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Pseudo PQ-injective systems over monoids

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Abstract

The purpose of this paper is to introduce a new kind of generalization of principally quasi injective S-systems over monoids (PQ-injective), (and hence generalized quasi injective), namely pseudo principally quasi injective S-systems over monoids. Several properties of this kind of generalization are discussed. Some of these properties are analogous to that notion of pseudo principally quasi injective class of general modules. Sufficient conditions are given for pseudo principally quasi injective S-systems to be principally quasi-injective and pseudo quasi principally injective S-systems. Characterizations of pseudo principally quasi injective S-systems are considered.

Keywords: Pseudo injective S-systems; Pseudo principally quasi injective S-systems over monoids; principally quasi injective S-systems; fully pseudo stable S-systems; fully stable S-systems.

1- Introduction and Preliminaries :

Throughout this paper , the basic S-system is a unitary right S-system with zero which is consists of a monoid with zero , a non-empty set M_s with a function $f : M \times S \rightarrow M$ such that $f(m,s) \mapsto ms$ and the following properties hold (1) $m \bullet 1 = m$.(2) $m(st) = (ms)t$ (3) $m0 = \Theta$ for all $m \in M$ and $s,t \in S$, where 0,1 is the zero , identity element of S and Θ is the zero element of M . In case a non-empty subset N of an S-system M_s such that $xs \in N$ satisfies for all $x \in N$ and $s \in S$, then N is called a **subsystem** of M_s. Let A_s and B_s be two S-systems . A mapping g: $A_s \rightarrow B_s$,such that $g(as) = g(a)s$ for all $a \in A_s$ and $s \in S$ is called an S-homomorphism [2] . An **S-congruence** ρ on a right S-system M_s is an equivalence relation on M_s such that whenever $(a,b) \in \rho$, then $(as, bs) \in \rho$ for all $s \in S$. The identity S-congruence on M_s will be denoted by I_M such that $(a,b) \in I_M$ if and only if $a = b$ [3] . The congruence ψ_M is called **singular** on M_s and it is defined by a ψ_M b if and only if $ax = bx$ for all x in some $\cap$-large right ideal of S [1] . For S-system M_s , $H \subset S$, $K \subset M\times M$,$T \subset M$, $J \subset S\times S$:(1)$\ell_M(H) = \{ (m,n) \in M \times M \mid mx = nx \text{ for all } x \in H \}(2)\gamma_s(K) = \{ s \in S \mid as = bs ,\text{for all } (a,b) \in K \}(3)\gamma_s(T) = \{(a,b) \in S \times S \mid ta = tb \text{ for all } t \in T \}(4)\ell_M(J) = \{a \in M \mid am = an \text{ for all } (m,n) \in J \}$ [4] .

If an S-system A_s is generated by one element , then it is called principal system and it is denoted by $A_s = \langle u \rangle$, where $u \in A$, then $A_s = uS$ ([5],P.63) .The authors defined that if for every $x \in M_s$, there is an S-homomorphism $f : M_s \rightarrow xS$ such that $x = f(x_1)$ for $x_1 \in M_s$,then an S-system M_s is called **principal self-generator** [6] . An S-system B_s is **a retract** of an S-system A_s if and only if there exists a subsystem W of A_s and epimorphism $f : A_s \rightarrow W$ such that $B_s \cong W$ and $f(w) = w$ for every $w \in W$ ([5],P.84) . An S-homomorphism f which maps an S-system M_s into an S-system N_s is said to be **split** if there exists S-homomorphism g which maps N_s into M_s such that $fg=1_N$ [3] .

Let M_s , N_s be a right S-systems . An S-system E is called **injective** if for every S-monomorphism $f : M_s \rightarrow N_s$ and every S-homomorphism $g : M_s \rightarrow E$, there is an S-homomorphism $h : N_s \rightarrow E$ such that $hf = g$ [10] . A right S-systems N_s is called **M-injective** if for each S-monomorphism f from S-system B_s into S-system M_s and every homomorphism $g : B_s \rightarrow N_s$, there is S-homomorphism

$h : M_s \rightarrow N_s$ such that $hf = g$. Thus N_s is **injective** if and only if N_s is M-injective for all S-system M_s [14] .

In [10], P.Berthiaume had studied injective S-systems. Then the concept of injectivity on S-systems is generalized to quasi injectivity by A.M.lopez, such that an S-system N_s is **quasi injective** if N_s is N-injective [1]. Also, in [13], T.Yan introduced the concept of pseudo injectivity as a generalization of quasi injectivity. An S-system M_s is called **pseudo-injective** if each S-monomorphism of a subsystem of M_s into M_s extends to an S-endomorphism of M_s . It is well known from above that every quasi injective S-system is pseudo injective, but the converse is not true in general and we gave an example which illustrated this fact.

At the same time, for another generalization of injectivity, we have : An S-system M_s is called **principal injective** system (**C-injective**) if for any S-system B_s , any principal subsystem C of B_s and any homomorphism f from C into M_s can be extended to S-homomorphism g from B_s into M_s [9] . As a proper generalization of quasi injective S-system , we introduced principally quasi injective S-system and some definitions relevant to our work . An S-system M_s is called **principally quasi injective** system (this means PQ-injective) if M_s is **PM-injective** [6].

