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Some identities involving degenerate Cauchy numbers and polynomials of the fourth kind

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Abstract

In this paper, we study the constant equations associated with the degenerate Cauchy polynomials of the fourth kind using the generating function and Riordan array. By using the generating function method and the Riordan array method, we establish some new constants between the degenerate Cauchy polynomials of the fourth kind and two types of Stirling numbers, Lab numbers, two types of generalized Bell numbers, Daehee numbers, Bernoulli numbers and polynomials.

Keywords

Cauchy polynomials; Generating functions; Riordan matrix; Degenerate Cauchy polynomials of the fourth kind; Stirling numbers; Lab numbers; Bell numbers; Daehee numbers.

1. Introduction

Recently, studying the degenerate forms of various special polynomials has become an active research field and has yielded many new combinatorial results; see, for instance, [2, 4, 5, 6]. We recall the generating function of C_n is $\sum_{n=0}^{\infty} C_n(x)\frac{t^n}{n!} = \frac{t}{\ln(1+t)}(1+t)^x$. In this paper, we discuss the generating function of the degenerate Cauchy polynomial of the fourth kind $C_{n,\lambda,4}(x)$. We refer to Pyo S S[2] for this topic. The definition of $C_{n,\lambda,4}(x)$ is

$$\sum_{n=0}^{\infty} C_{n,\lambda,4}(x)\frac{t^n}{n!} = \frac{\lambda t}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}, \tag{1}$$

From the generating function of $C_{n,\lambda,4}(x)$, we know that $C_{n,\lambda,4}(0) = C_{n,\lambda,4}$.$C_{n,\lambda,4}$ are called degenerate Cauchy numbers of the fourth kind.

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For convenience, let us recall some definitions. we give the definitions of the generating functions of several combinatorial sequences used in this paper[2, 3, 4, 5, 6, 7, 8, 9].

The generating function of the Cauchy number of the second kind is defined as follows:

$$\sum_{n=0}^{\infty} C_n^* \frac{t^n}{n!} = \frac{-t}{(1-t)\ln(1-t)}, \tag{2}$$

The generating function of the degenerate Cauchy polynomial of the first kind is defined as follows:

$$\sum_{n=0}^{\infty} C_{n,\lambda}(x) \frac{t^n}{n!} = \frac{\frac{1}{\lambda}\ln(1+\lambda t)}{\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))}(1+\frac{1}{\lambda}\ln(1+\lambda t))^x, \tag{3}$$

The generating function of the degenerate Cauchy polynomial of the second kind is defined as follows:

$$\sum_{n=0}^{\infty} C_{n,\lambda,2}(x) \frac{t^n}{n!} = \frac{t}{\ln(1+\frac{1}{\lambda}\ln(1+\lambda t))}(1+\frac{1}{\lambda}\ln(1+\lambda t))^x, \tag{4}$$

The generating function of the degenerate Cauchy polynomial of the third kind is defined as follows:

$$\sum_{n=0}^{\infty} C_{n,\lambda,3}(x) \frac{t^n}{n!} = \frac{\lambda((1+\lambda\ln(1+t))^{\frac{1}{\lambda}}-1)}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}, \tag{5}$$

The generating function of Stirling numbers of the first kind is defined as follows:

$$\sum_{n=k}^{\infty} s(n,k) \frac{t^n}{n!} = \frac{(\ln(1+t))^k}{k!}, \tag{6}$$

The generating function of unsigned Stirling numbers of the first kind is defined as follows:

$$\sum_{n=k}^{\infty} |s(n,k)| \frac{t^n}{n!} = \frac{(\ln\frac{1}{1-t})^k}{k!}, \tag{7}$$

The generating function of Stirling numbers of the second kind is defined as follows:

$$\sum_{n=k}^{\infty} S(n,k) \frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}, \tag{8}$$

The generating function of degenerate Stirling numbers of the first kind is defined as follows:

$$\sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!} = \frac{(\frac{1}{\lambda}[(1+t)^{\lambda}-1])^k}{k!}, \tag{9}$$

The generating function of degenerate Stirling numbers of the second kind is defined as follows:

$$\sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!} = \frac{((1+\lambda t)^{\frac{1}{\lambda}}-1)^k}{k!}, \tag{10}$$

The generating function of generalized Stirling numbers of the second kind is defined as follows:

$$\sum_{n=k}^{\infty} S_2(n,k,r) \frac{t^n}{n!} = e^{rt}\frac{(e^t-1)^k}{k!}, \tag{11}$$

The generating function of generalized Bell numbers of the first kind is defined as follows:

$$\sum_{n=k}^{\infty} B(n,k)\frac{t^n}{n!} = \frac{(e^{e^t-1}-1)^k}{k!}, \tag{12}$$

The generating function of generalized Bell numbers of the second kind is defined as follows:

$$\sum_{n=k}^{\infty} \beta(n,k)\frac{t^n}{n!} = \frac{(\ln(1+(\ln(1+t)))^k}{k!}, \tag{13}$$

The generating function of Lah numbers is defined as follows:

$$\sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!} = \frac{1}{k!}(\frac{-t}{1+t})^k, \tag{14}$$

The generating function of the classical Daehee polynomials is defined as follows:

$$\sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} = \frac{\ln(1+t)}{t}(1+t)^x, \tag{15}$$

The generating function of λ-Daehee numbers is defined as follows:

$$\sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!} = \frac{\lambda\ln(1+t)}{(1+t)^\lambda-1}, \tag{16}$$

The generating function of degenerate Daehee numbers is defined as follows:

$$\sum_{n=0}^{\infty} d_{n,\lambda}\frac{t^n}{n!} = \frac{\ln(1+t)}{\ln(1+\lambda t)^{\frac{1}{\lambda}}}, \tag{17}$$

The generating function of completely degenerate Daehee numbers is defined as follows:

$$\sum_{n=0}^{\infty} d^*_{n,\lambda}\frac{t^n}{n!} = \frac{(1+t)^\lambda-1}{\ln(1+\lambda t)}, \tag{18}$$

The generating function of degenerate Bernoulli numbers is defined as follows:

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}\frac{t^n}{n!} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}, \tag{19}$$

