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Monotonicity Results For Discrete Caputo-Fabrizio Fractional Operators

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Abstract

Nearly every theory in mathematics has a discrete equivalent that simplifies it theoretically and practically so that it may be used in modeling real-world issues. With discrete calculus, for instance, it is possible to find the "difference" of any function from the first order up to the n-th order. On the other hand, it is also feasible to expand this theory using discrete fractional calculus and make n any real number such that the 1⁄2-order difference is properly defined. This article is divided into five chapters, each of which develops the most straightforward discrete fractional variational theory while illustrating some fundamental concepts and features of discrete fractional calculus. It is also investigated how the idea may be applied to the development of tumors. The first section provides a succinct introduction to the discrete fractional calculus and several key mathematical concepts that are utilized often in the subject. We demonstrate in section 2 that if the Caputo-Fabrizio nabla fractional difference operator $({}_{\alpha-1}^{CFR}\nabla^{\alpha} y)(t)$ of order $0 < \alpha \leq 1$ and commencing at $\alpha - 1$ is positive for $t = a, a+1, \ldots$ then $y(t)$ is α-increasing. On the other hand, if $y(t)$is rising and $y(a) \geq 0$, then $({}_{\alpha-1}^{CFR}\nabla^{\alpha} y)(t) \geq 0$. Additionally, a result of monotonicity for the Caputo-type fractional difference operator is established. We show a fractional difference version of the mean-value theorem as an application and contrast it to the traditional discrete fractional instance.

Keywords: Discrete Fractional Calculus; Discrete Exponential Kernel; Caputo Fractional Difference; Riemann Fractional Difference; Discrete Fractional Mean Value Theorem.

1. Introduction

In numerous fields of engineering and research over the past ten years, the fractional calculus has been successfully applied [1],[2]. Discrete fractional calculus (DFC) was successfully developed using the fundamental ideas of this type of nonlocal calculus [3],[4]. This new direction, which was started more than 10 years ago, is in a state of steady evolution, and it has just recently started to be recognized as a potent instrument for uncovering hitherto unknown dynamics of intricate discrete dynamical systems. The discrete diffusion equation included in the discrete Reisz derivative was one of the most recent discoveries. The discrete diffusion equation is included inside the discrete Riesz derivative [5], [6]. Therefore, the discrete fractional calculus may be a natural development of conventional discrete ones. And Fabrizio, Caputo [7] On the basis of a nonsingular kernel, a different fractional derivative was presented. This operator's discrete variant was described in [8]. We think that the appearance of various forms of memory kernels improves the likelihood that various kinds of models will be appropriately developed when various types of memory emerge. Recent research has looked into discrete functions' discrete fractional operators to examine the monotonicity properties of such functions. While others looked into fractional difference operators of order $\alpha > 1$ [9],[10], some writers addressed the monotonicity analysis of fractional difference operators of orders $0 < \alpha < 1$, such as delta- or nabla-types [11]. These novel findings motivate us to compare the monotonicity results for this discrete fractional operator with a discrete exponential kernel to the discrete classical ones and discuss them in this thesis. We think that the fractional differences considered in this thesis result in novel kernels with new memories, which might be of diverse importance for applications. These kernels differ from the standard nabla fractional differences with kernels relying on the increasing factorial powers.

2. Preliminaries

Discrete fractional calculus's fundamental concepts and results are provided in the next chapter. The fractional sum and the fractional difference of a function f(x) to a random order α, starting from a, will be denoted by ∇_a^(-α) f(x) and ∇_a^(-α) f(x) respectively. Where α is a positive real number, and for a real number a, we demoted N_a={a,a+1,a+2,…}. Our recommendation for our readers is the reference [28] for further information on discrete fractional calculus concepts. Thenabla discrete exponential kernel may be expressed using the time scale notation as $(1-\alpha)^{t-\rho(s)}$ where $\rho(s) = s - 1$ [29].

