Scitech Research Organisation

Journal of Progressive Research in Mathematics
E-ISSN: 2395-0218

Convexity and Monotonicity Analyses for Discrete Fractional Operators with Discrete Exponential Kernes

[1]Şerif Amirov, [2]İhsan Hasan Saadullah
[1]Faculty of Science, Department of Mathematics, Karabük University, 78050, Karabük, Türkiye & samirov@karabuk.edu.tr
[2]Faculty of Science, Department of Mathematics, Karabük University, 78050, Karabük, Türkiye & ihsan.h.sitey@gmail.com

Received: May 3, 2024; Accepted: May 25, 2024; Published: June 17, 2024

Cite this article: Amirov, S., & Saadullah, I. H. (2024). Convexity and Monotonicity Analyses for Discrete Fractional Operators with Discrete Exponential Kernes. *Journal of Progressive Research in Mathematics*, *21*(1), 1-14. Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2225.

Abstract

For discrete fractional operators with exponential kernels, positivity, monotonicity, and convexity findings are taken into consideration in this paper. Our findings cover both sequential and non-sequential scenarios and show how fractional differences with other kinds of kernels and the exponential kernel example are comparable and different. This demonstrates that the qualitative information gathered in the exponential kernel case does not match other situations perfectly.

Keywords: Discrete Fractional Calculus; Exponential Kernel; Positivity Analysis; Monotonicity Analysis; Convexity Analysis.

1. Introduction and Preliminaries

1.1. Introduction

Numerous scholars from various disciplines, including mathematics, biology, physics, chemistry, engineering, even economics and social sciences, have been concentrating on the discrete fractional calculus field in recent years [1], [2], [3], [4]. A crucial effort has been made, particularly in the field of viscoelasticity, to use fractional mathematical models to accurately describe the behavior of materials.

In mathematics, the idea of monotonicity is crucial. Unfortunately, there are no monotonicity findings for fractional operators in the theory or applications of fractional calculus. The discrete fractional operators underwent a monotonicity study that was started by Dahal and Goodrich in [5] and Goodrich in [6]. For fractional orders between 0 and 1, however, monotonicity concerns are not taken into consideration. Since non-integer orders are the main focus of the first section of this paper, we are able to announce new definitions of monotonicity perceptions. Indicators of the mechanical properties of biomaterials are frequently linear differential equations created from physical spring and dashpot models. However, it has been shown that biological tissues exhibit more complicated performance, such as hysteresis, fatigue, and memory, which cannot be explained by combining perfect spring and dashpot combinations [2]. Since the tissues in the human body are naturally viscoelastic, it is important to incorporate correct viscoelastic when studying the mechanics of deformation [7]. The mechanical properties of living soft tissues create a unique combination of testing and modeling problems. To construct stress-strain correlations for viscoelastic materials, fractional calculus is employed.

It is acknowledged that the description of the characteristics of viscoelastic materials has long relied heavily on rheological constitutive equations with fractional derivatives [4]. First-order derivatives in the rheological constitutive equations must be replaced by fractional order derivatives. They are ideal for describing things with memory, such polymers or tissues, as the fractional derivative of a function depends on its whole history rather than on its instantaneous behavior [8]. We created discrete fractional rheological models for the reasons listed above. A material is described by a finite number of springs and dashpots in discrete models.

Although, there are now many different approaches to show a fractional sum and difference, they all have the important trait of being non-local. For instance, Riemann-Liouville definition, one of the most prominent fractional differences, states that the following approach is presented in the case of a backward or nabla difference:

$$(\nabla_a^v f)(t) = \sum_{s=a+1}^{t} H_{-v-1}\left(t, \rho(s)\right) f(s) (1.1)$$

for each $t \in N_{a+N}$,where

$$H_\mu(t,a) = \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)} \wedge \rho(s) = s-1.$$

The properties that we consider in this paper are given in two cases. Non-sequential which is single fractional difference operation for instance:

$$(\nabla_a^v f)(t) \geq 0.$$

Sequentially, that is a composition of fractional difference operators such as:

$$\left(\nabla_{a+1}^v \nabla_a^\mu f\right)(t).$$

This paper has been broadening this research to discrete fractional operators with exponential kernels. We do not need to impose these kinds of limitations on the parameters' locations based on the findings of this paper. Our qualitative findings, in particular, do not significantly differ from one regime to the next. Instead, they are valid over the complete range of allowed (μ,ν) parameters. This shows that, somewhat surprisingly, the results we may get for a discrete fractional operator with an exponential kernel are different from those for a discrete fractional operator with a Riemann-Liouville kernel.