A *Riordan array* [10] is a pair $(g(t), h(t))$ of formal power series with $h_0 = h(0) = 0$. It defines an infinite lower triangular array$(d_{n,k})_{n,k\in N}$ according to the rule

$$d_{n,k} = [t^n]g(t)(h(t))^k .$$

Hence we write $\{d_{n,k}\} = (g(t), h(t))$. Moreover, if $(g(t), h(t))$ is a Riordan array and $f(t)$ is the generating function of the sequence $\{f_k\}_{k\in N}$, i.e., $f(t) = \sum_{k=0}^{\infty} f_k t^k$, then we have

$$\sum_{k=0}^{\infty} d_{n,k}f_k = [t^n]g(t)f(h(t)) = [t^n]g(t)[f(y) \mid y = h(t)], \tag{20}$$

Furthermore, we give the inverse form of the Stiring number. Let f, g be functions defined on the set of positive integers, then

$$g_n = \sum_{k=0}^{n} s(n,k) f_k \iff f_n = \sum_{k=0}^{n} S(n,k) g_k, \tag{21}$$

$$g_n = \sum_{k=0}^{n} S_1(n,k;r) f_k \iff f_n = \sum_{k=0}^{n} S_2(n,k;r) g_k, \tag{22}$$

2. The relationship between the degenerate Cauchy polynomials of the fourth kind and some combination numbers and their polynomials

First, the degenerate Cauchy polynomials of the fourth kind are represented by the generating function method with some combinatorial numbers and a constant equation between the polynomials.

Theorem 1. *Let $n \geq 0$ be integers. Then*

$$C_{n,\lambda,4}(x) = \sum_{r+h+l=0}^{n} \sum_{k=0}^{h} \binom{n}{r,h,l} C_{r,\lambda,3}(x) \beta_{k,\lambda} s(h,k) C_l. \tag{23}$$

Proof. By applying (5), (6) and (19), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{n,\lambda,4}(x) \frac{t^n}{n!} = \frac{\lambda t}{\ln(1+\lambda \ln(1+t))} (1+\lambda \ln(1+t))^{\frac{x}{\lambda}} \\
&= \frac{\lambda((1+\lambda\ln(1+t))^{\frac{1}{\lambda}}-1)}{\ln(1+\lambda\ln(1+t))} \frac{\ln(1+t)}{(1+\lambda\ln(1+t))^{\frac{1}{\lambda}}-1} \frac{t}{\ln(1+t)} (1+\lambda\ln(1+t))^{\frac{x}{\lambda}} \\
&= \left(\sum_{n=0}^{\infty} C_{n,\lambda,3}(x) \frac{t^n}{n!}\right) \left(\sum_{k=0}^{\infty} \beta_{k,\lambda} \frac{\ln(1+t))^k}{k!}\right) \left(\sum_{n=0}^{\infty} C_n \frac{t^n}{n!}\right) \\
&= \left(\sum_{n=0}^{\infty} C_{n,\lambda,3}(x) \frac{t^n}{n!}\right) \left(\sum_{k=0}^{\infty} \beta_{k,\lambda} \sum_{n=k}^{\infty} s(n,k) \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} C_n \frac{t^n}{n!}\right) \\
&= \left(\sum_{n=0}^{\infty} C_{n,\lambda,3}(x) \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta_{k,\lambda} s(n,k) \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} C_n \frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \sum_{r+h+l=0}^{n} \sum_{k=0}^{h} \binom{n}{r,h,l} C_{r,\lambda,3}(x) \beta_{k,\lambda} s(h,k) C_l \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. □

Corollary 1. *The following relations hold:*

$$C_{n,\lambda,4} = \sum_{r+h+l=0}^{n} \sum_{k=0}^{h} \binom{n}{r,h,l} C_{r,\lambda,3} \beta_{k,\lambda} s(h,k) C_l. \tag{24}$$

Proof. Setting $x = 0$ in (23), we get (24). □

Theorem 2. *Let $n \geq 0$ be integers. Then*

$$C_{n,\lambda,4}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{n}{k}\binom{j}{l}C^*_{j-l}(\frac{x+\lambda}{\lambda})_j(-1)^{j-l}\lambda^j s(k,j)C_{n-k}. \tag{25}$$

Proof. By applying (2), (6), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}C_{n,\lambda,4}(x)\frac{t^n}{n!} = \frac{\lambda t}{\ln(1+\lambda\ln(1+t))}(1+\lambda\ln(1+t))^{\frac{x}{\lambda}}\\
&= \frac{\lambda\ln(1+t)}{(1+\lambda\ln(1+t))\ln(1+\lambda\ln(1+t))}\frac{t}{\ln(1+t)}(1+\lambda\ln(1+t))^{\frac{x+\lambda}{\lambda}}\\
&= \left(\sum_{j=0}^{\infty}C^*_j(-\lambda)^j\frac{(\ln(1+t))^j}{j!}\right)\left(\sum_{j=0}^{\infty}(\frac{x+\lambda}{\lambda})_j\lambda^j\frac{(\ln(1+t))^j}{j!}\right)\left(\sum_{n=0}^{\infty}C_n\frac{t^n}{n!}\right)\\
&= \left(\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}C^*_{j-l}(\frac{x+\lambda}{\lambda})_j(-\lambda)^{j-l}\lambda^l\frac{(\ln(1+t))^j}{j!}\right)\left(\sum_{n=0}^{\infty}C_n\frac{t^n}{n!}\right)\\
&= \left(\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{j}{l}C^*_{j-l}(\frac{x+\lambda}{\lambda})_j(-1)^{j-l}\lambda^j\sum_{n=j}^{\infty}s(n,j)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}C_n\frac{t^n}{n!}\right)\\
&= \left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{j}{l}C^*_{j-l}(\frac{x+\lambda}{\lambda})_j(-1)^{j-l}\lambda^j s(n,j)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}C_n\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{n}{k}\binom{j}{l}C^*_{j-l}(\frac{x+\lambda}{\lambda})_j(-1)^{j-l}\lambda^j s(k,j)C_{n-k}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. □

Corollary 2. *The following relations hold:*

$$C_{n,\lambda,4} = \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{n}{k}\binom{j}{l}C^*_{j-l}(1)_j(-1)^{j-l}\lambda^j s(k,j)C_{n-k}. \tag{26}$$

Proof. Setting $x=0$ in(25), we get (26). □

Theorem 3. *Let $n \geq 0$ be integers. Then*

$$C_{n,\frac{1}{\lambda},4}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\lambda^{k-n-j}(\lambda x)_j C_{n-k,\lambda,2}s(k,j). \tag{27}$$