The present work consists of two sections. The first one is devoted to introduce and investigate a new kind of generalization of principally quasi injective S-systems, namely pseudo principally quasi injective S-systems over monoids. Certain class of subsystems which inherit the property of pseudo principally quasi injective have been considered. Also, characterizations of this new class of S-systems was investigated . Example is given to illustrate that pseudo PQ-injective S-systems are not PQ-injective. Some known results on pseudo PQ-injective for general modules were generalized to S-systems. In the second section , we try to put some light on relation of pseudo PQ-injective S-systems with other classes of injectivity such as PQ-injective by using the concepts of fully stable, fully pseudo stable and pseudo M_s-projective and then we find conditions to versus pseudo PQ-injective S-systems with PQ-injective and pseudo QP-injective S-systems.

2-Pseudo Principally quasi Injective S-Systems:

(2-1)Definition: An S-system N_s is called **pseudo principally M-injective**(for short pseudo PM-injective) if for each S-monomorphism from a principal subsystem of an S-system M_s into N_s can be extended to S-homomorphism from M_s into N_s . An S-system M_s is called **pseudo principally quasi injective** if it is pseudo principally M-injective (if this is the case , we write M_s is pseudo PQ-injective).

(2-2) Remark and Example:

(1) Every PQ-injective (and hence quasi injective) S-system is pseudo PQ-injective . But the converse is not true in general , for example , let S be the monoid {1,a,b,0} with ab = a^2 = a and ba= b^2 =b . Now , consider S as a right S-system over itself , then the only non-trivial principal subsystems of S_s are aS ={a,0} and bS = {b,0}. It is easy to check that S_s is pseudo PQ- injective. But, when we take N={a,0} be principal subsystem of S_s and f be S-homomorphism defined by $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } x = a \end{cases}$, then this S-homomorphism cannot be extended to S-homomorphism g : $S_s \rightarrow S_s$. If not, that is there exists S-homomorphism g:$S_s \rightarrow S_s$ such that g(x) = f(x) ,∀ x ∈N , which is the trivial S-homomorphism(or zero homomorphism) ,since other extension is not S-homomorphism . Then , b = f(a) = g(a) = a(0) which implies that b = a(0) , and this is a contradiction .

(2) Retract of pseudo PQ-injective system is pseudo PM-injective.

Proof: Let M_s be pseudo PQ-injective S-system and N be a retract cyclic subsystem of M_s . Let A be principal subsystem of M_s and f : A → N be S-monomorphism. Define $\alpha(=j_N of)$: A → M_s, where j_N is the injection map of N into M_s , so α is S-monomorphism. Since M_s is pseudo PQ-injective system , so there exists S-homomorphism β : $M_s \rightarrow M_s$ such that $\beta o i_A = \alpha$, where i_A be the inclusion map of A into M_s. Now, let π_N be the projection map of M_s onto N. Then, define $\sigma(=\pi_N \beta)$:M_s → N . Thus we have that $\sigma o i_A = \pi_N o \beta o i_A = \pi_N o \alpha = \pi_N o j_N o f$ =f. Therefore , an S-homomophism σ is extends f and N is pseudo PM-injective S-system.

(2-3) Lemma: Every pseudo PM-injective subsystem of S-system M_s is a retract of M_s.

Proof: Let α be S-monomorphism from a principal subsystem N of S-system M_s into M_s and I_N be the identity map of N . Then, pseudo PM-injectivity of N implies that there exists S-homomorphism g : M_s → N such that $I_N = go\alpha$, hence α is a retraction. Therefore N ≅ α(N) is a retract of M_s.

(2-4)Proposition: Let M_s be S-system . If N_s is pseudo PM-injective , then N_s is pseudo PA-injective system for anyprincipal subsystem A of M_s.

Proof: Let X be principal subsystem of principal subsystem A of M_s , and let f be any S-monomorphism of X into S-system N_s . Let $i_X(i_A)$ be the inclusion map of X(A) into A (M_s) respectively . Since N_s is pseudo PM-injective, then there exists S-homomorphism g : $M_s \rightarrow N_s$ such that $goi_A o i_X$ = f . Define S-homomorphism h by $h(=goi_A)$: A → N , then,∀ x ∈ A we have $h(x) = h(i_X(x)) = (goi_A)(i_X(x)) =(goi_A o i_X)(x) = f(x)$, which implies that h extends f and N_s is pseudo PA-injective system.

(2-5) Theorem: Let M_1 and M_2 be two S-systems . If $M_1 \oplus M_2$ is pseudo PQ-injective .Then M_i isPM_j- injective (where i ,j = 1,2).

Proof: Let $M_1 \oplus M_2$ be pseudo PQ-injective . Let A be principal subsystem of M_2 , and f an S-homomorphism from A into M_1 . let j_1 and π_1 be the injection (and projection) map of M_1 into $M_1 \oplus M_2$(and $M_1 \oplus M_2$ onto M_1) . Define $\alpha : A \to M_1 \oplus M_2$ by $\alpha(a) = (f(a),a)$, $\forall a \in A$. It is clear that α is S-monomorphism. Since $M_1 \oplus M_2$ is pseudo PQ-injective , so by proposition (2-4) , $M_1 \oplus M_2$ is pseudo PM_2-injective . Hence , there exists S-homomorphism g from M_2 into $M_1 \oplus M_2$ such that goi = α . Now, put $h(=\pi_1 og) : M_2 \to M_1$ .Thus $\forall$ a ∈ A , we have hoi(a) $=\pi_1 ogoi(a) = \pi_1 o\alpha(a) = \pi_1(\alpha(a)) = \pi_1(f(a),a) = f(a)$. This means M_1 is PM_2-injectiveS-system.