We discuss the general structure of the remaining theses before we finish. First, we quickly go through the prerequisites for the remaining theses. The link between the sign of a suitable Caputo-Fabrizio fractional difference in the Caputo sense and, respectively, the positivity, monotonicity, and convexity of the function on which the difference works, is then discussed.

1.2. Preliminaries

It will be most important to our conclusions in the next sections to start by recalling a few basic results from the difference calculus. It is noted that in every part of this paper standard convention is followed for instance that $\sum_{k=m}^{n} a_k = 0$ whenever $n < m$ [9]. Moreover, we denote $N_a = \{a, a+1, a+2, \ldots\}$ for each $\in R$. The readers are advised to consult the sources [9],[10] for further details on both the discrete fractional calculus and the nabla difference calculus.

1.2.1. Backward Difference

Let $u: N_a \to R$ be the first order backward (nabla) difference of u is defined as follows [9]:

$$(\nabla u)(t) = u(t) - u(t-1), t \in N_{a+1},$$

and by using the following notation, we defined the N^{th}-order nabla difference of u:

$$(\nabla^N y)(t) = \left(\nabla(\nabla^{N-1} u)\right)(t), t \in N_{a+N},$$

where $N \in N_1$.

1.2.1.1. Caputo Fractional Difference

For the function u define on N_a and α be between 0 and 1, the α^{th}-order Caputo-Fabrizio in the Caputo sensenabla difference of u is introduced by [11]:

$$({}^{CFC}\nabla_a^{\alpha} u)(t) = B(\alpha) \sum_{s=a+1}^{t} (\nabla u)(s)(1-\alpha)^{t-s}, t \in N_{a+1},$$

and the function $\alpha \to B(\alpha)$ is a normalization constant with $B(0) = B(1) = 1$ and $B(\alpha) > 0$.

1.2.1.2. Higher Order Fractional Difference

Let $n \leq \alpha \leq n+1$ and u define on N_{a-n} . The α-order Caputo-Fabrizio in the Caputo sense of u is given by [12]:

$$({}^{CFC}\nabla_a^{\alpha} u)(t) = ({}^{CFC}\nabla_a^{\alpha-n} \nabla^n u)(t), t \in N_{a+1}.$$

2. Monotonicity and Convexity

2.1. α-Monotonicity

In this section several results have been proved, which establish a linking between the sign of suitable Caputo-Fabrizio operator in the Caputo sense fractional nabla difference and the function's positivity on which it performs. Moreover, there are also deduced some results regarding what it is termed, a perception which is given in an article that Goodrich and Lizama recently published [13]. It starts out by defining α-monotone increasing concept.

2.1.1. α-Monotone Increasing

Let α be between 0 and 1. Then, the function u defined on N_a is called α-monotone increasing if

$$u(t) \geq \alpha u(t-1), t \in N_{a+1}.$$

Note that in the above definition if $\alpha = 1$, then it is obtained $u(t) \geq u(t-1)$ for $t \in N_{a+1}$, which is 1-monotone increasing and represents monotonicity in the usual sense. And if $\alpha = 0$, then it is acquired $u(t) \geq 0$, for $t \in N_{a+1}$. So, it shows that 0-monotone increasing merely indicate that u is non-negative on N_a.

Lemma 2.1. *Let the function* u *be defined on* N_a *and* $\alpha \in (0,1)$. *If*

$$({}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0, for all t \in N_{a+1},$$

and $u(a) \geq 0$, *then* u *is positive and* α*-monotone increasing on* N_a.

Proof.

$({}^{CFC}\nabla_a^{\alpha} u)(t)$

$$B(\alpha)\sum_{s=a+1}^{t}(\nabla u)(s)(1-\alpha)^{t-s}$$

$$B(\alpha)\left[\sum_{s=a+1}^{t}[u(s)-u(s-1)](1-\alpha)^{t-s}\right]$$

$$B(\alpha)\left[\sum_{s=a+1}^{t}u(s)(1-\alpha)^{t-s}-\sum_{s=a+1}^{t}u(s-1)(1-\alpha)^{t-s}\right]$$

$$B(\alpha)\left[\sum_{s=a+1}^{t}u(s)(1-\alpha)^{t-s}-\sum_{s=a}^{t-1}u(s)(1-\alpha)^{t-s-1}\right]$$

$$B(\alpha)\left[\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}+u(t)(1-\alpha)^{t-t}-\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s-1}-u(a)(1-\alpha)^{t-a-1}\right]$$

$$B(\alpha)\left[\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}+u(t)(1-\alpha)^{0}-\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s-1}-u(a)(1-\alpha)^{t-a-1}\right]$$

$$B(\alpha)\left[u(t)-u(a)(1-\alpha)^{t-a-1}+\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}-\sum_{s=a+1}^{t-1}\frac{u(s)(1-\alpha)^{t-s}}{(1-\alpha)}\right]$$

$$B(\alpha)\left[u(t)-u(a)(1-\alpha)^{t-a-1}+\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}\left(1-\frac{1}{1-\alpha}\right)\right]$$

$$B(\alpha)\left[u(t)-u(a)(1-\alpha)^{t-a-1}-\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}\right] \quad (2.1)$$

