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Supremum and Infimum Operations on Fuzzy Sets

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Abstract

In this paper we have shown comparison between classical set and fuzzy set. Also we have discussed some basic definitions and properties related to fuzzy set that is used in this paper. Then we collected three theorems by reviewing some papers, which were unproved there and we proved these theorems. We have reviewed some research papers with proper references to do our work.

Keywords: Fuzzy set; Membership function; Supremum; Infimum.

1. Introduction

A well defined collection of objects or elements x out of some universe X, is termed as set A if and only if an element x shows this property that either it belongs to the set A and we can symbolically write $x \in A$, or it is excluded, and we write $x \notin A$. This conception can be described by using a characteristic function 1_A , which is a mapping of the form

$$1_A: X \to \{0,1\}$$

where, $$1_A(x) = \begin{cases} 0; if\ x \notin A \\ 1; if\ x \in A. \end{cases}$$

Against this background, fuzzy sets can be introduced as a generalization of conventional sets by allowing elements of a universe not only to entirely belong or not to belong to a specific set, but also to belong to the set to a certain grade. For the description of fuzzy sets, the characteristic function 1_A of a crisp set A can be generalized to a membership function μ_A for a fuzzy set [01] A, which is a mapping of the form

$$\mu_A: X \to [0,1]$$

represents a fuzzy measure from a set-theoretical point of view. In general, a fuzzy set A can thus be expressed by a set of pairs consisting of the elements x of a universe X and a certain grade of pre-assumed membership $\mu_A(x)$ of the form

$$A = \{(x, \mu_A(x)) | x \in X, \mu_A(x) \in [0,1]\}$$

Thus, a fuzzy set A in X is characterized by its membership function

$$\mu_A: X \to [0,1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

2. Some Basic Definitions and Operations

2.1. Definition [01]**:** Suppose $f, g \in I^X$ are fuzzy sets. Then the ***standard compliment*** of f is another fuzzy set $\overline{f}: X \to I$; defined by,

$$\overline{f}(x) = 1 - f(x), \forall x \in X.$$

2.2. Definition [01]: Suppose $f, g \in I^X$ are two fuzzy sets. Then the ***standard union*** of f and g is also a fuzzy set in I^X denoted by $f \cup g$ and is defined by,

$$(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x);\ \forall x \in X.$$

2.3. Definition [01]: Suppose $f, g \in I^X$ are two fuzzy sets. Then the ***standard intersection*** of f and g is also a fuzzy set in I^X denoted by $f \cap g$ and is defined by**,**

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x); \forall x \in X.$$

2.4. Definition [01]: Suppose $f, g \in I^X$ are fuzzy sets. Then we say that "***f is a subset of g***" denoted by " $f \subset g$" iff $f(x) \leq g(x), \forall x \in X$.

2.5. Definition [05]: Let X and Y be sets, and let $f: X \to Y$ be a function. For a fuzzy set μ in X, the ***image*** of μ under f is the fuzzy set $f(\mu)$ in Y defined, for $y \in Y$, by the rule

$$f(\mu)(y) = \begin{cases} \sup\{\mu(z): z \in f^{-1}(y)\} & if\ f^{-1}(y) \neq \emptyset \\ 0 & if\ f^{-1}(y) = \emptyset. \end{cases}$$

2.6. Definition [05]: Let X and Y be sets, and let $f: X \to Y$ be a function. For a fuzzy set ν in Y, the ***inverse image*** of ν under f is the fuzzy set $f^{-1}(\nu)$ in X by the rule

$$f^{-1}(\nu)(x) = \nu(f(x)) \text{ for } x \in X.$$

(i.e, $f^{-1}(\nu) = \nu \circ f$).

3. Some Basic Theorems

3.1. Theorem [05]: For fuzzy sets μ, μ_1, μ_2 and ν in X:

(i) $\nu \cap (\mu_1 \cup \mu_2) = (\upsilon \cap \mu_1) \cup (\upsilon \cap \mu_2)$
(ii) $\nu \cup (\mu_1 \cap \mu_2) = (\upsilon \cup \mu_1) \cap (\upsilon \cup \mu_2)$
(iii) $1 - (\mu_1 \cup \mu_2) = (1 - \mu_1) \cap (1 - \mu_2)$
(iv) $1 - (\mu_1 \cap \mu_2) = (1 - \mu_1) \cup (1 - \mu_2)$.