(2-6) Corollary: Let $\{M_i\}_{i\in I}$ be a family of S-systems , where I is a finite index set. If $\oplus_{i\in I} M_i$ is pseudo PQ-injective , then M_j is pseudo PM_K-injective system for all j,k ∈ I.

(2-7) Lemma: Let $\{N_i\}_{i\in I}$ be a family of S-systems , where I is a finite index set. Then , the direct product $\Pi_{i\in I} N_i$ is PM-injective if and only if N_i is PM-injective for every i ∈ I.

Proof :⇒) Assume that $N_s = \Pi_{i\in I} N_i$is PM-injective S-system. Let X be principal subsystem of M_s, f an S-homomorphism of X into N_i , and φ_i , π_i be the injection and projection map of N_i into N_s and $N_s$onto N_i respectively . Since N_s is PM-injective , so there exists S-homomorphism $g : M_s \to N_s$ such that goi = $\varphi_i$of , where i be the inclusion map of X into M_s . Then , define $h(=\pi_i og)$: $M_s \to N_i$ such that hoi = $\pi_i$ogoi = $\pi_i o\varphi_i$of = f . Thus N_i is PM-injective S-system.

⇐) Assume that N_i is PM-injective for each i ∈ I . Let X be principal subsystem of M_s,f an S-homomorphism of X into N_s andφ_i, π_i be the injection and projection maps of N_i into N_s and $N_s$onto N_i respectively . Since N_i is PM-injective S-system, so there exists S-homomorphism $\beta_i : M_s \to N_i$ such that $\beta_i$oi = $\pi_i$of , where i be the inclusion map of X into M_s . Now, define an S-homomorphism $\beta(=\varphi_i o\beta_i) : M_s \to N_s$, then βoi= $\varphi_i o\beta_i$oi = $\varphi_i o\pi_i$of = f . Therefore, N_s is PM-injective system.

(2-8) Corollary: For any integer n ≥2 , $M_s^n$is pseudo PQ-injective if and only if M_s is PQ-injective system.

Let M_s be S-system . For all element m ∈ M_s , with α∈ T=End(M_s) ,define :

$A_m = \{ n \in M_s \mid \gamma_s(n) = \gamma_s(m)\}$;

$S_{(\alpha,m)} = \{ \beta \in T \mid \ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)\}$;

$B_m = \{\alpha \in T \mid \ker\alpha \cap (mS \times mS) = I_{mS}$.

(2-9) Proposition: Let M_s be an S-system with T=End(M_s) , the following conditions are equivalent for an element m∈ M_s :

(1) M_s is pseudo principally injective (pseudo PM-injective) ,

(2) $A_m = B_m \bullet m$,

(3) If $A_m = A_n$, then $B_m \bullet m = B_n \bullet n$,

(4) For every S-monomorphism $\alpha : mS \to M_s$ and $\beta : mS \to M_s$, there existsσ ∈ T such that α = σoβ .

Proof: (1→2) Let $n \in A_m$, this implies $A_m = A_n$, hence $\alpha : mS \to M_s$ defined by α(ms) = ns , s ∈ S . Let $ms_1 = ms_2$, this implies $(s_1 , s_2) \in \gamma_s(m) = \gamma_s(n)$, then $ns_1 = ns_2$. Hence , $\alpha(ms_1) = \alpha(ms_1)$ and α is well-defined and for the reverse steps , we obtain that α is S-monomorphism, so by (1) , there exists an S-homomorphism β ∈ T extends α . Then , $\forall$ m ∈ M_s, we have β(m) = α(m) = n = β • m , so $\beta \in B_m$ [In fact , if (ms , mt) ∈ $\ker\beta \cap (mS \times mS)$,then β(ms) = β(mt) and ms = mt . So , $\ker\beta \cap (mS \times mS) = I_{mS}$]. Conversely , if $\beta \bullet m \in B_m \bullet m$, then $\beta \in B_m$, that is $\ker\beta \cap (mS \times mS) = I_{mS}$. It is obvious that $\gamma_s(m) \subseteq \gamma_s(\beta m)$, since for $(r,s) \in \gamma_s(m)$, we have mr = ms , since β is well-defined , so β(mr) = β(ms) . Thus , β(m)r = β(m)s which implies that (r,s) ∈ $\gamma_s(\beta m)$. Now, if β(mr) = β(ms) and (mr,ms) ∈ $\ker\beta \cap (mS \times mS) = I_{mS}$, then mr =ms and (r,s) ∈ $\gamma_s(m)$. Hence , $\gamma_s(\beta m) \subseteq \gamma_s(m)$. Then , $\gamma_s(\beta m) = \gamma_s(m)$. Therefore, βm ∈ A_m .