But$\left({}^{CFC}\nabla_a^{\alpha}u\right)(t)\geq 0, for all t\in N_{a+1}$, so we can write:

$$B(\alpha)\left[u(t)-u(a)(1-\alpha)^{t-a-1}-\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}\right]\geq 0,$$

$t\in N_{a+1}$

It is shown that:

$$u(t)\geq u(a)(1-\alpha)^{t-a-1}+\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s},$$

$t\in N_{a+1}$. (2.2)

Now, to show that u is positive, it is enough to show that $u(a+k)\geq 0 for any k\in N_0$, so we can use induction on k.

Taking $k=1$, which is $t=a+1$ in (2.2), and by setting $u(a)\geq 0$, we get:

$$u(a+1)\geq u(a)\geq 0.$$

Taking$k=2$,which is$t=a+2$ *in (3.2)* ,we have:

$$u(a+2) \geq u(a)(1-\alpha)^{a+2-a-1} + \frac{\alpha}{1-\alpha} \sum_{s=a+1}^{a+2-1} u(s)(1-\alpha)^{a+2-s}.$$

If we make it simpler*:*

$$u(a+2) \geq u(a)(1-\alpha) + \frac{\alpha}{1-\alpha} \sum_{s=a+1}^{a+1} u(s)(1-\alpha)^{a+2-s},$$

and by setting$u(a) \geq 0$, $u(a+1) \geq 0$and$0 < \alpha < 1$,we get

$$u(a+2) \geq u(a)(1-\alpha) + \alpha u(a+1) \geq 0.$$

From induction it is acquired $u(t) \geq 0,\ \forall t \in N_a$.
Now, to demonstrate u is α-monotone increasing on N_a. Arranging differently the terms in (2.2), we get

$$u(t) \geq \alpha u(t-1) + u(a)(1-\alpha)^{t-a-1} + \frac{\alpha}{1-\alpha} \sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s},$$

$t \in N_{a+1}$. (2.3)

since $u(t) \geq 0$, for all $t \in N_{a+1}$, from (2.3) we get

$$u(t) \geq \alpha u(t-1), t \in N_{a+1}.$$

By getting that u is α -monotone increasing on N_a. The proof is complete.

Theorem 2.1. *Let the function u be defined on$N_a$and$\alpha, \beta \in (0,1)$,such that$0 < \alpha + \beta \leq 1$.If*

$$\left({}^{CFC}\nabla_{a+1}^{\beta}\ {}^{CFC}\nabla_a^{\alpha} u \right)(t) \geq 0, for all t \in N_{a+2},$$

and $u(a+1) \geq u(a) \geq 0$, then u is positive and $\alpha + \beta$-monotone increasing on N_a.

Proof. Let

$$({}^{CFC}\nabla_a^{\alpha} u)(t) = f(t), for all t \in N_{a+1}.$$

So we can write

$$\left({}^{CFC}\nabla_{a+1}^{\beta}\ {}^{CFC}\nabla_a^{\alpha} u \right)(t) = \left({}^{CFC}\nabla_{a+1}^{\beta} f \right)(t).$$

Distinctly, by assumption, $\left({}^{CFC}\nabla_{a+1}^{\beta} f \right)(t), for all t \in N_{a+2}$. By definition of Caputo fractional difference we possess:

$$f(a+1) = ({}^{CFC}\nabla_a^{\alpha} u)(a+1) = B(\alpha) \sum_{s=a+1}^{a+1} (\nabla u)(s)(1-\alpha)^{a+1-s}$$
$$= B(\alpha)(\nabla u)(a+1)(1-\alpha)^{a+1-(a+1)} = B(\alpha)(\nabla u)(a+1) \geq 0.$$