Proof. By applying (4), (6), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}C_{n,\frac{1}{\lambda},4}(x)\frac{t^n}{n!} = \frac{\frac{1}{\lambda}t}{\ln(1+\frac{1}{\lambda}\ln(1+t))}(1+\frac{1}{\lambda}\ln(1+t))^{\lambda x}\\
&= \sum_{n=0}^{\infty}\frac{C_{n,\lambda,2}}{\lambda^n}\frac{t^n}{n!}\sum_{j=0}^{\infty}(\lambda x)_j\lambda^{-j}\frac{(\ln(1+t))^j}{j!}
\end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} \frac{C_{n,\lambda,2}}{\lambda^n} \frac{t^n}{n!}\right) \left(\sum_{j=0}^{\infty} (\lambda x)_j \lambda^{-j} \sum_{n=j}^{\infty} s(n,j) \frac{t^n}{n!}\right)$$

$$= \left(\sum_{n=0}^{\infty} \frac{C_{n,\lambda,2}}{\lambda^n} \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} \sum_{j=0}^{n} (\lambda x)_j \lambda^{-j} s(n,j) \frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \lambda^{k-n-j} (\lambda x)_j C_{n-k,\lambda,2} s(k,j) \frac{t^n}{n!} .$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. □

Theorem 4. *Let $n \geq 0$ be integers. Then*

$$C_{n,\lambda,4}(x+y) = \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{l=0}^{j} \binom{n}{k} S_{1,\lambda}(j,l) y^l s(k,j) C_{n-k,\lambda,4}(x). \tag{28}$$

Proof. By applying (6), (9), we get

$$\sum_{n=0}^{\infty} C_{n,\lambda,4}(x+y) \frac{t^n}{n!} = \frac{\lambda t}{\ln(1+\lambda \ln(1+t))} (1+\lambda \ln(1+t))^{\frac{x+y}{\lambda}}$$

$$= \frac{\lambda t}{\ln(1+\lambda \ln(1+t))} (1+\lambda \ln(1+t))^{\frac{x}{\lambda}} (1+\lambda \ln(1+t))^{\frac{y}{\lambda}}$$

$$= \left(\sum_{n=0}^{\infty} C_{n,\lambda,4}(x) \frac{t^n}{n!}\right) \left(\sum_{j=0}^{\infty} (\frac{y}{\lambda})_j \lambda^j \frac{(\ln(1+t))^j}{j!}\right)$$

$$= \left(\sum_{n=0}^{\infty} C_{n,\lambda,4}(x) \frac{t^n}{n!}\right) \left(\sum_{j=0}^{\infty} (\frac{y}{\lambda})_j \lambda^j \sum_{n=j}^{\infty} s(n,j) \frac{t^n}{n!}\right)$$

$$= \left(\sum_{n=0}^{\infty} C_{n,\lambda,4}(x) \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} \sum_{j=0}^{n} (\frac{y}{\lambda})_j \lambda^j s(n,j) \frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} (\frac{y}{\lambda})_j \lambda^j s(k,j) C_{n-k,\lambda,4}(x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} (y)_{j,\lambda} \lambda^{-j} \lambda^j s(k,j) C_{n-k,\lambda,4}(x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{l=0}^{j} \binom{n}{k} S_{1,\lambda}(j,l) y^l s(k,j) C_{n-k,\lambda,4}(x) \frac{t^n}{n!} .$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. □

Corollary 3. *The following relations hold:*

$$C_{n,\lambda,4}(1) = \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{l=0}^{j} \binom{n}{k} S_{1,\lambda}(j,l) s(k,j) C_{n-k,\lambda,4}. \tag{29}$$

Proof. Setting $x = 0$, $y = 1$ in (28), we get (29). □

Theorem 5. *Let $n \geq 0$ be integers. Then*

$$(-1)^n C_{n,\lambda,4}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}(-1)^{n-j-k}\lambda^j C_{n-k}C_j(\frac{x}{\lambda})\,|s(k,j)|\,. \tag{30}$$

Proof. By applying (7), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n C_{n,\lambda,4}(x)\frac{t^n}{n!} = \frac{-\lambda t}{\ln(1+\lambda\ln(1-t))}(1+\lambda\ln(1-t))^{\frac{x}{\lambda}}\\
&= \frac{\lambda\ln(1-t)}{\ln(1+\lambda\ln(1-t))}\frac{-t}{\ln(1-t)}(1+\lambda\ln(1-t))^{\frac{x}{\lambda}}\\
&= \left(\sum_{j=0}^{\infty}C_j(\frac{x}{\lambda})\lambda^j\frac{(\ln(1-t))^j}{j!}\right)\left(\sum_{n=0}^{\infty}C_n\frac{(-t)^n}{n!}\right)\\
&= \left(\sum_{j=0}^{\infty}C_j(\frac{x}{\lambda})\lambda^j(-1)^{-j}\sum_{n=j}^{\infty}|s(n,j)|\,\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n C_n\frac{t^n}{n!}\right)\\
&= \left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}C_j(\frac{x}{\lambda})\lambda^j(-1)^{-j}\,|s(n,j)|\,\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n C_n\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\lambda^j(-1)^{-j}C_j(\frac{x}{\lambda})\,|s(k,j)|\,(-1)^{n-k}C_{n-k}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}(-1)^{n-j-k}\lambda^j C_{n-k}C_j(\frac{x}{\lambda})|s(k,j)|\frac{t^n}{n!}\,.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. □

Corollary 4. *The following relations hold:*

$$(-1)^n C_{n,\lambda,4} = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}(-1)^{n-j-k}\lambda^j C_{n-k}C_j\,|s(k,j)|\,. \tag{31}$$

Proof. Setting $x = 0$ in (30), we get (31). □

Theorem 6. *Let $n \geq 0$ be integers. Then*

$$C_{n,\lambda,4}(x) - C_{n,\lambda,4} = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{k}{j}n\frac{\lambda^{k+1}}{j+1}(\frac{x}{\lambda})_{j+1}C_{k-j}s(n-1,k). \tag{32}$$