(2→3) Let $A_m = A_n$. Then , $A_m = B_m \bullet m$, $A_n = B_n \bullet n$. So , $B_m \bullet m = B_n \bullet n$.

(3→4) Let $\alpha : mS \to M_s$, $\beta : mS \to M_s$ be S-monomorphisms . Then , $\gamma_s(\beta m) = \gamma_s(\alpha m)$. Since , for $(s,t) \in \gamma_s(\beta m)$, then β(ms) = β(mt) . Since β is monomorphism, so ms = mt . Since α is well-defined , so α(ms) = α(mt) . This means $\gamma_s(\beta m) \subseteq \gamma_s(\alpha m)$. In similar way, we can find $\gamma_s(\alpha m) \subseteq \gamma_s(\beta m)$, thus $\gamma_s(\beta m) = \gamma_s(\alpha m)$, which implies $A_{\alpha m} = A_{\beta m}$,then by(3) $B_{\alpha m}\alpha m = B_{\beta m}\beta m$. Since $\ker I_M \cap \left(\alpha(mS) \times \alpha(mS)\right) = I_{\alpha(mS)}$, so $1_M \in B_{\alpha m}$. Then $\alpha m \in B_{\beta m}\beta m$, so there exists $\sigma \in B_{\beta m}$ such that α = σβ.

(4→1) Let $\beta = i_{mS}$ be the inclusion map of mS .

(2-10) Proposition : Let M_s be pseudo principally injective S-system with $T = End(M_s)$ .Then , for $\alpha \in T$, we have :$S_{(\alpha,m)} = B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$, $\forall m \in M_s$.

Proof : Let $\beta \in S_{(\alpha,m)}$, this means $\beta \in T$ and $\ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$. We claim that $\gamma_s(\alpha m) = \gamma_s(\beta m)$. In fact , if $(s,t) \in \gamma_s(\alpha m)$, then $\alpha(ms) = \alpha(mt)$ which implies $(ms, mt) \in \ker\alpha \cap (mS \times mS)$ and since$\ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$ by the proof . So , $(ms, mt) \in \ker\beta \cap (mS \times mS)$ which implies $\beta(ms) = \beta(mt)$ and then $\beta(m)s = \beta(m)t$. Thus $(s,t) \in \gamma_s(\beta m)$ .Hence , $\gamma_s(\alpha m) \subseteq \gamma_s(\beta m)$, similarly we have $\gamma_s(\beta m) \subseteq \gamma_s(\alpha m)$ and then we obtain $\gamma_s(\alpha m) = \gamma_s(\beta m)$. Then , we have $\beta \in A_{\alpha m}$. Since $A_{\alpha m} = B_{\alpha m}\alpha m$ (by proposition (2-9)) , so $\beta \in B_{\alpha m}\alpha m$ and since $\beta(ms) = \beta(mt)$, where $\beta \in T$, thus $\beta \in \ell_T(mS \times mS)$and then $\beta \in B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$. This means $S_{(\alpha,m)} \subseteq B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$ …(1) . Conversely , let $\beta \in B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$, so $\beta \in B_{\alpha m}\alpha$ or $\beta \in \ell_T(mS \times mS)$. If $\beta \in \ell_T(mS \times mS)$, so $\beta \in T$and $\beta(ms) = \beta(mt)$. If $\beta \in B_\alpha\alpha$, so there exists $\varphi \in B_\alpha$ such that $\beta = \varphi o \alpha$. Also , $\ker\varphi \cap (\alpha(mS) \times \alpha(mS)) =$ and $\ker\beta \cap (\alpha(mS) \times \alpha(mS)) = I_{\alpha(mS)}$. Now, if$(ms, mt) \in \ker\varphi\alpha \cap (mS \times mS)$, then $\varphi\alpha(ms) = \varphi\alpha(mt)$. Hence $(\alpha(ms), \alpha(mt)) \in \ker\varphi \cap (\alpha(mS) \times \alpha(mS)) = I_\alpha$. This implies that $(ms, mt) \in \ker\alpha \cap (mS \times mS)$. Thus, $\ker\beta \cap (mS \times mS) \subseteq \ker\alpha \cap (mS \times mS)$ (1). If $(ms, mt) \in \ker\alpha \cap (mS \times mS)$, so $\alpha(ms) = \alpha(mt)$, since $\varphi \in T$ and it is well-defined , so $\varphi\alpha(ms) = \varphi\alpha(mt)$ which implies $\beta(ms) = \beta(mt)$ and then $(ms, mt) \in \ker\beta \cap (mS \times mS)$.Thus,$\ker\alpha \cap (mS \times mS) \subseteq \ker\beta \cap (mS \times mS)$…(2) .From (1) and (2) , we have $\ker\alpha \cap (mS \times mS) = \ker\beta \cap (mS \times mS)$ and then $\beta \in S_{(\alpha,m)}$.

(2-11) Proposition: Let M_s be pseudo principally injective S-systemwith $T = End(M_s)$ and $\alpha \in T$, $m \in M_s$. Then:

$\alpha \in B_m$ if and only if $B_m = B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$.