According to Lemma 2.1, f is positive and β-monotone increasing on N_{a+1}.
So

$$f(t) \geq 0, forallt \in N_{a+1},$$

$$f(t) \geq \beta f(t-1), forallt \in N_{a+2}, (2.4)$$

since

$$f(t) = (\ {}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0, forallt \in N_{a+1},$$

and $u(a) \geq 0$, again from Lemma 2.1, u is positive and α-monotone increasing on N_a. That is,

$$u(t) \geq 0, forallt \in N_a$$

and

$$u(t) \geq \alpha u(t-1), t \in N_{a+1}. (2.5)$$

Now, from (2.4) it holds

$$f(t) \geq \beta f(t-1), forallt \in N_{a+2}.$$

It is shown that

$$0 \leq f(t) - \beta f(t-1) = (\ {}^{CFC}\nabla_a^{\alpha} u)(t) - \beta(\ {}^{CFC}\nabla_a^{\alpha} u)(t-1)$$

and by applying (2.1) we get:

$$B(\alpha)\left[u(t) - u(a)(1-\alpha)^{t-a-1} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1} u(s)(1-\alpha)^{t-s}\right]$$
$$- B(\alpha)\beta\left[u(t-1) - u(a)(1-\alpha)^{t-a-2} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s-1}\right]$$
$$B(\alpha)\left[u(t) - u(a)(1-\alpha)^{t-a-1} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s} - \alpha u(t-1) - \beta u(t-1)\right.$$
$$\left. + \beta u(a)(1-\alpha)^{t-a-2} + \beta\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s-1}\right]$$
$$B(\alpha)\left[u(t) - \beta u(t-1) - u(a)(1-\alpha)^{t-a-1} + \beta u(a)(1-\alpha)^{t-a-2} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s}\right.$$
$$\left. - \alpha u(t-1) + \beta\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s-1}\right]$$
$$B(\alpha)\left[u(t) - \beta u(t-1) - u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)\right.$$
$$\left. - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)[(1-\alpha)^{t-s} - \beta(1-\alpha)^{t-s-1}] - \alpha u(t-1)\right]$$

$$B(\alpha)\left[u(t)-\beta u(t-1)-u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)-\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2}u(s)(1-\alpha)^{t-s-1}(1-\alpha-\beta)\right.$$
$$\left.-\alpha u(t-1)\right]$$
$$B(\alpha)\left[u(t)-\beta u(t-1)-\alpha u(t-1)-u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)\right.$$
$$\left.-\frac{\alpha(1-\alpha-\beta)}{(1-\alpha)^2}\sum_{s=a+1}^{t-2}u(s)(1-\alpha)^{t-s}\right], (2.6)$$

Since $B(\alpha) > 0$, from (2.6) it is gained:

$$u(t)-\beta u(t-1)-\alpha u(t-1)-u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)-\frac{\alpha(1-\alpha-\beta)}{(1-\alpha)^2}\sum_{s=a+1}^{t-2}u(s)(1-\alpha)^{t-s}$$
$$\geq 0, t \in N_{a+2}.$$

The following is also considered

$$u(t)-\beta u(t-1)-\alpha u(t-1) \geq u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)+\frac{\alpha(1-\alpha-\beta)}{(1-\alpha)^2}\sum_{s=a+1}^{t-2}u(s)(1-\alpha)^{t-s}, t$$
$$\in N_{a+2}. (2.7)$$

Since $0 < \alpha \leq 1, 0 < \alpha + \beta \leq 1$, and $(t) \geq 0$, from (2.7) we can write:

$$u(t)-\beta u(t-1)-\alpha u(t-1) \geq 0, t \in N_{a+2}.$$

Also we get:

$$u(t) \geq (\beta+\alpha)u(t-1), t \in N_{a+2}.$$

Hence the proof is completed.

Theorem 2.2. *Let the function* $\boldsymbol{u}$ *is defined on* $\boldsymbol{N_a}$ *and* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbf{0}, \mathbf{1})$*, such that* $\mathbf{0} < \boldsymbol{\alpha} + \boldsymbol{\beta} \leq \mathbf{1}$*. If*

$$\left({}^{CFC}\nabla_{a+1}^{\beta} \; {}^{CFC}\nabla_a^{\alpha} u\right)(t) \geq 0, forallt \in N_{a+2}.$$

and $u(a+1) \geq u(a) \geq 0$*, then u is positive and* α*-monotone increasing on* N_a*.*

Proof. It is demonstrated by the theorem mentioned above.

2.2. Monotonicity and α-Convexity

Once more, it was during a conference of monotonicity-type results. For both non-sequential and sequential Caputo-Fabrizio, fractional differences are found in the Caputo sense. It begins with two fundamental lemmas.