Proof. By applying (6), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(C_{n,\lambda,4}(x) - C_{n,\lambda,4}\right)\frac{t^n}{n!} = \frac{\lambda t}{\ln(1+\lambda\ln(1+t))}((1+\lambda\ln(1+t))^{\frac{x}{\lambda}}-1)\\
&= \frac{\lambda\ln(1+t)}{\ln(1+\lambda\ln(1+t))}\frac{t}{\ln(1+t)}((1+\lambda\ln(1+t))^{\frac{x}{\lambda}}-1)
\end{aligned}$$

$$
\begin{aligned}
&= t\left(\sum_{k=0}^{\infty} C_k \lambda^k \frac{(\ln(1+t))^k}{k!}\right)\left(\sum_{k=1}^{\infty} (\frac{x}{\lambda})_k \lambda^k \frac{(\ln(1+t))^{k-1}}{k!}\right) \\
&= t\left(\sum_{k=0}^{\infty} C_k \lambda^k \frac{(\ln(1+t))^k}{k!}\right)\left(\sum_{k=0}^{\infty} (\frac{x}{\lambda})_{k+1} \frac{\lambda^{k+1}}{k+1} \frac{(\ln(1+t))^k}{k!}\right) \\
&= t\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k+1}}{j+1}(\frac{x}{\lambda})_{j+1}C_{k-j}\frac{(\ln(1+t))^k}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k+1}}{j+1}(\frac{x}{\lambda})_{j+1}C_{k-j}\sum_{n=k}^{\infty} s(n,k)\frac{t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k+1}}{j+1}(\frac{x}{\lambda})_{j+1}C_{k-j}s(n,k)\frac{t^{n+1}}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{k}{j}n\frac{\lambda^{k+1}}{j+1}(\frac{x}{\lambda})_{j+1}C_{k-j}s(n-1,k)\frac{t^{n}}{n!}\,.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. □

Theorem 7. *Let $n \geq 0$ be integers. Then*

$$
\sum_{m=0}^{n} C_{m,\lambda,4}(x)s(n,m) = \sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum_{j=0}^{k}\binom{n}{k}\lambda^i C_j C_i(\frac{x}{\lambda})\beta(n-k,i)s(k,j). \tag{33}
$$

Proof. By applying (6), (13), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} C_{n,\lambda,4}(x)\frac{(\ln(1+t))^n}{n!} = \frac{\lambda\ln(1+t)}{ln(1+\lambda\ln(1+\ln(1+t)))}(1+\lambda\ln(1+\ln(1+t)))^{\frac{x}{\lambda}} \\
&= \frac{\lambda\ln(1+\ln(1+t))}{\ln(1+\lambda\ln(1+\ln(1+t)))}\frac{\ln(1+t)}{\ln(1+\ln(1+t))}(1+\lambda\ln(1+\ln(1+t)))^{\frac{x}{\lambda}} \\
&= \sum_{i=0}^{\infty} C_i(\frac{x}{\lambda})\frac{\lambda^i(\ln(1+\ln(1+t)))^i}{i!}\sum_{j=0}^{\infty} C_j\frac{(\ln(1+t))^j}{j!} \\
&= \left(\sum_{i=0}^{\infty} C_i(\frac{x}{\lambda})\lambda^i\sum_{n=i}^{\infty}\beta(n,i)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} C_j\sum_{n=j}^{\infty} s(n,j)\frac{t^n}{n!}\right) \\
&= \left(\sum_{n=0}^{\infty}\sum_{i=0}^{n}\lambda^i C_i(\frac{x}{\lambda})\beta(n,i)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{j=0}^{n} C_j s(n,j)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{n-k}\sum_{j=0}^{k}\binom{n}{k}\lambda^i C_j C_i(\frac{x}{\lambda})\beta(n-k,i)s(k,j)\frac{t^n}{n!}
\end{aligned}
$$

$$
\sum_{m=0}^{\infty} C_{m,\lambda,4}(x)\frac{(ln(1+t))^m}{m!} = \sum_{m=0}^{\infty} C_{m,\lambda,4}(x)\sum_{n=m}^{\infty} s(n,m)\frac{t^n}{n!}
$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_{m,\lambda,4}(x) s(n,m) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. □

Corollary 5. *The following relations hold:*

$$\sum_{m=0}^{n} C_{m,\lambda,4} s(n,m) = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \sum_{j=0}^{k} \binom{n}{k} \lambda^i C_j C_i \beta(n-k,i) s(k,j). \tag{34}$$

Proof. Setting $x = 0$ in (33), we get (34). □

Corollary 6. *The following relations hold:*

$$C_{n,\lambda,4}(x) = \sum_{k=0}^{n} \sum_{h=0}^{k} \sum_{i=0}^{k-h} \sum_{j=0}^{h} \binom{k}{h} S(n,k) \lambda^i C_j C_i (\frac{x}{\lambda}) \beta(k-h,i) s(h,j). \tag{35}$$

Proof. By applying(21), we get (35). □

Corollary 7. *The following relations hold:*

$$C_{n,\lambda,4} = \sum_{k=0}^{n} \sum_{h=0}^{k} \sum_{i=0}^{k-h} \sum_{j=0}^{h} \binom{k}{h} S(n,k) \lambda^i C_j C_i \beta(k-h,i) s(h,j). \tag{36}$$

Proof. Setting $x = 0$ in (35), we get (36). □

Theorem 8. *Let $n \geq 0$ be integers. Then*

$$\sum_{r+h+l=0}^{n} \sum_{m=0}^{h} \binom{n}{r,h,l} \frac{\lambda^{-m} S(h,m)}{(r+1)(m+1)} = \sum_{m=0}^{n} \sum_{l=0}^{m} C_{m,\lambda,4}(x) S(m,l) S(n,m) \lambda^{-m}. \tag{37}$$

Proof. By applying (8),let $t = e^{\frac{e^t-1}{\lambda}} - 1$,we get

$$\sum_{n=0}^{\infty} C_{n,\lambda,4}(x) \frac{(e^{\frac{e^t-1}{\lambda}} - 1)^n}{n!} = \frac{\lambda(e^{\frac{e^t-1}{\lambda}} - 1)}{t} e^t$$

$$= \frac{\lambda}{t} \left(\sum_{m=0}^{\infty} \frac{(\frac{e^t-1}{\lambda})^m}{m!} - 1 \right) e^t = \frac{\lambda}{t} \sum_{m=1}^{\infty} \frac{(\frac{e^t-1}{\lambda})^m}{m!} e^t$$

$$= \frac{1}{t} \sum_{m=0}^{\infty} \frac{\lambda^{-m}(e^t-1)^{m+1}}{(m+1)!} e^t = \frac{1}{t}(e^t-1) \sum_{m=0}^{\infty} \frac{\lambda^{-m}}{m+1} \frac{(e^t-1)^m}{m!} e^t$$