Proof :⇒) Let $\alpha \in B_m$ and $f \in S_{(\alpha,m)}$, so $\ker f \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$, but $\ker\alpha \cap (mS \times mS) = i_{mS}$, hence $\ker f \cap (mS \times mS) = i_{mS}$, which implies $f \in B_m$ .Thus , $S_{(\alpha,m)} = B_m$, so by proposition (2-10)$B_m = B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$

⇐) Assume that $B_m = B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$ and $\alpha \in T$, $\alpha \notin B_m$. Then , we have $\ker\alpha \cap (mS \times mS) \neq I_{mS}$, so there exists $(ms, mt) \in \ker\alpha \cap (mS \times mS)$withms $\neq$ mt , then $\alpha(ms) = \alpha(mt)$. Since$1_M \in B_m$, so $\ker I_M \cap (mS \times mS) = I_{mS}$. But , since $S_{(\alpha,m)} = B_m = B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$, hence $I_M \in S_{(\alpha,m)}$, and then $\ker\alpha \cap (mS \times mS) = \ker I_M \cap (mS \times mS)$. Thus , $\ker\alpha \cap (mS \times mS) = I_{mS}$ which implies ms = mt and this is a contradiction with ms ≠ mt . So $\alpha \in B_m$ implies a contradiction.

Recall that **$Soc_N(M_s)$** represent **homogeneous component** of $Soc(M_s)$ containing N. Thus, we denote $Soc_N(M_s) := \cup \{X \text{ be subsystem of } M_s \mid X \cong N\}$[6].

(2-12) Proposition: Let M_s be pseudo principally injective S-system with T= End(M_s). Then :

(1) If N is a simple subsystem of M_s , then $Soc_N(M_s) = TN$.

(2) If nS is a simple S-system , $n \in M_s$. Then , Tn is a simple T- system .

(3) $Soc(M_s) = Soc({}_TM)$.

Proof :(1) Let $N_1 \subseteq Soc_N(M_s)$, and $f : N \to N_1$ be an isomorphism , where $N_1 \subseteq M_s$. If N = nS , then $\gamma_s(n) = \gamma_s(f(n))$. Since , if $(s,t) \in \gamma_s(n)$, then ns = nt , since f is well-defined , so f(ns) = f(nt) . This implies f(n)s = f(n)t and $(s,t) \in \gamma_s(f(n))$, so $\gamma_s(n) \subseteq \gamma_s(f(n))$. Conversely , let $(s,t) \in \gamma_s(f(n))$, so f(ns) = f(nt) . Since f is monomorphism , so ns = nt . This implies that $(s,t) \in \gamma_s(n)$, so $\gamma_s(f(n)) \subseteq \gamma_s(n)$. Thus $\gamma_s(f(n)) = \gamma_s(n)$, which implies $B_n \bullet n = B_{fn} \bullet fn$ by proposition(2-9) . Thus fn$\in B_n \bullet n \subseteq Tn \subseteq TN$. Hence, if g is an extension of f to T , we have $N_1 = f(nS) = g(nS) \in T$. Thus $Soc_N(M_s) \subseteq TN$. The other inclusion always holds , this means $TN \subseteq Soc_N(M_s)$, since for $\alpha \in TN$, α:N →N be identity map and since N ≅ N and N be subsystem of M_s , so $\alpha(N) = N \subseteq Soc_N(M_s)$ which implies $TN \subseteq Soc_N(M_s)$.Therefore , $Soc_N(M_s) = TN$.

(2) Let $\alpha \in T$, $\alpha : M_s \to M_s$, since M_s is pseudo principally injective , so α_1 $(=\alpha_{|nS})$: $nS \to M_s$ is S-monomorphism . Since nS is simple subsystem of M_s , so $\alpha_1 : nS \to \alpha_1(nS)$ is an S-isomorphism . Thus , let $\sigma : \alpha_1(nS) \to nS$ be its inverse . For $\Theta \neq \alpha n \in Tn$ and if $g \in T$ extends σ , then $g(\alpha_1(n)) = \sigma(\alpha_1(n)) = n \in T\alpha n$. Therefore , Tn$\subseteq$ Tαn . Then , Tn= Tαn whence Tαn$\subseteq$Tn , such that if we take $\beta\alpha n \in T\alpha n$, and $\beta \in T$, then ,since $\beta \in T$ and $\alpha \in T$,so $\beta\alpha \in T$. Thus , $\beta\alpha n \in Tn$ and Tαn ⊆ Tn.

(3) This follows by (2).

(2-13) Proposition : Let M_s be pseudo principally injective S-system with $T = End(M_s)$. Then:

(1) If N and K are isomorphic principal subsystem ofM_s and K is a retract of M_s , then N is also a retract of M_s .

