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The Galerkin Approximation to Forward-Backward Stochastic Partial Differential Equations

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Abstract

In this paper, the authors utilized the Galerkin approximation scheme approach to solve a class of fully coupled forward-backward stochastic partial differential equations in an infinite dimensional functional setup.

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Keywords

Forward-backward equations, stochastic differential equations, partial differential equations, Galerkin approximation.

1. Introduction

Stochastic partial differential equations (SPDEs) arise naturally in many fields of science and engineering

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where the systems of interest are subject to uncertainty [4]. Finding analytical solutions to SPDEs can be challenging due to their complex and random nature. As a result, numerical methods are often employed to approximate the solutions to these equations. Forward-backward stochastic differential equations (FBSDEs) are a class of stochastic differential equations that arise in a wide range of applications, including finance, engineering, and physics. They consist of a system of coupled equations, where the forward equation describes the evolution of a process over time, and the backward equation describes the evolution of a related process in reverse time [14]. One of the challenges in solving FBSDEs is that they typically involve high-dimensional systems and nonlinearities, making analytical solutions difficult to obtain. As a result, numerical methods are often employed to approximate the solutions [5, 8]. One approach to approximating FBSDEs is through Galerkin approximations, which involve projecting the equations onto a finite-dimensional subspace. In recent years, there has been increasing interest in the development of efficient numerical methods for solving FBSDEs and Galerkin approximations. These methods have been applied in a range of applications, including finance, engineering, and physics [6, 3, 1, 2, 9].

In this paper we consider the following forward-backward stochastic partial differential equations (FBSPDEs):

$$\begin{cases} \partial_t \mathbf{u}(t,x) = a \displaystyle\sum_{i,j=1}^{d} \mathbf{u}_{x_i x_j}(t,x)dt + b(t,x,\mathbf{u}(t,x),\mathbf{v}(t,x))dt \\ \qquad\qquad\quad +\sigma(t,x,\mathbf{u}(t,x),\mathbf{v}(t,x))dW(t,x) \\ \partial_t \mathbf{v}(t,x) = -h \displaystyle\sum_{i,j=1}^{d} \mathbf{v}_{x_i x_j}(t,x)dt + k(t,x,\mathbf{u}(t,x),\mathbf{v}(t,x))dt \\ \qquad\qquad\quad -Z(t,x)dW(t,x) \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \text{ and } \mathbf{v}(T,x) = g(\mathbf{u}(T,x)), \ t \in [0,T], \ x \in G, \end{cases} \tag{1.1}$$

where a and h are positive constants, and G is a bounded domain in $\mathbb{R}^d$ with smooth boundary conditions. FBSPDEs can be viewed as a natural extension of FBSDEs. In light of the nonlinear Feynman-Kac formula, or the Four Step Scheme [10], it is not hard to imagine that the solution of a backward SPDE could be a crucial device for solving an FBSDE with random coefficients [12, 13]. It has been shown that the solvability of a large class of non-Markovian FBSDEs is almost equivalent to the solvability of the corresponding backward stochastic partial differential equations (BSPDEs) [11]. Therefore the solvability of FBSPDEs could be considered as part of the effort for a full understanding of the solvability of general strongly coupled FBSDEs with random coefficients. In addition, the interesting structure of FBSPDEs can be used to describe many natural phenomena. For instance, an application to the reaction-diffusion models is provided in [15]. In the context of FBSPDEs, Galerkin approximation methods involve expanding the solution of the equation in a finite-dimensional basis of functions and then solving the resulting system of ordinary differential equations numerically. This approach allows for efficient and accurate computation of the solution, even for high-dimensional problems.

The rest of the paper is organized as follows. In Section 2, we introduce the notations and formulate the Galerkin approximations. Assumptions and a preliminary result are provided in Section 3. In Section 4, we apply the Galerkin approximation scheme to establish the main result.

2 Projections and Estimates

Denote $\langle\cdot,\cdot\rangle$ the inner product of $L^2(G)$. Let $|\cdot|$ be the norm of $L^2(G)$ and $\|\cdot\|$ be the norm of $H_0^1(G)$. They are given as follows:

$$|\mathbf{u}| \triangleq \Big(\int_G |\mathbf{u}|^2 dx\Big)^{\frac{1}{2}},$$

and

$$\|\mathbf{u}\| \triangleq \Big(\int_G |\nabla\mathbf{u}|^2 dx\Big)^{\frac{1}{2}}.$$

For notational simplicity, the norm $|\cdot|$ inside the integral signs is also used to denote the standard norm on $\mathbb{R}^n$, $n\in\mathbb{N}$.