$$= \frac{1}{t} \left(\sum_{n=1}^{\infty} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{\lambda^{-m}}{m+1} \sum_{n=m}^{\infty} S(n,m) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right)$$

$$= \left(\sum_{n=0}^{\infty} \frac{1}{n+1} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\lambda^{-m} S(n,m)}{m+1} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{r+h+l=0}^{n} \sum_{m=0}^{h} \binom{n}{r,h,l} \frac{\lambda^{-m} S(h,m)}{(r+1)(m+1)} \frac{t^n}{n!}$$

$$\sum_{m=0}^{\infty} C_{m,\lambda,4}(x)\frac{(e^{\frac{e^t-1}{\lambda}}-1)^m}{m!} = \sum_{m=0}^{\infty} C_{m,\lambda,4}(x)\sum_{m=l}^{\infty} S(m,l)\frac{(\frac{e^t-1}{\lambda})^m}{m!}$$
$$= \sum_{m=0}^{\infty} C_{m,\lambda,4}(x)\sum_{m=l}^{\infty} S(m,l)\sum_{n=m}^{\infty} S(n,m)\lambda^{-m}\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m} C_{m,\lambda,4}(x)S(m,l)S(n,m)\lambda^{-m}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. $\square$

Corollary 8. *The following relations hold:*

$$\sum_{r+h+l=0}^{n}\sum_{m=0}^{h}\binom{n}{r,h,l}\frac{\lambda^{-m}S(h,m)}{(r+1)(m+1)} = \sum_{m=0}^{n}\sum_{l=0}^{m} C_{m,\lambda,4}S(m,l)S(n,m)\lambda^{-m}. \tag{38}$$

Proof. Setting $x=0$ in (37), we get(38). $\square$

Theorem 9. *Let $n \geq 0$ be integers. Then*

$$\sum_{k=0}^{n}\binom{n}{k}D_{n-k,\lambda}C_{k,\lambda,4} = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}C_j\lambda^{k-n+j}s(k,j)\beta_{n-k,\frac{1}{\lambda}}. \tag{39}$$

Proof. By applying (6), (16)and (19), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}D_{n-k,\lambda}C_{k,\lambda,4}\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} C_{n,\lambda,4}\frac{t^n}{n!}\right)$$
$$= \frac{\lambda\ln(1+t)}{(1+t)^\lambda-1}\frac{\lambda t}{\ln(1+\lambda\ln(1+t))} = \frac{\lambda t}{(1+t)^\lambda-1}\frac{\lambda\ln(1+t)}{\ln(1+\lambda\ln(1+t))}$$
$$= \sum_{j=0}^{\infty} C_j\lambda^j\frac{(\ln(1+t))^j}{j!}\sum_{n=0}^{\infty}\beta_{n,\frac{1}{\lambda}}\lambda^n\frac{t^n}{n!}$$
$$= \left(\sum_{j=0}^{\infty} C_j\lambda^j\sum_{n=j}^{\infty} s(n,j)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\beta_{n,\frac{1}{\lambda}}\lambda^n\frac{t^n}{n!}\right)$$
$$= \left(\sum_{n=0}^{\infty}\sum_{j=0}^{n} C_j\lambda^j s(n,j)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\beta_{n,\frac{1}{\lambda}}\lambda^n\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}C_j\lambda^{n-k+j}s(k,j)\beta_{n-k,\frac{1}{\lambda}}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation is easy to prove the theorem. $\square$

Second, the identity between degenerate Cauchy polynomials of the fourth kind and degenerate Stiring number, degenerate Daehee number, generalized Stiring number, Lah number, generalized Bell number and their polynomials is studied by using the method of Riordan matrix.

Theorem 10. *Let $n \geq 0$ be integers. Then*

$$\sum_{k=0}^{n} S_{2,\lambda}(n,k)C_{k,\lambda,4}(x) = \sum_{r+h+l=0}^{n}\sum_{j=0}^{r}\binom{n}{r,h,l}C_j(\frac{x}{\lambda})s(r,j)\lambda^{r+h}d^*_{h,\frac{1}{\lambda}}d_{l,\lambda}. \tag{40}$$

Proof. For $\left\{\frac{k!}{n!}S_{2,\lambda(n,k)}\right\} = (1,(1+\lambda t)^{\frac{1}{\lambda}}-1)$, by applying (20), we get

$$\begin{aligned}
&\sum_{k=0}^{n} S_{2,\lambda}(n,k)C_{k,\lambda,4}(x) = n!\sum_{k=0}^{n}\frac{k!}{n!}S_{2,\lambda}(n,k)\frac{C_{k,\lambda,4}(x)}{k!}\\
&= n![t^n]\frac{\lambda y}{\ln(1+\lambda\ln(1+y))}(1+\lambda\ln(1+y))^{\frac{x}{\lambda}}(y=(1+\lambda t)^{\frac{1}{\lambda}}-1))\\
&= n![t^n]\frac{\lambda((1+\lambda t)^{\frac{1}{\lambda}}-1)}{\ln(1+\ln(1+\lambda t))}(1+\ln(1+\lambda t))^{\frac{x}{\lambda}}\\
&= n![t^n]\frac{\ln(1+\lambda t)}{\ln(1+\ln(1+\lambda t))}\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\ln(1+t)}\frac{\lambda\ln(1+t)}{\ln(1+\lambda t)}(1+\ln(1+\lambda t))^{\frac{x}{\lambda}}\\
&= n![t^n]\left(\sum_{j=0}^{\infty}C_j(\frac{x}{\lambda})\frac{(\ln(1+\lambda t))^j}{j!}\right)\left(\sum_{n=0}^{\infty}d^*_{n,\frac{1}{\lambda}}\lambda^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}d_{n,\lambda}\frac{t^n}{n!}\right)\\
&= n![t^n]\left(\sum_{j=0}^{\infty}C_j(\frac{x}{\lambda})\sum_{n=j}^{\infty}s(n,j)\lambda^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}d^*_{n,\frac{1}{\lambda}}\lambda^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}d_{n,\lambda}\frac{t^n}{n!}\right)\\
&= n![t^n]\left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}C_j(\frac{x}{\lambda})s(n,j)\lambda^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}d^*_{n,\frac{1}{\lambda}}\lambda^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}d_{n,\lambda}\frac{t^n}{n!}\right)\\
&= n![t^n]\sum_{n=0}^{\infty}\sum_{r+h+l=0}^{n}\sum_{j=0}^{r}\binom{n}{r,h,l}C_j(\frac{x}{\lambda})s(r,j)\lambda^{r+h}d^*_{h,\frac{1}{\lambda}}d_{l,\lambda}\frac{t^n}{n!}\\
&= \sum_{r+h+l=0}^{n}\sum_{j=0}^{r}\binom{n}{r,h,l}C_j(\frac{x}{\lambda})s(r,j)\lambda^{r+h}d^*_{h,\frac{1}{\lambda}}d_{l,\lambda}.
\end{aligned}$$