2.1. Caputo Fractional Difference

Definition 2.1.[8] The Caputo-Fabrizio in the Caputo sense nabla difference of f can be defined as follows for $0 < \alpha < 1$ and f defined on $\mathbb{N}_a$:

$$({}^{CFC}\nabla_a^\alpha f)(t) = \frac{B(\alpha)}{1-\alpha} \sum_{s=a+1}^{t} (\nabla_s f)(s)(1-\alpha)^{t-\rho(s)}$$

$$= B(\alpha) \sum_{s=a+1}^{t} (\nabla_s f)(s)(1-\alpha)^{t-s} \tag{1}$$

where $B(\alpha)$ is a normalizing positive constant which depends on α and sustaining $B(0) = B(1) = 1$.

2.2. Riemann Fractional Difference

Definition 2.2. [8] For $0 < \alpha < 1$ and f defined on $\mathbb{N}_a$, the Caputo-Fabrizio in the Riemann sensenabla difference of f can be defined by:

$$({}^{CFR}\nabla_a^\alpha f)(t) = \frac{B(\alpha)}{1-\alpha} \nabla_t \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-\rho(s)}$$

$$= B(\alpha)\nabla_t \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-s} \tag{2}$$

wherever $B(\alpha)$ is a normalizing positive constant depending on α and satisfying $B(0) = B(1) = 1$.

2.3. Fractional Sum

Definition 2.3. [8] For $0 < \alpha < 1$ and f defined on $\mathbb{N}_a$, the fractional sum of f can be defined by:

$$({}^{CF}\nabla_a^{-\alpha} f)(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \sum_{s=a+1}^{t} f(s)ds \tag{3}$$

It was shown that $({}^{CF}\nabla_a^{-\alpha} \quad {}^{CF}\nabla_a^\alpha f)(t)$. Also, it was shown that $({}^{CF}\nabla_a^\alpha \quad {}^{CF}\nabla_a^{-\alpha} f)(t)$.

The following statement and lemma include several elements that are crucial to moving forward.

Remark 2.4. [12] The association between Riemann and Caputo kind fractional difference is given by

$$({}^{CFC}\nabla_a^\alpha f)(t) = ({}^{CFR}\nabla_a^\alpha f)(t) - \frac{B(\alpha)}{1-\alpha} f(a)(1-\alpha)^{t-a}$$

Lemma 2.5. For $\alpha \in (0,1)$ and g defined on $\mathbb{N}_a$, there are.

(i)$({}^{CF}\nabla_a^{-\alpha}(1-\alpha)^t)(t) = \frac{(1-\alpha)^{a+1}}{B(\alpha)}$

(ii) $\nabla_s(1-\alpha)^{t-s} = \alpha(1-\alpha)^{t-s}$

(iii) $({}^{CF}\nabla_a^{-\alpha}\nabla g)(t) = (\nabla^{CF}\nabla_a^{-\alpha} g)(t) - \frac{\alpha}{B(\alpha)} g(a)$

(iv) $\nabla(1-\alpha)^t = -\alpha(1-\alpha)^{t-1}$

(v) $({}^{CFR}\nabla_a^\alpha(1-\alpha)^t)(t) = B(\alpha)(1-\alpha)^{t-1}[1-\alpha(t-a)]$

Proof

The proof of (i):

$$({}^{CF}\nabla_a^{-\alpha}(1-\alpha)^t)(t)=\frac{1-\alpha}{B(\alpha)}(1-\alpha)^t+\frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}(1-\alpha)^s ds$$

Since $\alpha \in (0,1)$ we can apply geometric series, and we get

$$=\frac{1-\alpha}{B(\alpha)}(1-\alpha)^t+\frac{\alpha}{B(\alpha)}(1-\alpha)^{a+1}\frac{1-(1-\alpha)^{t+1-(a+1)}}{1-(1-\alpha)}$$
$$=\frac{1-\alpha}{B(\alpha)}(1-\alpha)^t+\frac{\alpha}{B(\alpha)}(1-\alpha)^{a+1}\frac{1-(1-\alpha)^{t-a}}{\alpha}$$
$$=\frac{1}{B(\alpha)}[(1-\alpha)^{t+1}+(1-\alpha)^{a+1}-(1-\alpha)^{t+1}]$$
$$=\frac{(1-\alpha)^{a+1}}{B(\alpha)}$$