Lemma 2.2. *Let the function* u *be defined on* N_a *and* $\alpha \in (0,1)$*. If*

$$\nabla({}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0, forallt \in N_{a+2},$$

and $(\nabla u)(a+1) \geq 0$*, then*

$$(\nabla u)(t) \geq 0, forallt \in N_{a+1}.$$

Proof. First, by definition of Caputo fractional difference, for all $t \in N_{a+2}$ we have

$$\nabla ({}^{CFC}\nabla_a^\alpha u)(t) = \nabla\left[B(\alpha)\sum_{s=a+1}^{t}(\nabla u)(s)(1-\alpha)^{t-s}\right]$$
$$B(\alpha)\left[\sum_{s=a+1}^{t}(\nabla u)(s)(1-\alpha)^{t-s} - \sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-1-s}\right]$$
$$B(\alpha)\left[\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s} + (\nabla u)(t) - \sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-1-s}\right]$$
$$B(\alpha)\left[(\nabla u)(t) + \sum_{s=a+1}^{t-1}(\nabla u)(s)((1-\alpha)^{t-s} - (1-\alpha)^{t-1-s})\right]$$
$$B(\alpha)\left[(\nabla u)(t) - \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s}\right], (2.8)$$

Since $\nabla({}^{CFC}\nabla_a^\alpha u)(t) \geq 0$, we have

$$B(\alpha)\left[(\nabla u)(t) - \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s}\right] \geq 0$$

and since $B(\alpha) \geq 0$, it is written:

$$(\nabla u)(t) - \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s} \geq 0.$$

Also it has the same meaning with this:

$$(\nabla u)(t) \geq \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s}. (2.9)$$

Now, to show that $(\nabla u)(t) \geq 0$ for all $t \in N_{a+1}$, it is enough to show that $(\nabla u)(a+k) \geq 0, \forall k \in N_1$. We use induction on k. $(\nabla u)(a+1) \geq 0$ is given. Taking $t = a+2$ in (2.9) , we have:

$$(\nabla u)(a+2) \geq \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{a+1}(\nabla u)(s)(1-\alpha)^{a+2-s}$$

By simplifying it, we get:

$$(\nabla u)(a+2) \geq \alpha(\nabla u)(a+1) \geq 0.$$

Taking $t = a+3$ in (2.9), it becomes

$(\nabla u)(a+3) \geq \alpha(\nabla u)(a+2) + (\nabla u)(a+1)(1-\alpha) \geq 0.$

Following the same procedure, it is obtained $(\nabla u)(t) \geq 0$, for all $t \in N_{a+1}$. Hence the proof has been established.

Lemma 2.3. *Let the function u be defined on N_a and $\alpha \in (0,1)$. If*

$$\left({}^{CFC}\nabla_{a+1}^{\alpha}\nabla u\right)(t) \geq 0, for all t \in N_{a+2},$$

and $(\nabla u)(a+1) \geq 0$, *then*

$$(\nabla u)(t) \geq 0, for all t \in N_{a+1}.$$

Proof. By definition of Caputo fractional difference, for all $t \in N_{a+2}$, we have

$$\begin{aligned}
\left({}^{CFC}\nabla_{a+1}^{\alpha}\nabla u\right)(t) &= B(\alpha)\sum_{s=a+2}^{t}(\nabla^2 u)(s)(1-\alpha)^{t-s} \\
& B(\alpha)\left[\sum_{s=a+2}^{t}(\nabla u)(s)(1-\alpha)^{t-s} - \sum_{s=a+2}^{t}(\nabla u)(s-1)(1-\alpha)^{t-s}\right] \\
& B(\alpha)\left[\sum_{s=a+2}^{t}(\nabla u)(s)(1-\alpha)^{t-s} - \sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s-1}\right] \\
B(\alpha)&\left[\sum_{s=a+2}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s} + (\nabla u)(t) - \sum_{s=a+2}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s-1} - (\nabla u)(a+1)(1-\alpha)^{t-a-2}\right] \\
& B(\alpha)\left[(\nabla u)(t) + \sum_{s=a+2}^{t-1}(\nabla u)(s)[(1-\alpha)^{t-s} - (1-\alpha)^{t-s-1}] - (\nabla u)(a+1)(1-\alpha)^{t-a-2}\right] \\
& B(\alpha)\left[(\nabla u)(t) - \frac{\alpha}{1-\alpha}\sum_{s=a+2}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s} - (\nabla u)(a+1)(1-\alpha)^{t-a-2}\right], (2.10)
\end{aligned}$$

nevertheless, because $B(\alpha) \geq 0$, and $\left({}^{CFC}\nabla_{a+1}^{\alpha}\nabla u\right)(t) \geq 0, for all t \in N_{a+2}$, so from (2.10), we get

$$(\nabla u)(t) \geq \frac{\alpha}{1-\alpha}\sum_{s=a+2}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s} + (\nabla u)(a+1)(1-\alpha)^{t-a-2}. (2.11)$$

Now, to show that $(\nabla u)(t) \geq 0$ for all $t \in N_{a+1}$, it is enough to show that $(\nabla u)(a+k) \geq 0$, for each $k \in N_1$. We apply induction on k. Given that $(\nabla u)(a+1) \geq 0$. Using $t = a+2$ in (2.11) , it has got:

$$(\nabla u)(a+2) \geq (\nabla u)(a+1) \geq 0$$

Taking $t = a+3$ in (2.9), there we get

$(\nabla u)(a+3) \geq \alpha(\nabla u)(a+2) + (\nabla u)(a+1)(1-\alpha) \geq 0.$

Continuing in this way, we arrive at $(\nabla u)(t) \geq 0$, for all $t \in N_{a+1}$. The proof is completed.