□

Corollary 9. *The following relations hold:*

$$\sum_{k=0}^{n} S_{2,\lambda}(n,k)C_{k,\lambda,4} = \sum_{r+h+l=0}^{n}\sum_{j=0}^{r}\binom{n}{r,h,l}C_j s(r,j)\lambda^{r+h}d^*_{h,\frac{1}{\lambda}}d_{l,\lambda}. \tag{41}$$

Proof. Setting $x=0$ in (40), we get(41). □

Theorem 11. *Let $n \geq 0$ be integers. Then*

$$\sum_{k=0}^{n} S_2(n,k,r)C_{k,\lambda,4}(x) = \sum_{i+j+l=0}^{n}\binom{n}{i,j,l}\frac{r^i\lambda^l}{j+1}C_l(\frac{x}{\lambda}). \tag{42}$$

Proof. For $\left\{\frac{k!}{n!}S_2(n,k,r)\right\} = (e^{rt}, e^t-1)$, by applying (20), we get

$$
\begin{aligned}
&\sum_{k=0}^{n} S_2(n,k,r)C_{k,\lambda,4}(x) = n!\sum_{k=0}^{n}\frac{k!}{n!}S_2(n,k,r)\frac{C_{k,\lambda,4}(x)}{k!}\\
&= n![t^n]e^{rt}\frac{\lambda y}{\ln(1+\lambda\ln(1+y))}(1+\lambda\ln(1+y))^{\frac{x}{\lambda}}(y=e^t-1)\\
&= n![t^n]e^{rt}\frac{\lambda(e^t-1)}{\ln(1+\lambda t)}(1+\lambda t)^{\frac{x}{\lambda}}\\
&= n![t^n]\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)\left(\sum_{n=1}^{\infty}\frac{t^{n-1}}{n!}\right)\left(\sum_{n=0}^{\infty}C_n(\frac{x}{\lambda})\lambda^n\frac{t^n}{n!}\right)\\
&= n![t^n]\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}C_n(\frac{x}{\lambda})\lambda^n\frac{t^n}{n!}\right)\\
&= n![t^n]\sum_{n=0}^{\infty}\sum_{i+j+l=0}^{n}\binom{n}{i,j,l}\frac{r^i\lambda^l}{j+1}C_l(\frac{x}{\lambda})\frac{t^n}{n!}\\
&= \sum_{i+j+l=0}^{n}\binom{n}{i,j,l}\frac{r^i\lambda^l}{j+1}C_l(\frac{x}{\lambda}).
\end{aligned}
$$

□

Corollary 10. *The following relations hold:*

$$\sum_{k=0}^{n} S_2(n,k,r)C_{k,\lambda,4} = \sum_{i+j+l=0}^{n}\binom{n}{i,j,l}\frac{r^i\lambda^l}{j+1}C_l. \tag{43}$$

Proof. Setting $x=0$ in (42), we get(43). □

Corollary 11. *The following relations hold:*

$$C_{n,\lambda,4}(x) = \sum_{k=0}^{n}\sum_{i+j+l=0}^{k}\binom{k}{i,j,l}S_1(n,k,r)\frac{r^i\lambda^l}{j+1}C_l(\frac{x}{\lambda}). \tag{44}$$

Proof. By applying (22), we get (44). □

Corollary 12. *The following relations hold:*

$$C_{n,\lambda,4} = \sum_{k=0}^{n}\sum_{i+j+l=0}^{n}\binom{k}{i,j,l}S_1(n,k,r)\frac{r^i\lambda^l}{j+1}C_l. \tag{45}$$

Proof. Setting $x=0$ in (44), we get(45). □

Theorem 12. *Let $n\geq 0$ be integers. Then*

$$\sum_{k=0}^{n} L(n,k)C_{k,\lambda,4}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}C_j(\frac{x}{\lambda})(-1)^{n-k+j}\lambda^j s(k,j)C^*_{n-k}. \tag{46}$$

Proof. For $\left\{\frac{k!}{n!}L(n,k)\right\} = (1,\frac{-t}{1+t})$, by applying (20), we get

$$
\begin{aligned}
&\sum_{k=0}^{n} L(n,k)C_{k,\lambda,4}(x) = n!\sum_{k=0}^{n}\frac{k!}{n!}L(n,k)\frac{C_{k,\lambda,4}(x)}{k!}\\
&= n![t^n]\frac{\lambda y}{\ln(1+\lambda\ln(1+y))}(1-\lambda\ln(1+y))^{\frac{x}{\lambda}}(y=\frac{-t}{1+t})\\
&= n![t^n]\frac{-\lambda t}{(1+t)\ln(1-\lambda\ln(1+t))}(1-\lambda\ln(1+t))^{\frac{x}{\lambda}}\\
&= n![t^n]\frac{-\lambda\ln(1+t)}{ln(1-\lambda\ln(1+t))}\frac{t}{(1+t)\ln(1+t)}(1-\lambda\ln(1+t))^{\frac{x}{\lambda}}\\
&= n![t^n]\left(\sum_{j=0}^{\infty}C_j(\frac{x}{\lambda})(-\lambda)^j\frac{(\ln(1+t))^j}{j!}\right)\left(\sum_{n=0}^{\infty}(-1)^nC_n^*\frac{t^n}{n!}\right)\\
&= n![t^n]\left(\sum_{j=0}^{\infty}C_j(\frac{x}{\lambda})(-\lambda)^j\sum_{n=j}^{\infty}s(n,j)\frac{t^n}{n!})\right)\left(\sum_{n=0}^{\infty}(-1)^nC_n^*\frac{t^n}{n!}\right)\\
&= n![t^n]\left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}C_j(\frac{x}{\lambda})(-\lambda)^js(n,j)\frac{t^n}{n!})\right)\left(\sum_{n=0}^{\infty}(-1)^nC_n^*\frac{t^n}{n!}\right)\\
&= n![t^n]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}C_j(\frac{x}{\lambda})(-\lambda)^js(k,j)(-1)^{n-k}C_{n-k}^*\frac{t^n}{n!}\\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}C_j(\frac{x}{\lambda})(-1)^{n-k+j}\lambda^js(k,j)C_{n-k}^*\,.
\end{aligned}
$$

