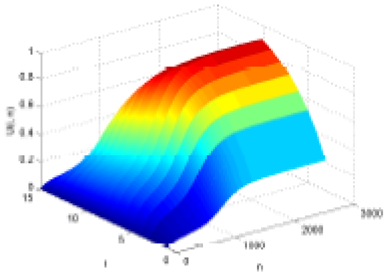
Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.809359	2657	-	-
32	0.810080	10299	-	-
64	0.810269	39820	2	1.93
128	0.810317	153904	9	1.97
256	0.810329	592219	66	2.00

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.821012	2636	-	-
32	0.812994	10194	-	-
64	0.810997	3937	1	2.01
128	0.810499	151904	10	2.00
256	0.810374	585059	74	1.99

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where I=16. In figures 1 and 2 we can appreciate that the discrete solution is nondecreasing and reaches the value one at the middle node. In figures 3 and 4 we see that the approximation of $u(x,T)$ is nondecreasing and reaches the value one at the middle node. Here, T is the quenching time of the solution u. In figures 5 and 6 we observe that the approximation of $\|u(.,t)\|_\infty$ is also nondecreasing and reaches the value one at the time $t \approx 0.81$.



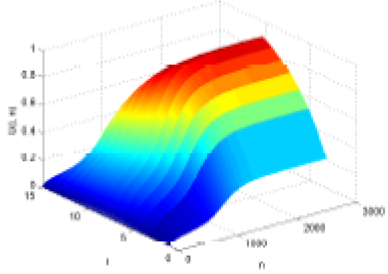
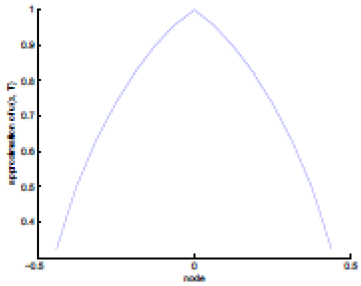


Figure 1: Evolution of the discrete solution(Explicit scheme)

Figure 2: Evolution of the discrete solution(Implicit scheme)



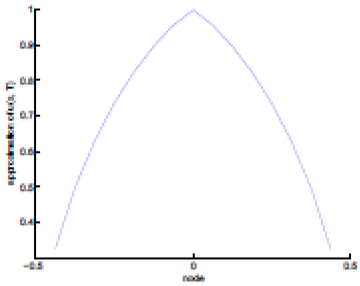
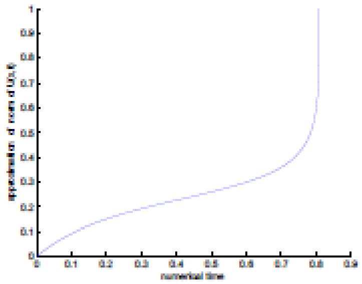


Figure 3: Profil of the approximation of $u(x,T)$ where, T is the quenching time (Explicit scheme)

Figure 4: Profil of the approximation of $u(x,T)$ where, T is the quenching time (Implicit scheme)



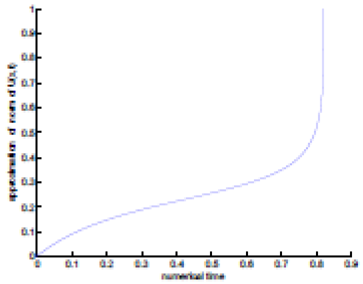
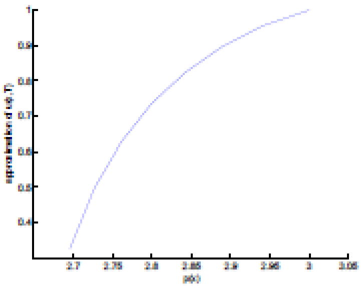


Figure 5: Profil of the approximation of $\|u(.,t)\|_\infty$ (Explicit scheme)

Figure 6: Profil of the approximation of $\|u(.,t)\|_\infty$ (Implicit scheme)



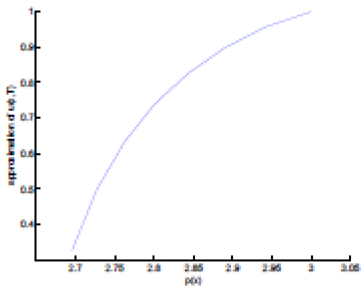


Figure 7: Graph of U against $p(x)$, T is the quenching Time (Explicit scheme)

Figure 8: Graph of U against $p(x)$, T is the quenching Time (Implicit scheme)

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