The proof of (ii)

$$\nabla_s(1-\alpha)^{t-s}=(1-\alpha)^{t-s}-(1-\alpha)^{t-(s-1)}$$
$$=(1-\alpha)^{t-s}-(1-\alpha)^{t-s+1}$$
$$=(1-\alpha)^{t-s+1}[\frac{1}{(1-\alpha)}-1]$$
$$=(1-\alpha)^{t-s+1}[\frac{1-1+\alpha}{(1-\alpha)}]$$
$$=\alpha(1-\alpha)^{t-s}.$$

The proof of (iii)

$$({}^{CF}\nabla_a^{-\alpha}\nabla g)(t)=\frac{1-\alpha}{B(\alpha)}\nabla g(t)+\frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}\nabla g(s)ds$$

But note that $\sum_{s=a+1}^{t}\nabla g(s)ds=g(t)-g(a)$, and from $\nabla\sum_{s=a+1}^{t}g(s)ds=g(t)$, we can write

$$({}^{CF}\nabla_a^{-\alpha}\nabla g)(t)=[\frac{1-\alpha}{B(\alpha)}\nabla g(t)+\frac{\alpha}{B(\alpha)}\nabla\sum_{s=a+1}^{t}g(s)ds]-\frac{\alpha}{B(\alpha)}g(a)$$
$$=\nabla[\frac{1-\alpha}{B(\alpha)}g(t)+\frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}g(s)ds]-\frac{\alpha}{B(\alpha)}g(a)$$
$$=(\nabla{}^{CF}\nabla_a^{-\alpha}g)(t)-\frac{\alpha}{B(\alpha)}g(a)$$

The proof of (iv)

$$\nabla(1-\alpha)^t=(1-\alpha)^t-(1-\alpha)^{t-1}$$
$$=(1-\alpha)^{t-1}[(1-\alpha)-1]$$
$$=(1-\alpha)^{t-1}[(1-\alpha)-1]$$
$$=-\alpha(1-\alpha)^{t-1}$$

The proof of (v)

$$({}^{CFR}\nabla_a^{\alpha}(1-\alpha)^t)(t)=B(\alpha)\nabla_t\sum_{s=a+1}^{t}(1-\alpha)^s(1-\alpha)^{t-s}$$
$$=B(\alpha)\nabla_t(1-\alpha)^t\sum_{s=a+1}^{t}1$$
$$=B(\alpha)\nabla_t[(1-\alpha)^t(t-a)]$$
$$=B(\alpha)[(1-\alpha)^t(t-a)-(1-\alpha)^{t-1}(t-1-a)]$$
$$=B(\alpha)(1-\alpha)^{t-1}[(1-\alpha)(t-a)-(t-1-a)]$$
$$=B(\alpha)(1-\alpha)^{t-1}[t-a-t\alpha+a\alpha-t+a+1]$$
$$=B(\alpha)(1-\alpha)^{t-1}[1-t\alpha+a\alpha]$$
$$=B(\alpha)(1-\alpha)^{t-1}[1-\alpha(t-a)]$$

2. The Monotonicity Results

3.1. Increase

Definition 3.1. [13] Let y be a function defined on $\mathbb{N}_{\boldsymbol{a}}$ so that satisfying $y(a) \geq 0$. Then y is named an α-increasing function on $\mathbb{N}_{\boldsymbol{a}}$if

$$y(t+1) \geq \alpha y(t) \quad for\ all\ t \in \mathbb{N}_{\boldsymbol{a}}$$

Theorem 3.2. Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$, $\alpha \in (0,1)$, and $({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) \geq 0,\ t \in \mathbb{N}_{\boldsymbol{a-1}}$. Then $y(t)$ is α-increasing.