□

Corollary 13. *The following relations hold:*

$$\sum_{k=0}^{n} L(n,k)C_{k,\lambda,4} = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}C_j(-1)^{n-k+j}\lambda^js(k,j)C_{n-k}^*. \tag{47}$$

Proof. Setting $x=0$ in (46), we get(47). □

Theorem 13. *Let $n \geq 0$ be integers. Then*

$$\sum_{k=0}^{n}\sum_{j=0}^{k} S(n,k)L(k,j)C_{j,\lambda,4}(x) = \sum_{k=0}^{n}\binom{n}{k}\lambda^kC_k(\frac{x}{\lambda})\frac{(-1)^n}{n-k+1}. \tag{48}$$

Proof. By applying (46), we get

$$\sum_{k=0}^{n} L(n,k)C_{n,\lambda,4}(x) = n![t^n]\frac{-\lambda t}{(1+t)\ln(1-\lambda\ln(1+t))}(1-\lambda\ln(1+t))^{\frac{x}{\lambda}}.$$

For $\left\{\frac{k!}{n!}S(n,k)\right\} = (1, e^t-1)$, by applying (20), we get

$$\sum_{k=0}^{n}\sum_{j=0}^{k} S(n,k)L(k,j)C_{j,\lambda,4}(x) = n!\sum_{k=0}^{n}\frac{k!}{n!}S(n,k)\frac{\sum_{j=0}^{k}L(k,j)C_{j,\lambda,4}(x)}{k!}$$

$$= n![t^n]\frac{-\lambda y}{(1+y)\ln(1-\lambda\ln(1+y))}(1-\lambda\ln(1+y))^{\frac{x}{\lambda}}(y=e^t-1) = n![t^n]\frac{-\lambda(e^t-1)}{e^t\ln(1-\lambda t)}(1-\lambda t)^{\frac{x}{\lambda}}$$
$$= n![t^n]\frac{-\lambda}{ln(1-\lambda t)}(1-e^{-t})(1-\lambda t)^{\frac{x}{\lambda}} = n![t^n]\frac{-\lambda t}{\ln(1-\lambda t)}(-\frac{1}{t})(e^{-t}-1)(1-\lambda t)^{\frac{x}{\lambda}}$$
$$= n![t^n]\left(\sum_{n=0}^{\infty}C_n(\frac{x}{\lambda})(-\lambda)^n\frac{t^n}{n!}\right)\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{n-1}}{n!}\right) = n![t^n]\left(\sum_{n=0}^{\infty}C_n(\frac{x}{\lambda})(-\lambda)^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\frac{t^n}{n!}\right)$$
$$= n![t^n]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\lambda^kC_k(\frac{x}{\lambda})\frac{(-1)^n}{n-k+1}\frac{t^n}{n!} = \sum_{k=0}^{n}\binom{n}{k}\lambda^kC_k(\frac{x}{\lambda})\frac{(-1)^n}{n-k+1}.$$

□

Corollary 14. *The following relations hold:*

$$\sum_{k=0}^{n}\sum_{j=0}^{k}S(n,k)L(k,j)C_{j,\lambda,4} = \sum_{k=0}^{n}\binom{n}{k}\lambda^kC_k\frac{(-1)^n}{n-k+1}. \tag{49}$$

Proof. Setting $x=0$ in (48), we get(49). □

Corollary 15. *The following relations hold:*

$$\sum_{j=0}^{n}L(n,j)C_{j,\lambda,4}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\lambda^lC_l(\frac{x}{\lambda})\frac{(-1)^k}{k-l+1}s(n,k). \tag{50}$$

Proof. By applying (22), we get (50). □

Corollary 16. *The following relations hold:*

$$\sum_{j=0}^{n}L(n,j)C_{j,\lambda,4} = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\lambda^lC_l\frac{(-1)^k}{k-l+1}s(n,k). \tag{51}$$

Proof. Setting $x=0$ in (50), we get(51). □

Theorem 14. *Let $n \geq 0$ be integers. Then*

$$\sum_{k=0}^{n}B(n,k)C_{n,\lambda,4}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{j+1}C_{k-j}(\frac{x}{\lambda})S(n,k). \tag{52}$$

Proof. For $\left\{\frac{k!}{n!}B(n,k)\right\} = (1, e^{e^t-1}-1)$, by applying (20), we get

$$\sum_{k=0}^{n}B(n,k)C_{k,\lambda,4}(x) = n!\sum_{k=0}^{n}\frac{k!}{n!}B(n,k)\frac{C_{k,\lambda,4}(x)}{k!}$$
$$= n![t^n]\frac{\lambda y}{\ln(1+\lambda\ln(1+y))}(1+\lambda\ln(1+y))^{\frac{x}{\lambda}}(y=e^{e^t-1}-1)$$
$$= n![t^n]\frac{\lambda(e^{e^t-1}-1)}{\ln(1+\lambda(e^t-1))}(1+\lambda(e^t-1))^{\frac{x}{\lambda}}$$

$$
\begin{aligned}
&= n![t^n](e^{e^t-1}-1)\frac{\lambda(e^t-1)}{\ln(1+\lambda(e^t-1))}\frac{1}{e^t-1}(1+\lambda(e^t-1))^{\frac{x}{\lambda}} \\
&= n![t^n]\left(\sum_{k=1}^{\infty}\frac{(e^t-1)^{k-1}}{k!}\right)\left(\sum_{k=0}^{\infty}C_k(\frac{x}{\lambda})\lambda^k\frac{(e^t-1)^k}{k!}\right) \\
&= n![t^n]\left(\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{(e^t-1)^k}{k!}\right)\left(\sum_{k=0}^{\infty}C_k(\frac{x}{\lambda})\lambda^k\frac{(e^t-1)^k}{k!}\right) \\
&= n![t^n]\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{j+1}C_{k-j}(\frac{x}{\lambda})\frac{(e^t-1)^k}{k!} \\
&= n![t^n]\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{j+1}C_{k-j}(\frac{x}{\lambda})\sum_{n=k}^{\infty}S(n,k)\frac{t^n}{n!} \\
&= n![t^n]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{j+1}C_{k-j}(\frac{x}{\lambda})S(n,k)\frac{t^n}{n!} \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{j+1}C_{k-j}(\frac{x}{\lambda})S(n,k)\,.
\end{aligned}
$$