Define the following operator

$$\mathcal{L}\mathbf{u} \triangleq -\sum_{i,j=1}^{d} \mathbf{u}_{x_i x_j}$$

for any $\mathbf{u}\in L^2(G)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a family of nondecreasing unbounded positive numbers such that for each k, λ_k is an eigenvalue of the operator $\mathcal{L}$. For every $k\in\mathbb{N}$, let $\mathbf{e}_k\in H_0^1(G)$ be a corresponding eigenfunction such that $\{\mathbf{e}_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $L^2(G)$. Let $\{q_i\}_{i=1}^{\infty}$ be a family of positive numbers such that $\sum_{i=1}^{\infty} q_i<\infty$. Let our Wiener process W to be defined as

$$W(t,x)\triangleq\sum_{i=1}^{\infty}\sqrt{q_i}B^i(t)\mathbf{e}_i(x),$$

where $\{B^i(t)\}$ is a sequence of iid Brownian motions in $\mathbb{R}$. For any $\mathbf{u}\in L^2(0,T;L^2(G))$, $i\in\mathbb{N}$, $t\in[0,T]$ and $x\in G$, let $\langle\mathbf{u}(t,x),\mathbf{e}_i(x)\rangle=u_i(t)$, and we denote $\mathbf{u}^N(t,x)\triangleq\sum_{i=1}^{N}u_i(t)\mathbf{e}_i(x)$ and $\hat{\mathbf{u}}^N(t)\triangleq\begin{pmatrix}u_1(t)\\ \vdots\\ u_N(t)\end{pmatrix}\in\mathbb{R}^N$, any $N\in\mathbb{N}$. Clearly one has $\langle\mathcal{L}\mathbf{u},\mathbf{e}_i\rangle=\lambda_i u_i$. Define $\lambda^N\triangleq\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_N\end{pmatrix}$ and $\mathbf{e}^N\triangleq\begin{pmatrix}\mathbf{e}_1 & 0 & \cdots & 0\\ 0 & \mathbf{e}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{e}_N\end{pmatrix}$.

Then $\lambda^N\mathbf{e}^N\hat{\mathbf{u}}^N=\begin{pmatrix}\lambda_1u_1\mathbf{e}_1\\ \lambda_2u_2\mathbf{e}_2\vdots\\ \lambda_Nu_N\mathbf{e}_N\end{pmatrix}$.

Let $Z^N(t,x) \triangleq \sum_{i=1}^N \langle Z(t,x), \mathbf{e}_i(x)\rangle \mathbf{e}_i(x)$ and $W^N(t,x) \triangleq \sum_{i=1}^N \sqrt{q_i} B^i(t)\mathbf{e}_i(x)$. Since

$$
\begin{aligned}
&\langle \int_t^T Z^N(s,x) dW^N(s,x), \mathbf{e}_i(x)\rangle \\
=&\langle \int_t^T dW^N(s,x), Z^{N*}(s,x)(\mathbf{e}_i(x))\rangle \\
=&\langle \sum_{k=1}^N \int_t^T \sqrt{q_k}\mathbf{e}_k(x) dB^k(s), Z^{N*}(s,x)(\mathbf{e}_i(x))\rangle \\
=&\sum_{k=1}^N \int_t^T \langle \mathbf{e}_k(x), Z^{N*}(s,x)(\mathbf{e}_i(x))\rangle \sqrt{q_k} dB^k(s),
\end{aligned}
$$

we define $\hat{Z}^N(t)$ as

$$
\begin{pmatrix}
\langle \mathbf{e}_1(x), (Z^N(s,x))^*(\mathbf{e}_1(x))\rangle, & \langle \mathbf{e}_2(x), (Z^N(s,x))^*(\mathbf{e}_1(x))\rangle, & \cdots, & \langle \mathbf{e}_N(x), (Z^N(s,x))^*(\mathbf{e}_1(x))\rangle \\
\langle \mathbf{e}_1(x), (Z^N(s,x))^*(\mathbf{e}_2(x))\rangle, & \langle \mathbf{e}_2(x), (Z^N(s,x))^*(\mathbf{e}_2(x))\rangle, & \cdots, & \langle \mathbf{e}_N(x), (Z^N(s,x))^*(\mathbf{e}_2(x))\rangle \\
\vdots & \vdots & \vdots & \vdots \\
\langle \mathbf{e}_1(x), (Z^N(s,x))^*(\mathbf{e}_N(x))\rangle, & \langle \mathbf{e}_2(x), (Z^N(s,x))^*(\mathbf{e}_N(x))\rangle, & \cdots, & \langle \mathbf{e}_N(x), (Z^N(s,x))^*(\mathbf{e}_N(x))\rangle
\end{pmatrix},
$$

$$
\hat{Q}^N \triangleq \begin{pmatrix}
q_1, & 0, & \cdots, & 0 \\
0, & q_2, & \cdots, & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0, & 0, & \cdots, & q_N
\end{pmatrix}
$$

and $\hat{W}^N(t) \triangleq \sqrt{\hat{Q}^N} \cdot \begin{pmatrix} B^1(t) \\ \vdots \\ B^N(t) \end{pmatrix}$.