(2) Every pseudo principally injective has C_2 –condition

Proof: It is obvious that (1) implies (2) , so it is enough to prove (1) . Let N be a subsystem of M_s and i be the inclusion map of N into M_s. It is enough to prove that inclusion map split . Let $\alpha : N \to K$ be an S-isomorphism . Since K is a retract of M_s , so there exists S-homomorphims $\pi : M_s \to K$ and $j : K \to M_s$ projection and injection map respectively . Let i_1 be the inclusion map of N into M_s and α^{-1} be the inverse map of α (since α is S-isomorphism) .Since M_s is pseudo principally injective ,so there exists S-homomorphism $\overline{\alpha}:M_s \to M_s$ which is extension of α(this means $\overline{\alpha} o i = j o \alpha$).Now , define $\sigma(=\alpha^{-1}\pi\overline{\alpha}) : M_s \to N$. If n∈ N , write α(n) = k ∈ K , hence $\sigma n = \alpha^{-1}(\pi\overline{\alpha}(n)) \in N$, then $\sigma n = \alpha^{-1}(\pi\overline{\alpha}(n)) = \alpha^{-1}(\pi\alpha(n)) = \alpha^{-1}(\pi(k)) = \alpha^{-1}(k) = \alpha^{-1}(\alpha(n)) = n$ .Thus , σn = n and inclusion split , since $\sigma o i = I_N$.

Recall that an S-system $M_s$is called **principally self-generator** if every x ∈ M_s, there is an S-homomorphism f : $M_s \to xS$ such that $x = f(x_1)$ for $x_1 \in M_s$ [6] .

(2-14) Lemma: Let M_s be principally self-generator. Then, every principal subsystem is of the form mS, where $\gamma_s(m_0) \subseteq \gamma_s(m)$ and $M_s = m_0S$.

Proof: Let $M_s = m_0S$ be a principal S-system and nS be a principal subsystem of M_s, since M_s is self –generator , then for n ∈ M_s , there is an S-homomorphism $\alpha : M_s \to nS$, so $n = \alpha(m_1)$ for some $m_1 \in M_s$. Then , $nt = \alpha(m_1)t = \alpha(m_1t) = \alpha(m_0st)$, which implies that α is onto . Thus, Im α = nS = $\alpha(m_0)S$ =mS where m = $\alpha(m_0)$. Now , $\forall (s,t) \in \gamma_s(m_0)$ implies $m_0s = m_0t$ and then $ms = \alpha(m_0)s = \alpha(m_0s) = \alpha(m_0t) = \alpha(m_0)t = mt$. This means $m \in \ell_M(\gamma_s(m_0))$ which implies that $\gamma_s(m_0) = \gamma_s\left(\ell_M(\gamma_s(m_0))\right) \subseteq \gamma_s(m)$.

(2-15) Proposition: Let M_s be a principal system which is a principal self-generator and let T =End(M_s) . The following conditions are equivalent:

(1) M_s is pseudo principally injective;

(2) $S_{(\alpha,m)} = B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$ for all α ∈ T and all m ∈ M_s ;

(3) If $A_{\alpha m} = A_{\beta m}$, then $\beta \in B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$.

Proof: (1→2) By proposition (2-10).

(2→3) Let $A_{\alpha m} = A_{\beta m}$, then $\gamma_s(\alpha m) = \gamma_s(\beta m)$. Let (x,y) ∈ kerα , so α(x) = α(y) where x,y ∈ M_s = mS . Let $x = ms_1$, and $y = ms_2$, then $\alpha(m)s_1 = \alpha(m)s_2$, so $(s_1,s_2) \in \gamma_s(\alpha(m)) = \gamma_s(\beta(m))$. This implies $\beta(m)s_1 = \beta(m)s_2$ and then $\beta(ms_1) = \beta(ms_2)$, this means β(x) = β(y) and (x,y) ∈ kerβ . Thus kerα ⊆ kerβ . For the other direction , let (x,y) ∈ kerβ , so β(x) = β(y) since x, y ∈M_s = mS . Let $x = ms_1$, and $y = ms_2$. Thus $\beta(m)s_1 = \beta(m)s_2$ and then $(s_1,s_2) \in \gamma_s(\beta(m)) = \gamma_s(\alpha(m))$. This implies $\alpha(m)s_1 = \alpha(m)s_2$,then $\alpha(ms_1) = \alpha(ms_2)$, so α(x) = α(y) which implies (x,y) ∈ kerα , thus kerα = kerβ . So , $\ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$ which implies $S_{(\alpha,m)} = S_{(\beta,m)}$, so by (2) , we have $B_{\alpha m}\alpha \cup \ell_T(mS \times mS) = B_{\beta m}\beta \cup \ell_T(mS \times mS)$. Since $1_M \in B_{\beta(m)}$. This means $\beta = 1_M \bullet \beta \in B_{\beta m}\beta$, so $\beta \in B_{\beta m}\beta \cup \ell_T(mS \times mS) = B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$, this implies $\beta \in B_{\alpha m}\alpha \cup \ell_T(mS \times mS)$. Also , $\alpha \in B_{\beta m}\beta \cup \ell_T(mS \times mS)$.

(3→1) Assume that f : mS → M_s be an S-homomorphism. Since M_s is principal, so there exists $m_0 \in M_s$ such that $M_s = m_0S$ and $\alpha : M_s \to mS$ with $\alpha(m_0) = m$, where $\gamma_s(m_0) \subseteq \gamma_s(m)$. Again since M_s is principal self-generator , so there exists β : M_s → f(m)S such that :f(m) = $\beta(m_0)$, where $M_s = m_0S$ …(1) .