Proof.

$$({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) = B(\alpha)\nabla_t \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s}$$

$$= B(\alpha)[\sum_{s=a}^{t} y(s)(1-\alpha)^{t-s} - \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s-1}]$$

$$= B(\alpha)[\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} + y(t) - \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s-1}]$$

$$= B(\alpha)[y(t) + \sum_{s=a}^{t-1} y(s)((1-\alpha)^{t-s} - (1-\alpha)^{t-s-1})]$$

$$= B(\alpha)[y(t) + \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s-1}\,(1-\alpha-1)]$$

$$= B(\alpha)[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}] \tag{4}$$

But $({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) \geq 0$, we have

$$B(\alpha)\left[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}\right] \geq 0$$

Since $(\alpha) \geq 0$, we get

$$y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \geq 0$$

It follows

$$y(t) \geq \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}. \tag{5}$$

Putting $t = a$ for (5) we get $(a) \geq 0$, put $t = a+1$ for into (5) , we get

$$y(a+1) \geq \frac{\alpha}{1-\alpha} y(a)(1-\alpha)$$

It follows

$$y(a+1) \geq \alpha y(a).$$

And hence $y(a+1) \geq \alpha y(a) \geq 0$, we will proceed by induction, we get $y(a+k) \geq 0$, for all $k \in \mathbb{N}_{\boldsymbol{0}}$ which is the same with $y(t) \geq 0$ for all $t \in \mathbb{N}_{\boldsymbol{a}}$.

Now replacing t with $t+1$ in (5) we get

$$y(t+1) \geq \frac{\alpha}{1-\alpha}\sum_{s=a}^{t} y(s)(1-\alpha)^{t-s+1}$$

Also, we have

$$y(t+1) \geq \alpha y(t) + \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1}$$

And since $\alpha \in (0,1)$ and $y(t) \geq 0$for all $\in \mathbb{N}_{\boldsymbol{a}}$, we can write

$$y(t+1) \geq \alpha y(t) + \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1} \geq \alpha y(t)$$
$$y(t+1) \geq \alpha y(t)$$

which completes the proof.

Theorem 3.3. Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$, $\alpha \in (0,1)$, and $({}^{CFC}\nabla^{\alpha}_{a-1}y)(t) \geq -\frac{B(\alpha)}{1-\alpha} y(a-1)(1-\alpha)^{t-a+1}$, $t \in \mathbb{N}_{\boldsymbol{a-1}}$,then y(t) is α-increasing.

Proof. By assumption, we have

$$({}^{CFC}\nabla^{\alpha}_{a-1}y)(t) + \frac{B(\alpha)}{1-\alpha} y(a-1)(1-\alpha)^{t-a+1} \geq 0$$

and from Remark 2.4, we get

$$({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) \geq 0, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}$$

and from Theorem 3.2, we get

$y(t)$ is α-increasing, hence the proof is complete.

Theorem 3.4. Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$ satisfying $y(a) \geq 0$ and increasing on $\mathbb{N}_{\boldsymbol{a}}$. Then, for $\alpha \in (0,1)$

$$({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) \geq 0, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}$$

Proof. From (4), we have

$$({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) = B(\alpha)[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}]$$

and since $B(\alpha) \geq 0$ so to show that $({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) \geq 0$ we need to prove that.

$$y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \geq 0$$

$$y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}$$
$$= y(t) - \alpha y(t-1) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-2} y(s)(1-\alpha)^{t-s}$$
$$= y(t) - \alpha y(t-1)$$
$$-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-2}[y(s) - y(t-1) + y(t-1)](1-\alpha)^{t-s}$$
$$= y(t) - \alpha y(t-1)$$
$$-\frac{\alpha}{1-\alpha}\left[\sum_{s=a}^{t-2}\big(y(s) - y(t-1)\big)(1-\alpha)^{t-s} + \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s}\right] \qquad (6)$$