Lemma 2.4. *Let the function* u *be defined on* N_a , $\mu \in (0,1)$ *and* $\nu \epsilon (1,2)$. *If*

$$\left({}^{CFC}\nabla_{a+2}^{\nu}\ {}^{CFC}\nabla_{a}^{\mu} u\right)(t) \geq 0, for all t \in N_{a+3},$$

and $(\nabla u)(a+2) \geq (\nabla u)(a+1) \geq 0$, *then*

$$(\nabla u)(t) \geq 0, for all t \in N_{a+1}.$$

Proof. Let

$$\left({}^{CFC}\nabla_a^{\mu} u\right)(t) = v(t), for all t \in N_{a+1}.$$

Consider

$$\left({}^{CFC}\nabla_{a+2}^{\nu} v\right)(t) = \left({}^{CFC}\nabla_{a+2}^{\nu-1} \nabla v\right)(t)$$

and given that $\left({}^{CFC}\nabla_{a+2}^{\nu-1} \nabla v\right)(t) \geq 0, for all t \in N_{a+3}$, from (2.8) we have:

$$\nabla v(a+2) = \left(\nabla \middle| \ \middle| CFC\nabla_a^{\mu} u\right)(a+2) = B(\mu)\Bigg[(\nabla u)(a+2) \frac{-\mu}{1-\mu}\sum_{s=a+1}^{a+1}(\nabla u)(s)(1-\mu)^{a+2-s}\Bigg]$$
$$B(\mu)[(\nabla u)(a+2) - \mu(\nabla u)(a+1)]$$
$$B(\mu)[(\nabla u)(a+2) - (\nabla u)(a+1)] \geq 0$$

Then, from Lemma 2.3, we have:

$$\nabla(v)(t) = \left(\nabla \middle| \ \middle| CFC\nabla_{a+1}^{\mu} u\right)(t) \geq 0, (2.12)$$

and since $(\nabla u)(a+1) \geq 0$, from Lemma 2.2, we get

$$\nabla(u)(t) \geq 0, for all t \in N_{a+1}.$$

So the proof is finished.

Lemma 2.5. *Let the function u be defined on N_a , $\mu \in (1,2)$ and $\nu \epsilon (0,1)$. If*

$$\left({}^{CFC}\nabla_{a+2}^{\nu}\ {}^{CFC}\nabla_{a+1}^{\mu} u\right)(t) \geq 0, for all t \in N_{a+3},$$

and $(\nabla u)(a+2) \geq (\nabla u)(a+1) \geq 0$, *then*

$$(\nabla u)(t) \geq 0, for all t \in N_{a+1}.$$

Proof. Let

$$\left({}^{CFC}\nabla_{a+1}^{\mu} u\right)(t) = v(t), for all t \in N_{a+1}.$$

Consider

$$\left({}^{CFC}\nabla_{a+2}^{\nu} v\right)(t) = \left({}^{CFC}\nabla_{a+2}^{\nu-1} v\right)(t)$$

and given that $\left({}^{CFC}\nabla_{a+2}^{\nu} v\right)(t) \geq 0, for all t \in N_{a+3}$, from (2.10) we have:

$$v(a+2) = \left({}^{CFC}\nabla_{a+1}^{\mu} u\right)(a+2) = \left({}^{CFC}\nabla_{a+1}^{\mu-1} \nabla u\right)(a+2) = B(\mu-1)\Bigg[(\nabla u)(a+2) \frac{-\mu-1}{2-\mu}\sum_{s=a+1}^{a+1}(\nabla u)(s)(2-\mu)^{a+2-s} - (\nabla u)(a+1)(2-\mu)^{a+2-a-2}\Bigg]$$
$$B(\mu-1)[(\nabla u)(a+2) - \mu(\nabla u)(a+1)]$$
$$B(\mu-1)[(\nabla u)(a+2) - (\nabla u)(a+1)] \geq 0.$$

Then, from Lemma 2.2 we have

$$v(t) \geq 0, for all t \in N_{a+2},$$

which is the same with

$$\left({}^{CFC}\nabla_{a+1}^{\mu} u\right)(t) \geq 0, forallt \in N_{a+2}, (2.13)$$

Thus, we obtain

$$0 \leq \left({}^{CFC}\nabla_{a+1}^{\mu} u\right)(t) = \left({}^{CFC}\nabla_{a+1}^{\mu-1} \nabla u\right)(t), forallt \in N_{a+2},$$

and since $(\nabla u)(a+1) \geq 0$, from Lemma 2.3, we get

$$\nabla(u)(t) \geq 0, forallt \in N_{a+1}.$$

So the proof is completed.