Since f is S-monomorphism , so $\gamma_s(f(m)) = \gamma_s(m)$. In fact , since , if $(s,t) \in \gamma_s(f(m))$, so f(ms) = f(mt) , since f is monomorphism , so ms = mt which implies $(s,t) \in \gamma_s(m)$ and then $\gamma_s(f(m)) \subseteq \gamma_s(m)$. For the other direction , let $(s,t) \in \gamma_s(m)$, so ms =mt . Since f is well-defined , so f(ms) = f(mt) . Thus f(m)s = f(m)t which implies $(s,t) \in \gamma_s(f(m))$ and then $\gamma_s(m) \subseteq \gamma_s(f(m))$. Thus , $\gamma_s(f(m)) = \gamma_s(m)$. This implies $\gamma_s(\beta(m_0)) = \gamma_s(\alpha(m_0))$. This means kerα = kerβ . In fact , for (x,y) ∈ kerα , this implies α(x) = α(y) where where x,y∈ $M_s = m_0S$. Let $x = m_0s_1$, and $y = m_0s_2$, then $\alpha(m_0s_1) = \alpha(m_0s_2)$ which implies $\alpha(m_0)s_1 = \alpha(m_0)s_2$, so $(s_1,s_2) \in \gamma_s(\alpha(m_0)) = \gamma_s(\beta(m_0))$ by the proof . This implies $\beta(m_0)s_1 = \beta(m_0)s_2$ and then $\beta(m_0s_1) = \beta(m_0s_2)$, this means β(x) = β(y) and (x,y) ∈ kerβ . Thus kerα ⊆ kerβ . Similarly for other direction , thus kerα = kerβ . So , $\ker\alpha \cap (m_0S \times m_0S) = \ker\beta \cap (m_0S \times m_0S)$ which implies $S_{(\alpha,m_0)} = S_{(\beta,m_0)}$ and $A_{\alpha m_0} = A_{\beta m_0}$, so by (3) we have $\beta \in B_{\alpha m_0}\alpha \cup \ell_T(m_0S \times m_0S)$. Thus , either $\beta \in B_{\alpha m_0}\alpha$ or $\beta \in \ell_T(m_0S \times m_0S)$. If $\beta \in B_{\alpha m_0}\alpha$, then there exists S-homomorphism $\varphi \in B_{\alpha m_0}$ which implies φ ∈ T and β = φα . Thus , $\varphi(m) = \varphi(\alpha(m_0)) = \beta(m_0)$ and by (1) $\beta(m_0) = f(m)$, so $\varphi_{|mS} = f$, so M_s is pseudo principally injective system . If $\beta \in \ell_T(m_0S \times m_0S)$, so $\beta \in \ell_T(M_s \times M_s)$ which implies β ∈ Tand $\forall (x,y) \in M_s \times M_s$, we have $\beta(x) = \beta(y) \forall (x,y) \in M_s$. This implies $\ker\beta = M_s \times M_s$ and then β = 0 which implies

f = 0 and this is a contradiction.

3- Relation Between Pseudo PQ-Injective S-Systems With Other Classes of Injectivity:

It is well known that each PQ-injective system is pseudo PQ-injective . To show under which conditions the converse is true , we need the following concepts and some propositions and lemmas.

Recall that a subsystem N of an S-system M_s is called (pseudo)stable if $f(N) \subseteq N$ for each S-homomorphism (S-monomorphism) $f : N \to M_s$. An S-system M_s is called fully (pseudo) stable if each subsystem of M_s is (pseudo) stable [12] ,[8] . It is clear that every stable subsystem is pseudo stable and hence every fully stable S-system is fully pseudo stable . It was proved that

every fully pseudo stable S-system is pseudo PQ-injective.

Recall that an S-system M_s is **multiplication** if each subsystem of M_s is of the form MI , for some right ideal I of S . This is equivalent to saying that every principal subsystem is of this form [11] .

(3-1) Proposition : Let M_s be multiplication S-system . Then , M_s is fully pseudo stable if and only if M_s is pseudo PQ-injective S-system.

Proof: Let mS be principal subsystem of an S-system M_s and $\alpha : mS \to M_s$ be an S-monomorphism , where $m \in M_s$. Then , since M_s is pseudo PQ-injective , so α extends to an S-homomorphism $\beta : M_s \to M_s$. Since M_s is multiplication system , so there is an ideal I of S such that mS = MI . Hence , $\alpha(mS) = \beta(mS) = \beta(MI) = \beta(M)\, I \subseteq MI = mS$. ThusM_s is fully pseudo stable.

Now, we give under which conditions on pseudo PQ-injective systems to be PQ-injective . But , before this we need the following propositions :

(3-2) Proposition[8]: An S-system M_s is fully stable if and only if M_s is fully pseudo-stable and $xS \cong Hom(xS , M_s)$ for each x in M_s .

(3-3) Proposition[6] : Let S be a commutative monoid and M_s be a multiplication S-system . Then M_s is fully stable if and only ifM_s is PQ-injective S-system .

(3-4) Proposition : Let M_s be multiplication S-system , where S is a commutative monoid and $xS \cong Hom(xS , M_s)$ for each x in M_s . If M_s is pseudo PQ-injective system , then M_s is PQ-injective .

Proof: Assume that M_s is pseudo PQ-injective system . Since M_s is multiplication system , so M_s is fully pseudo stable by proposition (3-1) . Since $xS \cong Hom(xS , M_s)$, so by proposition (3-2) , M_s is fully stable system . Again since M_s is multiplication system , so by proposition(3-3) M_s is PQ-injective system .