Since y is increasing, it indicates that $y(t) \geq y(t-1) \geq y(t-2) \geq \cdots \geq y(a) \geq 0$,so we get

$$y(t) - \alpha y(t-1) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s}$$
$$= y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(t-1)(1-\alpha)^{t-s}$$
$$= y(t) - y(t-1) + y(t-1) - \frac{\alpha}{1-\alpha} y(t-1)\sum_{s=a}^{t-1}(1-\alpha)^{t-s}$$
$$\geq y(t-1) - \frac{\alpha}{1-\alpha} y(t-1)\sum_{s=a}^{t-1}(1-\alpha)^{t-s}$$

$$= y(t-1)\left[1-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}(1-\alpha)^{t-s}\right]$$
$$= y(t-1)\left[1-\frac{\alpha}{1-\alpha}\sum_{s=0}^{t-1-a}(1-\alpha)^{t-(s+a)}\right]$$
$$= y(t-1)\left[1-\frac{\alpha}{1-\alpha}\sum_{s=0}^{(t-a)-1}(1-\alpha)^{t-a-s}\right]$$
$$= y(t-1)\left[1-\frac{\alpha}{1-\alpha}(1-\alpha)^{t-a}\sum_{s=0}^{(t-a)-1}\left(\frac{1}{1-\alpha}\right)^{s}\right]$$

By using geometric series, we have

$$= y(t-1)\left[1-\frac{\alpha}{1-\alpha}(1-\alpha)^{t-a}\left(\frac{1-(\frac{1}{1-\alpha})^{t-a}}{1-\frac{1}{1-\alpha}}\right)\right]$$
$$= y(t-1)\left[1-\frac{\alpha}{1-\alpha}(1-\alpha)^{t-a}\left(\frac{1-(1-\alpha)^{a-t}}{\frac{-\alpha}{1-\alpha}}\right)\right]$$
$$= y(t-1)[1-(1-\alpha)^{t-a}((1-\alpha)^{a-t}-1)]$$
$$= y(t-1)[1-(1-(1-\alpha)^{t-a})]$$
$$= y(t-1)(1-\alpha)^{t-a} \geq 0, \qquad (7)$$

which completes the proof.

Theorem 3.5. Let y be a function defined on $\mathbb{N}_{a-1}$ satisfy $y(a) \geq 0$ and be strictly increasing on $\mathbb{N}_{a}$. Then, for $\alpha \in (0,1)$

$$({}^{CFR}\nabla_{a-1}^{\alpha}y)(t) > 0, \qquad t \in \mathbb{N}_{a-1}$$

Proof. Similar to the previous theorem, this one can be proven.

3. α-Decrease

Definition 4.1. [13] Let y be a function defined on $\mathbb{N}_{a}$,so that satisfying $y(a) \geq 0$. Then y is named an α-decreasing function on $\mathbb{N}_{a}$if

$$y(t+1) \leq \alpha y(t) \quad for\ all\ t \in \mathbb{N}_{a}$$

Theorem 4.2. Let y be a function defined on $\mathbb{N}_{a-1}$, $\alpha \in (0,1)$, and

$$({}^{CFR}\nabla_{a-1}^{\alpha}y)(t) \leq 0, \qquad t \in \mathbb{N}_{a-1}$$

Then $y(t)$ is α-decreasing.

Proof. From (2.1) we have

$$({}^{CFR}\nabla_{a-1}^{\alpha}y)(t) \ = B(\alpha)\left[y(t)-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}y(s)(1-\alpha)^{t-s}\right]$$

But given that $({}^{CFR}\nabla_{a-1}^{\alpha}y)(t) \leq 0$, so we have.

$$B(\alpha)\left[y(t)-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}y(s)(1-\alpha)^{t-s}\right] \leq 0$$

Since $B(\alpha) \geq 0$, we get.

$$y(t)-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}y(s)(1-\alpha)^{t-s} \leq 0$$