Proof:

(i) $\forall\, x \in X;$

$$\begin{aligned} \big(\nu \cap (\mu_1 \cup \mu_2)\big)(x) &= \nu(x) \wedge (\mu_1 \cup \mu_2)(x) \\ &= \nu(x) \wedge \{\mu_1(x) \vee \mu_2(x)\} \\ &= \big(\nu(x) \wedge \mu_1(x)\big) \vee \big(\nu(x) \wedge \mu_2(x)\big) \\ &= \big((\nu \cap \mu_1)(x)\big) \vee \big((\nu \cap \mu_2)(x)\big) \end{aligned}$$

$$= \big((\nu \cap \mu_1) \cup (\nu \cap \mu_2)\big)(x)$$

$$\therefore \nu \cap (\mu_1 \cup \mu_2) = (\upsilon \cap \mu_1) \cup (\upsilon \cap \mu_2).$$

(ii) $$\big(\nu \cup (\mu_1 \cap \mu_2)\big)(x) = \nu(x) \vee (\mu_1 \cap \mu_2)(x)$$

$$= \nu(x) \vee \{\mu_1(x) \wedge \mu_2(x)\}$$

$$= \big(\nu(x) \vee \mu_1(x)\big) \wedge \big(\nu(x) \vee \mu_2(x)\big)$$

$$= \big((\nu \cup \mu_1)(x)\big) \wedge \big((\nu \cup \mu_2)(x)\big)$$

$$= \big((\nu \cup \mu_1) \cap (\nu \cup \mu_2)\big)(x).$$

$$\therefore \nu \cup (\mu_1 \cap \mu_2) = (\upsilon \cup \mu_1) \cap (\upsilon \cup \mu_2).$$

(iii) $$\big(1 - (\mu_1 \cup \mu_2)\big)(x) = 1(x) - (\mu_1 \cup \mu_2)(x)$$

$$= 1 - \vee \{\mu_1(x), \mu_2(x)\}$$

$$= \wedge \left(1 - \{\mu_1(x), \mu_2(x)\}\right)$$

[By De Morgan's law]

$$= \big(1 - \mu_1(x)\big) \wedge \big(1 - \mu_2(x)\big)$$

$$= (1 - \mu_1)(x) \wedge (1 - \mu_2)(x)$$

$$= \big((1 - \mu_1) \cap (1 - \mu_2)\big)(x)$$

$$\therefore 1 - (\mu_1 \cup \mu_2) = (1 - \mu_1) \cap (1 - \mu_2).$$

(iv) $$\big(1 - (\mu_1 \cap \mu_2)\big)(x) = 1(x) - (\mu_1 \cap \mu_2)(x)$$

$$= 1 - \wedge \big(\mu_1(x), \mu_2(x)\big)$$

$$= \vee \Big(1 - \big(\mu_1(x), \mu_2(x)\big)\Big)$$ [By De Morgan's law]

$$= \big(1 - \mu_1(x)\big) \vee \big(1 - \mu_2(x)\big)$$

$$= (1 - \mu_1)(x) \vee (1 - \mu_2)(x)$$

$$= \big((1 - \mu_1) \cup (1 - \mu_2)\big)(x)$$

$$\therefore 1 - (\mu_1 \cap \mu_2) = (1 - \mu_1) \cup (1 - \mu_2).$$

3.2. Theorem: For fuzzy sets $\mu_i (i \in I)$:

(i) $\upsilon \cap (\bigcup_{i \in I} \mu_i) = \bigcup_{i \in I} (\upsilon \cap \mu_i)$,

(ii) $\upsilon \cup (\bigcap_{i\in I} \mu_i) = \bigcap_{i\in I}(\upsilon \cap \mu_i)$,

(iii) $1 - \bigcup_{i\in I} \mu_i = \bigcap_{i\in I}(1 - \mu_i)$,

(iv) $1 - \bigcap_{i\in I} \mu_i = \bigcup_{i\in I}(1 - \mu_i)$.

Proof:

(i) $(\upsilon \cap (\bigcup_{i\in I} \mu_i))(x) = \wedge \{\nu(x), \bigcup_{i\in I} \mu_i(x)\}$

$$= \bigcup_{i\in I}(\nu(x) \wedge \mu_i(x))$$

$$= \bigcup_{i\in I}(\upsilon \cap \mu_i)(x)$$

$$\therefore \upsilon \cap (\bigcup_{i\in I} \mu_i) = \bigcup_{i\in I}(\upsilon \cap \mu_i)$$

(ii) $(\upsilon \cup (\bigcap_{i\in I} \mu_i))(x) = \vee \{\nu(x), \bigcap_{i\in I} \mu_i(x)\}$

$$= \bigcap_{i\in I}(\nu(x) \vee \mu_i(x))$$

$$= \bigcap_{i\in I}(\upsilon \cup \mu_i)(x)$$

$$\therefore \upsilon \cup (\bigcap_{i\in I} \mu_i) = \bigcap_{i\in I}(\upsilon \cup \mu_i).$$