2.2.1. α- Convex

Let $\alpha \in (1,2)$.It is mentioned that a function u is defineed on N_a is called α-convex if

$$u(t) - \alpha u(t-1) + (\alpha-1)u(t-2) \geq 0, t \in N_{a+2}$$

Lemma 2.6. *Let the function* u *is defined on* N_a *and* $\alpha \in (1,2)$. *If*

$$\left({}^{CFC}\nabla_{a+1}^{\alpha} u\right)(t) \geq 0, forallt \in N_{a+2},$$

and

$$u(a+1) \geq u(a) \geq 0.$$

Then u *is monotone increasing and positive on* N_a. *Furthermore, u is* α*-convex on* N_a.

Proof. From the properties we can say

$$0 \leq \left({}^{CFC}\nabla_{a+1}^{\alpha} u\right)(t) = \left({}^{CFC}\nabla_{a+1}^{\alpha-1} \nabla u\right)(t), forallt \in N_{a+2}.$$

and since $\nabla u(a+1) = u(a+1) - u(a) \geq 0$, from Lemma 2.3 it follows that

$$(\nabla u)(t) \geq 0, forallt \in N_{a+1},$$

which means that u is monotone increasing and positive on N_a.

The next demonstration is that u is α-convex. From (2.11)

$$(\nabla u)(t) \geq \frac{\alpha-1}{2-\alpha} \sum_{s=a+2}^{t-1} (\nabla u)(s)(2-\alpha)^{t-s} + (\nabla u)(a+1)(2-\alpha)^{t-a-2}$$

$$(\alpha-1)(\nabla u)(t-1) + \frac{\alpha-1}{2-\alpha} \sum_{s=a+2}^{t-2} (\nabla u)(s)(2-\alpha)^{t-s} + (\nabla u)(a+1)(2-\alpha)^{t-a-2}, (2.14)$$

Since $\alpha \in (1,2)$ and $(\nabla u)(t) \geq 0$,for all $t \in N_{a+1}$, from (2.14) it is written

$$(\nabla u)(t) \geq (\alpha-1)(\nabla u)(t-1).$$

If it is simplified, it will result the following :

$$u(t) - u(t-1) \geq (\alpha-1)[u(t-1) - u(t-2)].$$

Alsoit results in:

$$u(t) - \alpha u(t-1) + (\alpha - 1)u(t-2) \geq 0.$$

It is indicated that u is α-convex, hence the proof is completed.

2.3. Convexity

Some convexity-type results are reported in this section. The conclusion is that there is a link between the sign of the non-sequential difference $({}^{CFC}\nabla^{\alpha}_{a+2}u)(t) \geq 0$, and the following lemma, and the convexity of u. This is basically the type of result assumed in Goodrich [6] as well as Jia, Erbe, and Peterson [14].

Lemma 2.7. *Let $\alpha \in (2,3)$ and the function u be defined on N_a. If*

$$({}^{CFC}\nabla^{\alpha}_{a+2}u)(t) \geq 0, for all t \in N_{a+3}$$

and

$$(\nabla^2 u)(a+2) \geq 0$$

Then, u is convex on N_{a+2}.

Proof. We start with the following substitution

$$(\nabla u)(t) = v(t), for all t \in N_{a+1}.$$

Consider

$$({}^{CFC}\nabla^{\alpha}_{a+2}u)(t) = ({}^{CFC}\nabla^{\alpha-2}_{a+2}\nabla^2 u)(t) = ({}^{CFC}\nabla^{\alpha}_{a+2}\nabla v)(t)$$

Given that$({}^{CFC}\nabla^{\alpha}_{a+2}\nabla v)(t) \geq 0, for all t \in N_{a+3}$. *Since*

$$(\nabla v)(a+2) = (\nabla^2 u)(a+2) \geq 0.$$

from Lemma 2.3., it follows that

$$(\nabla v)(t) = (\nabla^2 u)(t) \geq 0.$$

which is

$$\nabla[u(t) - u(t-1)] \geq 0.$$

Also, there we have

$$[u(t) - u(t-1)] - [u(t-1) - u(t-2)] \geq 0.$$

Now, we get

$$u(t) - 2u(t-1) + u(t-2) \geq 0, for all t \in N_{a+2}.$$

The proof is completed.