It follows

$$y(t) \leq \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}y(s)(1-\alpha)^{t-s}. \qquad (8)$$

Putting $t = a$ for (8) we get $B(a) \leq 0$, put $t = a+1$ for into (8), we get

$$y(a+1) \leq \frac{\alpha}{1-\alpha} y(a)(1-\alpha)$$

It follows

$$y(a+1) \leq \alpha y(a)$$

and hence $y(a+1) \leq \alpha y(a) \leq 0$,we will proceed by induction. We get $y(a+k) \leq 0$, for all $k \in \mathbb{N}_{\mathbf{0}}$ which is the same with $y(t) \leq 0$ for all $t \in \mathbb{N}_{\boldsymbol{a}}$.

Now replacing t with $t+1$ in (8), we get

$$y(t+1) \leq \frac{\alpha}{1-\alpha} \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s+1}$$

Also, we have

$$y(t+1) \leq \alpha y(t) + \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1}$$

And since $\alpha \in (0,1)$ and $y(t) \leq 0$for all $t \in \mathbb{N}_{\boldsymbol{a}}$so we can write

$$y(t+1) \leq \alpha y(t) + \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1} \leq \alpha y(t)$$

$$y(t+1) \leq \alpha y(t)$$

which completes the proof.

Theorem 4.3. Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$ satisfy $y(a) \geq 0$ and be decreasing on $\mathbb{N}_{\boldsymbol{a}}$. Then, for $\alpha \in (0,1)$

$$({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) \leq 0, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}$$

Proof. From (4), we have

$$({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) = B(\alpha)[y(t) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}]$$

and since $B(\alpha) \geq 0$ so to show that $({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) \leq 0$ we need to prove that $y(t) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \leq 0$, now from (6) we have

$$y(t) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}$$

$$= y(t) - \alpha y(t-1)$$

$$- \frac{\alpha}{1-\alpha} \left[\sum_{s=a}^{t-2} \big(y(s) - y(t-1)\big)(1-\alpha)^{t-s} + \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s} \right].$$

Since y is increasing, it indicates that $y(t) \leq y(t-1) \leq$

$y(t-2) \leq \cdots \leq y(a) \leq 0$, so we get

From (7) we have

$$y(t) - \alpha y(t-1) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s} \leq y(t-1)(1-\alpha)^{t-a} \leq 0$$

which completes the proof.

4. Application

We know that $({}^{CF}\nabla_a^{-\alpha} \ {}^{CFR}\nabla_a^{\alpha} y)(t) = y(t)$. Nevertheless, the following result, delivers an initial condition y(a), will be a instrument to prove our fractional difference mean value theorem.

Theorem 5.1. For $\in (0,1)$, we have

$$({}^{CF}\nabla_a^{-\alpha} \ {}^{CFR}\nabla_{a-1}^{\alpha} y)(t) = y(t) - \alpha y(a) \tag{9}$$

Proof. From definition, we have

$$({}^{CF}\nabla_a^{-\alpha} \ {}^{CFR}\nabla_{a-1}^{\alpha} y)(t) = {}^{CF}\nabla_a^{-\alpha} \left[B(\alpha) \nabla_t \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s} \right]$$

$$= B(\alpha) {}^{CF}\nabla_a^{-\alpha} \nabla_t \left[y(a)(1-\alpha)^{t-a} + \sum_{s=a+1}^{t} y(s)(1-\alpha)^{t-s} \right]$$

$$= \left[B(\alpha)^{CF}\nabla_a^{-\alpha}\nabla_t y(a)(1-\alpha)^{t-a} + \ {}^{CF}\nabla_a^{-\alpha} B(\alpha)\nabla_t \sum_{s=a+1}^{t} y(s)(1-\alpha)^{t-s}\right]$$
$$= B(\alpha)^{CF}\nabla_a^{-\alpha}\nabla_t y(a)(1-\alpha)^{t-a} + \ {}^{CF}\nabla_a^{-\alpha} \ {}^{CFR}\nabla_a^{\alpha} y)(t)$$
$$= B(\alpha)y(a)(1-\alpha)^{-a\,CF}\nabla_a^{-\alpha}\nabla_t(1-\alpha)^t + y(t)$$