(iii) $(1 - \bigcup_{i\in I} \mu_i)(x) = 1 - \bigcup_{i\in I} \mu_i(x)$

$$= \bigcap_{i\in I}(1 - \mu_i)(x)$$

$$\therefore 1 - \bigcup_{i\in I} \mu_i = \bigcap_{i\in I}(1 - \mu_i).$$

(iv) $(1 - \bigcap_{i\in I} \mu_i)(x) = 1 - \bigcap_{i\in I} \mu_i\,(x)$

$$= \bigcup_{i\in I}(1 - \mu_i)(x)$$

$$\therefore 1 - \bigcap_{i\in I} \mu_i = \bigcup_{i\in I}(1 - \mu_i).$$

3.3. Theorem [06] [07]: Let f be a function from X to Y. If α and $\alpha_i, i \in I$, are fuzzy sets in X and if β and $\beta_j, j \in I$, are fuzzy sets in Y, then the following relations are valid:

(i) $f\left(f^{-1}(\beta)\right) = \beta$ when f is onto Y.
(ii) $f(\cap\, \alpha_i) \subseteq \cap\, f(\alpha_i)$.
(iii) $f^{-1}(\cap\, \beta_i) = \cap\, f^{-1}(\beta_i)$.
(iv) $f(\cup\, \alpha_i) = \cup\, f(\alpha_i)$.
(v) $f^{-1}(\cup\, \beta_i) = \cup\, f^{-1}(\beta_i)$.
(vi) $f(f^{-1}(\beta) \cap \alpha) = \beta \cap f(\alpha)$.

Proof: These results are all consequences of the definition of $f(\alpha)$ and $f^{-1}(\beta)$.

(i) $f\left(f^{-1}(\beta)\right)(y) = \vee \{f^{-1}(\beta)(x): f(x) = y\}$

$$= \vee \left\{\beta\left(f(x)\right): f(x) = y\right\} = \beta(y)$$

$$\therefore f\left(f^{-1}(\beta)\right) = \beta\,.$$

(ii) $f(\cap\, \alpha_i)(y) = \vee_x \cap_i \{\alpha_i(x): f(x) = y\}$

$$\leq \cap_i \vee_x \{\alpha_i(x): f(x) = y\}$$

$$= \cap f(\alpha_i)(y).$$

$$\therefore f(\cap \alpha_i) \subseteq \cap f(\alpha_i).$$

(iii) $$f^{-1}(\cap \beta_i)(x) = \{\cap \beta_i(f(x)): x \in X\}$$

$$= \cap \{\beta_i(f(x)): x \in X\}$$

$$= \cap f^{-1}(\beta_i)(x)$$

$$\therefore f^{-1}(\cap \beta_i) = \cap f^{-1}(\beta_i)\,.$$

(iv) $$f(\cup \alpha_i)(y) = \vee_x \cup_i \{\alpha_i(x): f(x) = y\}$$

$$= \cup_i \vee_x \{\alpha_i(x): f(x) = y\}$$

$$= \cup f(\alpha_i)(y).$$

$$\therefore f(\cup \alpha_i) = \cup f(\alpha_i).$$

(v) $$f^{-1}(\cup \beta_i)(x) = \{\cup \beta_i(f(x)): x \in X\}$$

$$= \cup \{\beta_i(f(x)): x \in X\}$$

$$= \cup f^{-1}(\beta_i)(x)$$

$$\therefore f^{-1}(\cup \beta_i) = \cup f^{-1}(\beta_i).$$

(vi) $$f(f^{-1}(\beta) \cap \alpha)(y) = \vee \{(f^{-1}(\beta) \cap \alpha)(x): f(x) = y\}$$

$$= \vee \{\beta(f(x)) \cap \alpha(x): f(x) = y\}$$

$$= \beta(y) \wedge (\vee \{\alpha(x): f(x) = y\})$$

$$= (\beta \cap f(\alpha))(y).$$

$$\therefore f(f^{-1}(\beta) \cap \alpha) = \beta \cap f(\alpha).$$

Conclusion

Since its inception in 1965, as a generalization of classical set theory, fuzzy set has been applied to many mathematical areas such as algebra, graph theory, control theory, analysis, and operations research, topology and so on. In addition, it has been applied in practice in various disciplines such as control, data processing, engineering, management, stock market, medicine and so on.

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