3. Conclusion

In this paper, it is investigated some positivity, monotonicity, and convexity results for discrete Caputo-Fabrizio fractional operators in the context of discrete fractional calculus. Also, it is considered the connections of these results to the non-negativity of both non-sequential and sequential Caputo-Fabrizio fractional differences of Caputo type. Finally, it is found that there are some significant dissimilarities between this type of fractional difference and, for instance, the more well-known Riemann-Liouville type.

4. References

1. Kilbas, Anatoliĭ Aleksandrovich, Hari M. Srivastava, and Juan J. Trujillo. *Theory and applications of fractional differential equations*. Vol. 204. Elsevier, 2006.

2. Magin, R. "Begell House Inc." *Fractional Calculus in Bioengineering* (2006).

3. Podlubny, I. "Fractional Differential Equations Academic Press, San Diego, 1999.".

4. Smit, W., and H. De Vries. "Rheological models containing fractional derivatives." *Rheologica Acta* 9, no. 4 (1970): 525-534.

5. Dahal, Rajendra, and Christopher S. Goodrich. "A monotonicity result for discrete fractional difference operators." *Archiv der Mathematik* 102, no. 3 (2014): 293-299.

6. Goodrich, Christopher S. "A convexity result for fractional differences." *Applied Mathematics Letters* 35 (2014): 58-62.

7. Jamison, C. E., R. D. Marangoni, and A. A. Glaser. "Viscoelastic properties of soft tissue by discrete model characterization." (1968): 239-247.

8. Coussot, Cecile. "Fractional derivative models and their use in the characterization of hydropolymer and in-vivo breast tissue viscoelasticity." (2008).

9. Kelley, Walter G., and Allan C. Peterson. *Difference equations: an introduction with applications*. Academic press, 2001.

10. Goodrich, Christopher, and Allan C. Peterson. *Discrete fractional calculus*. Vol. 1350. Berlin: Springer, 2015.

11. Abdeljawad, Thabet, and Dumitru Baleanu. "Monotonicity results for fractional difference operators with discrete exponential kernels." *Advances in Difference Equations* 2017, no. 1 (2017): 1-9.

12. Abdeljawad, Thabet, Qasem M. Al-Mdallal, and Mohamed A. Hajji. "Arbitrary order fractional difference operators with discrete exponential kernels and applications." *Discrete Dynamics in Nature and Society* 2017 (2017).

13. Goodrich, Christopher, and Carlos Lizama. "A transference principle for nonlocal operators using a convolutional approach: Fractional monotonicity and convexity." *Israel Journal of Mathematics* 236, no. 2 (2020): 533-589.

14. Baoguo, Jia, Lynn Erbe, and Allan Peterson. "Convexity for nabla and delta fractional differences." *Journal of Difference Equations and Applications* 21, no. 4 (2015): 360-373.

15. Abdeljawad, Thabet. "On Riemann and Caputo fractional differences." *Computers & Mathematics with Applications* 62, no. 3 (2011): 1602-1611.

16. Abdeljawad, Thabet, and Dumitru Baleanu. "Monotonicity results for fractional difference operators with discrete exponential kernels." *Advances in Difference Equations* 2017, no. 1 (2017): 1-9.

17. Anastassiou, George A. "Nabla discrete fractional calculus and nabla inequalities." *Mathematical and Computer Modelling* 51, no. 5-6 (2010): 562-571.

18. Ferreira, Rui AC. "A discrete fractional Gronwall inequality." *Proceedings of the American Mathematical Society* (2012): 1605-1612.

19. Ferreira, Rui AC. "Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one." *Journal of Difference Equations and Applications* 19, no. 5 (2013): 712-718.

20. Dahal, Rajendra, and Christopher S. Goodrich. "A uniformly sharp convexity result for discrete fractional sequential differences." *Rocky Mountain Journal of Mathematics* 49, no. 8 (2019): 2571-2586.

21. Jia, Baoguo, Lynn Erbe, and Allan Peterson. "Some relations between the Caputo fractional difference operators and integer-order differences." (2015).

22. Baoguo, Jia, Lynn Erbe, and Allan Peterson. "Monotonicity and convexity for nabla fractional q-differences." *Dynam. Systems Appl* 25, no. 1-2 (2016): 47-60.

23. Atici, Ferhan M., and Paul Eloe. "Discrete fractional calculus with the nabla operator." *Electronic Journal of Qualitative Theory of Differential Equations [electronic only]* 2009 (2009): Paper-No.

24. Anastassiou, George A. "Nabla discrete fractional calculus and nabla inequalities." *Mathematical and Computer Modelling* 51, no. 5-6 (2010): 562-571.