From part (iv)of lemma2.5 we have

$$= -\alpha B(\alpha)y(a)(1-\alpha)^{-a\,CF}\nabla_a^{-\alpha}(1-\alpha)^{t-1} + y(t)$$

and from part (i) of Lemma 2.5 also we get

$$= -\alpha B(\alpha)y(a)(1-\alpha)^{-a}\frac{(1-\alpha)^a}{B(\alpha)} + y(t) = y(t) - \alpha y(a)$$

The proof is complete.

Theorem 5.2. Let f and g be functions defined on $\mathbb{N}_{\boldsymbol{a,b}} = \{a, a+1, \dots, b-1, b\}$ where $a < b$ with $a \equiv b\ (mod\ 1)$. Suppose that g is strictly increasing and $\alpha \in (0,1)$. Then, $\exists\ s_1 s_2 \in \mathbb{N}_{\boldsymbol{a,b}}$ such that

$$\frac{({}^{CFR}\nabla_{a-1}^{\alpha} f)(s_1)}{({}^{CFR}\nabla_{a-1}^{\alpha} g)(s_1)} \le \frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)} \le \frac{({}^{CFR}\nabla_{a-1}^{\alpha} f)(s_2)}{({}^{CFR}\nabla_{a-1}^{\alpha} g)(s_2)} \qquad (10)$$

Proof. We employ contradiction, letting (10) be untrue either then.

$$\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)} > \frac{({}^{CFR}\nabla_{a-1}^{\alpha} f)(t)}{({}^{CFR}\nabla_{a-1}^{\alpha} g)(t)},\ for\ all\ \ t \in \mathbb{N}_{\boldsymbol{a,b}} \qquad (11)$$

Or

$$\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)} < \frac{({}^{CFR}\nabla_{a-1}^{\alpha} f)(t)}{({}^{CFR}\nabla_{a-1}^{\alpha} g)(t)},\ for\ all\ \ t \in \mathbb{N}_{\boldsymbol{a,b}} \qquad (12)$$

Given that g is strictly increasing, so by Theorem 3.5 we have $({}^{CFR}\nabla_{a-1}^{\alpha} g)(t) > 0$, hence from (11) we get

$$\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)}({}^{CFR}\nabla_{a-1}^{\alpha} g)(t) > ({}^{CFR}\nabla_{a-1}^{\alpha} f)(t).$$

Now take the fractional sum for both sides it becomes

$$\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)}({}^{CF}\nabla_a^{-\alpha}\ \ {}^{CFR}\nabla_{a-1}^{\alpha} g)(t) > ({}^{CF}\nabla_a^{-\alpha}\ \ {}^{CFR}\nabla_{a-1}^{\alpha} f)(t).$$

And from (9), we have

$$\frac{f(b)-\alpha f(a)}{g(b)-\alpha g(a)} g(t) - \alpha g(a) > f(t) - \alpha f(a)$$

By set t=b, we obtain

$$f(b) - \alpha f(a) > f(b) - \alpha f(a)$$

which is a contradiction, and we can show that (2.9) can also result by contradiction. The proof is complete.

5. Conclusion

This paper presents some new alpha-monotonicity analysis results for discrete Caputo-Fabrizio fractional differences in the sense of Riemann-Liouville and Caputo operators. The monotonicity of the function (increasing or decreasing) has been obtained from the positivity or negativity of the discrete Caputo-Fabrizio fractional operator. As a result, we provide the connection between the Riemann-Liouville and Caputo senses of the operators so that we may get the relevant conclusions using Caputo operators. In addition, a discrete mean value theorem is given to show the established results.

6. Author Contributions

All the authors equally contributed to this work.

7. Conflict of Interest

All the authors declare no conflict of interest.

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