□

Corollary 17. *The following relations hold:*

$$
\sum_{k=0}^{n}B(n,k)C_{n,\lambda,4}=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{j+1}C_{k-j}S(n,k). \tag{53}
$$

Proof. Setting $x=0$ in (52), we get(53). □

Theorem 15. *Let $n\geq 0$ be integers. Then*

$$
\sum_{k=0}^{n}\beta(n,k)C_{k,\lambda,4}(x)=\sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\lambda^iC_i(\frac{x}{\lambda})C_{k-j}s(j,i)s(l,k)s(n,l). \tag{54}
$$

Proof. For $\left\{\frac{k!}{n!}\beta(n,k)\right\}=(1,\ln(1+\ln(1+t)))$, by applying (20), we get

$$
\begin{aligned}
&\sum_{k=0}^{n}\beta(n,k)C_{k,\lambda,4}(x)=n!\sum_{k=0}^{n}\frac{k!}{n!}\beta(n,k)\frac{C_{k,\lambda,4}(x)}{k!} \\
&= n![t^n]\frac{\lambda y}{\ln(1+\lambda\ln(1+y))}(1+\lambda\ln(1+y))^{\frac{x}{\lambda}}(y=\ln(1+\ln(1+t)) \\
&= n![t^n]\frac{\lambda\ln(1+ln(1+t))}{\ln(1+\lambda\ln(1+\ln(1+\ln(1+t))))}(1+\lambda\ln(1+\ln(1+\ln(1+t))))^{\frac{x}{\lambda}} \\
&= n![t^n]\frac{\lambda\ln(1+\ln(1+\ln(1+t)))}{\ln(1+\lambda\ln(1+\ln(1+\ln(1+t))))}\frac{\ln(1+\ln(1+t))}{\ln(1+\ln(1+\ln(1+t)))}(1+\lambda\ln(1+\ln(1+\ln(1+t))))^{\frac{x}{\lambda}} \\
&= n![t^n]\left(\sum_{i=0}^{\infty}C_i(\frac{x}{\lambda})\lambda^i\frac{(\ln(1+\ln(1+\ln(1+t))))^i}{i!}\right)\left(\sum_{k=0}^{\infty}C_k\frac{(ln(1+ln(1+t)))^k}{k!}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= n![t^n]\left(\sum_{i=0}^{\infty} C_i(\frac{x}{\lambda})\lambda^i \sum_{k=i}^{\infty} s(k,i)\frac{(\ln(1+\ln(1+t)))^k}{k!}\right)\left(\sum_{k=0}^{\infty} C_k \frac{(\ln(1+\ln(1+t)))^k}{k!}\right)\\
&= n![t^n]\left(\sum_{k=0}^{\infty}\sum_{i=0}^{k} C_i(\frac{x}{\lambda})\lambda^i s(k,i)\frac{(\ln(1+\ln(1+t)))^k}{k!}\right)\left(\sum_{k=0}^{\infty} C_k \frac{(\ln(1+\ln(1+t)))^k}{k!}\right)\\
&= n![t^n]\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j} C_i(\frac{x}{\lambda})\lambda^i C_{k-j} s(j,i)\frac{(\ln(1+\ln(1+t)))^k}{k!}\\
&= n![t^n]\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j} C_i(\frac{x}{\lambda})\lambda^i C_{k-j} s(j,i)\sum_{l=k}^{\infty} s(l,k)\frac{(\ln(1+t))^l}{l!}\\
&= n![t^n]\sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j} C_i(\frac{x}{\lambda})\lambda^i C_{k-j} s(j,i)s(l,k)\sum_{n=l}^{\infty} s(n,l)\frac{t^n}{n!}\\
&= n![t^n]\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\lambda^i C_i(\frac{x}{\lambda}) C_{k-j} s(j,i)s(l,k)s(n,l)\frac{t^n}{n!}\\
&= \sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\lambda^i C_i(\frac{x}{\lambda}) C_{k-j} s(j,i)s(l,k)s(n,l)\,.
\end{aligned}
$$

□

Corollary 18. *The following relations hold:*

$$
\sum_{k=0}^{n}\beta(n,k)C_{k,\lambda,4} = \sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}\binom{k}{j}\lambda^i C_i C_{k-j} s(j,i)s(l,k)s(n,l). \tag{55}
$$

Proof. Setting $x = 0$ in (54), we get(55). □

Theorem 16. *Let $n \geq 0$ be integers. Then*

$$
\sum_{k=0}^{n}\sum_{j=0}^{k} B(k,j)\beta(n,k)C_{j,\lambda,4} = \lambda^n C_{n,\frac{1}{\lambda},2}. \tag{56}
$$

Proof. By applying (53), we get

$$
\sum_{k=0}^{n} B(n,k)C_{n,\lambda,4}(x) = n![t^n]\frac{\lambda(e^{e^t-1}-1)}{\ln(1+\lambda(e^t-1))}(1+\lambda(e^t-1))^{\frac{x}{\lambda}}.
$$

For $\left\{\frac{k!}{n!}\beta(n,k)\right\} = (1, \ln(1+\ln(1+t)))$, by applying (20), we get

$$
\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{k} B(k,j)\beta(n,k)C_{j,\lambda,4} = n!\sum_{k=0}^{n}\frac{k!}{n!}\beta(n,k)\frac{\sum_{j=0}^{k} B(k,j)C_{j,\lambda,4}}{k!}\\
&= n![t^n]\frac{\lambda(e^{e^y-1}-1)}{\ln(1+\lambda(e^y-1))}(y=\ln(1+\ln(1+t))) = n![t^n]\frac{\lambda t}{\ln(1+\lambda\ln(1+t))}\\
&= n![t^n]\sum_{n=0}^{\infty}\lambda^n C_{n,\frac{1}{\lambda},2}\frac{t^n}{n!} = \lambda^n C_{n,\frac{1}{\lambda},2}\,.
\end{aligned}
$$

□

Corollary 19. *The following relations hold:*

$$\sum_{k=0}^{n}\sum_{j=0}^{k} B(k,j)\beta(n,k)C_{j,\lambda,4} = \lambda^n C_{n,\frac{1}{\lambda},2}. \tag{57}$$

Proof. Setting $x = 0$ in (56), we get(57). □

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