It is clear that every quasi injective system is pseudo PQ-injective system (and hence PQ-injective) , but the converse is not true in general . For the converse , we need the following proposition :

(3-5)Proposition[6]: Let M_s be multiplication S-system . If M_s is PQ-injective, then M_s is quasi injective .

(3-6) Proposition : Let M_s be multiplication S-system , where S is a commutativemonoid and $xS \cong Hom(xS , M_s)$ for each x in M_s . If M_s is pseudo PQ-injective S-system , then M_s is quasi injective .

Proof: By proposition (3-4) and proposition (3-5) .

At the same time, we can give another conditions to versus pseudo PQ-injective S-systems with PQ-injective , but we need the following concept:

(3-7) Proposition : Let M_s be a cog-reversible nonsingular S-system with $\ell_M(s) = \Theta$, $\forall\, s \in S$ .IfM_s is pseudo PQ-injective , then M_s is PQ-injective.

Proof : Let N be principal subsystem of S-system M_s and f be S-homomorphism from N into M_s . If f is S-monomorphism , then there is nothing to prove . So assume f is not S-monomorphism . Then , by using the proof of theorem(3.2.17) , we get the required . This means thatM_s is PQ-injective S-system .

The following proposition explain under which conditions on pseudo PQ-injective system to beingpseudo QP-injective and the proof is similar to proposition(2-22) in [7] by replacing S-homomorphisms by S-monomorphism.

(3-8) Proposition : Let M_s be an S-system which is principal and principal self-generator. Then , M_s is pseudo PQ-injective S-system if and only ifM_s is pseudo QP-injective .

Proof :⇐) Let N be cyclic subsystem of M_s and f be S-monomorphism from N into M_s. Since M_s is principal self-generator, so there exists some $\alpha: M_s \to mS$, such that $m = \alpha(m_1)$, $\forall\, m \in M_s$. This means α is S-epimorphism , thus N is M_s-cyclic subsystem of M_s . Since M_s is pseudo QP-injective system , so f can be extended to S-homomorphism $g : M_s \to M_s$, such that goi = f , where i be the inclusion map of N into M_s, therefore M_s is pseudo PQ-injective system .

⇒)Let N be M_s -cyclic subsystem of an S-system M_s , so there exists an S-epimorphism $\alpha : M_s \rightarrow N$. Since M_s is principal, so N is principal . Let f be S-monomorphism from N into M_s . Since M_s is pseudo PQ-injective system , so f can be extended to S-homomorphism g from M_s into M_s such that goi = f , where i be the inclusion map of N into M_s . Thus M_s is pseudo QP-injective system .

REFERENCES

[1] A . M .Lopez , Jr. and J. K. Luedeman, (1979) , Quasi-injective S-systems and their S-endomorphism semi group, *CzechoslovakMath .J*., 29(1), pp.97-104.

[2] A.M.Lopez, Jr. and J.K.Luedeman , (1976) , The Bicommutator of the injective hull of a non- singular semigroup, *Semigroup Forum*, 12 , pp.71-77.

[3] C.V. Hinkle and Jr. , (1974) ,The extended centralizer of an S-set , *Pacific journal of mathematics* , Vol.53, No.1, pp 163-170.

[4] K.Jupil , (2008), PI-S-systems , *J. of Chungcheong Math. Soc.*, 21 , no.4,pp.591-599.

[5] M . Kilp , U . Knauer and A .V . Mikhalev , (2000) , Monoids acts and categories . Walter de Gruyter , Berlin , New York.

[6] M.S. Abbas and A.Shaymaa ,(2015), principally quasi injective system over monoid , *journal of advances in mathematics* ,Vol .(10),No.(1) , pp.3152-3162.

[7] M.S.Abbas and A.Shaymaa , (2015) , quasi principally injective systems over monoids , *journal of advances in mathematics* ,Vol .(10),No.(5) , pp.3493-3502.

[8] M.S. Abbas and A. A.Mustafa , (2015) , fully pseudo stable systems , *journal of advances in mathematics* ,Vol .(10),No.(3) , pp. 3356-3361.

[9] M.Ershad , M.Roueentan and M.A.Naghipoor , (2012) , On C-injectivity of acts over monoids, *22nd Iranian Algebra seminar* , pp. 94-97.

[10] P . Berthiaume , (1967) , The injective envelope of S-sets ,*Canad . Math. Bull*., 10, pp.261 – 273.

[11] [11] R.Mohammad and E.Majid , (2014) ,Strongly duo and duo right S-acts , *Italian journal of pure and applied mathematics*,32, pp.143-154.

[12] R.B.Hiba , (2014) , On fully stable acts . MSC thesis , College of Science , Univ. of Al-Mustansiriyah.

[13] T.Yan , (2011) , Generalized injective S-acts on a monoid , *Advances in mathematics* , Vol.40,No.4,pp. 421-432.

[14] T.Yan , A.Javed , X.FEI and G.XIAO,(2007), Monoids characterized by their injectivity and projectivity classes , *Advances in mathematics* , Vol.36,No.3 ,pp. 321-326 .