Let $b^N \triangleq \sum_{i=1}^N \langle b, \mathbf{e}_i\rangle \mathbf{e}_i$ and $\hat{b}^N(t, \hat{\mathbf{u}}^N, \hat{\mathbf{v}}^N) \triangleq \begin{pmatrix} \langle b^N(t,x,\mathbf{u}^N,\mathbf{v}^N), \mathbf{e}_1(x)\rangle \\ \vdots \\ \langle b^N(t,x,\mathbf{u}^N,\mathbf{v}^N), \mathbf{e}_N(x)\rangle \end{pmatrix}$. Similarly, we can define k^N and $\hat{k}^N$.

Let $\sigma^N \triangleq \sum_{i=1}^N \langle \sigma, \mathbf{e}_i\rangle \mathbf{e}_i$ and define $\hat{\sigma}^N(t, \hat{\mathbf{u}}^N, \hat{\mathbf{v}}^N)$ as

$$
\begin{pmatrix}
\langle \sigma^N(\mathbf{e}_1), \mathbf{e}_1\rangle, & \langle \sigma^N(\mathbf{e}_2), \mathbf{e}_1\rangle, & \cdots, & \langle \sigma^N(\mathbf{e}_N), \mathbf{e}_1\rangle \\
\langle \sigma^N(\mathbf{e}_1), \mathbf{e}_2\rangle, & \langle \sigma^N(\mathbf{e}_2), \mathbf{e}_2\rangle, & \cdots, & \langle \sigma^N(\mathbf{e}_N), \mathbf{e}_2\rangle \\
\vdots & \vdots & \vdots & \vdots \\
\langle \sigma^N(\mathbf{e}_1), \mathbf{e}_N\rangle, & \langle \sigma^N(\mathbf{e}_2), \mathbf{e}_N\rangle, & \cdots, & \langle \sigma^N(\mathbf{e}_N), \mathbf{e}_N\rangle
\end{pmatrix},
$$

where for notational convenience, we breviate $\sigma^N(t, x, \mathbf{u}^N, \mathbf{v}^N)$ as σ^N.

For the initial condition, we define $\mathbf{u}_0^N(x) \triangleq \sum_{i=1}^N \langle \mathbf{u}_0^N(x), \mathbf{e}_i(x)\rangle \mathbf{e}_i(x)$ and

$$
\hat{\mathbf{u}}_0^N \triangleq \begin{pmatrix} \langle \mathbf{u}_0^N(x), \mathbf{e}_1(x)\rangle \\ \vdots \\ \langle \mathbf{u}_0^N(x), \mathbf{e}_N(x)\rangle \end{pmatrix}.
$$

For the terminal condition, one defines

$$
\mathbf{v}^N(T,x) = g^N(\mathbf{u}^N(T,x)) \triangleq \sum_{i=1}^N \langle g(\mathbf{u}^N(T,x)), \mathbf{e}_i(x)\rangle \mathbf{e}_i(x)
$$

and

$$\hat{\mathbf{v}}^N(T) = \hat{g}^N(\hat{\mathbf{u}}^N(T)) \triangleq \begin{pmatrix} \langle g(\mathbf{u}^N(T,x)), \mathbf{e}_1(x)\rangle \\ \langle g(\mathbf{u}^N(T,x)), \mathbf{e}_2(x)\rangle \\ \cdots \\ \langle g(\mathbf{u}^N(T,x)), \mathbf{e}_N(x)\rangle \end{pmatrix}.$$

Now we are able to define a projected system as follows:

$$\begin{cases} \partial_t \mathbf{u}^N = -a\mathcal{L}\mathbf{u}^N dt + b^N(t,x,\mathbf{u}^N,\mathbf{v}^N)dt + \sigma^N(t,x,\mathbf{u}^N,\mathbf{v}^N)dW^N \\ \partial_t \mathbf{v}^N = h\mathcal{L}\mathbf{v}^N dt + k^N(t,x,\mathbf{u}^N,\mathbf{v}^N)dt - Z^N dW^N \\ \mathbf{u}^N(0,x) = \mathbf{u}_0^N(x), \text{ and } \mathbf{v}^N(T,x) = g^N(\mathbf{u}^N(T,x)), \ t \in [0,T], \ x \in G, \end{cases} \tag{2.1}$$

and an equivalent system in $\mathbb{R}^N$:

$$\begin{cases} d\hat{\mathbf{u}}^N = -a\lambda^N \hat{\mathbf{u}}^N dt + \hat{b}^N(t,\hat{\mathbf{u}}^N,\hat{\mathbf{v}}^N)dt + \hat{\sigma}^N(t,\hat{\mathbf{u}}^N,\hat{\mathbf{v}}^N)d\hat{W}^N \\ d\hat{\mathbf{v}}^N = h\lambda^N \hat{\mathbf{v}}^N dt + \hat{k}^N(t,\hat{\mathbf{u}}^N,\hat{\mathbf{v}}^N)dt - \hat{Z}^N d\hat{W}^N \\ \hat{\mathbf{u}}^N(0) = \hat{\mathbf{u}}_0^N, \text{ and } \hat{\mathbf{v}}^N(T) = \hat{g}^N(\hat{\mathbf{u}}^N(T)), \ t \in [0,T]. \end{cases} \tag{2.2}$$

3 Assumptions

Let us assume the following assumptions through out this paper.

(A.1) Suppose that for every $t > 0$, b, σ and k are continuous, and they are $\mathcal{F}_t$-progressively measurable processes such that for any u and $v \in L^2(G)$, they are in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;L^2(G)))$ and $L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$, respectively. The function g is linear and continuous, and for every $u \in L^2(G)$, $g(u)$ is in $L^2_{\mathcal{F}_T}(\Omega; L^2(G))$.

(A.2) There exists a constant $c_1 > 0$, such that for every $t > 0$ and u, $v \in L^2(G)$,

$$\begin{aligned} &|b(t,\cdot,u,v)| + \|\sigma(t,\cdot,u,v)\|_{L_Q} + |k(t,\cdot,u,v)| \\ \le &|b(t,\cdot,0,0)| + \|\sigma(t,\cdot,0,0)\|_{L_Q} + |k(t,\cdot,0,0)| + c_1|u| + c_1|v| \end{aligned}$$

and

$$|g(u)| \le |g(0)| + c_1|u|, \quad \text{P-a.s.}$$

(A.3) There exists a constant $c_2 > 0$, such that for every u, u', v, $v' \in L^2(G)$, and z, $z' \in L_Q$,

$$\begin{aligned} &- a\langle \mathcal{L}(u-u'), v-v'\rangle + \langle b(t,\cdot,u,v) - b(t,\cdot,u',v'), v-v'\rangle \\ &+ \langle \sigma(t,\cdot,u,v) - \sigma(t,\cdot,u',v'), z-z'\rangle_{L_Q} \end{aligned}$$

$$
\begin{aligned}
&+h\langle\mathcal{L}(v-v'),u-u'\rangle+\langle k(t,\cdot,u,v)-k(t,\cdot,u',v'),u-u'\rangle\\
\le&-c_2|u-u'|^2-c_2|v-v'|^2-c_2\|z-z'\|^2_{L_Q}
\end{aligned}
$$

and

$$
\langle g(u)-g(u'),u-u'\rangle\le -c_2|u-u'|^2,\quad \text{P-a.s.}
$$

(A.4) For any u, u', v, $v'\in H_0^1(G)$, and some constants δ and β, and nonpositive constants η and α, the monotonicity conditions hold:

$$
\begin{aligned}
&-2a\langle\mathcal{L}(u-u'),u-u'\rangle+2\langle b(t,x,u,v)-b(t,x,u',v'),u-u'\rangle\\
&+\|\sigma(t,x,u,v)-\sigma(t,x,u',v')\|^2_{L_Q}\le\delta|u-u'|^2+\eta|v-v'|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&-h\langle\mathcal{L}(v-v'),v-v'\rangle-\langle k(t,x,u,v)-k(t,x,u',v'),v-v'\rangle\\
\le&\alpha|u-u'|^2+\beta|v-v'|^2.
\end{aligned}
$$

Under the assumptions, the following result is easy to obtain.

Theorem 3.1. *Assume that (A.1)-(A.3) hold. System (2.2) has a unique adapted solution* $(\hat{\mathbf{u}}^N,\hat{\mathbf{v}}^N,\hat{Z}^N)$ *such that*

$$
\begin{aligned}
&E(\sup_{t\in[0,T]}|\hat{\mathbf{u}}^N(t)|^2)+E\int_0^T\langle\lambda^N\hat{\mathbf{u}}^N(t),\hat{\mathbf{u}}^N(t)\rangle dt\\
+&E(\sup_{t\in[0,T]}|\hat{\mathbf{v}}^N(t)|^2)+E\int_0^T\langle\lambda^N\hat{\mathbf{v}}^N(t),\hat{\mathbf{v}}^N(t)\rangle dt\\
+&E\int_0^T tr(\hat{Z}^N(t)\hat{Q}^N(\hat{Z}^N(t))^*)dt\le K
\end{aligned}
$$

for some constant K, independent of $N\in\mathbb{N}$.

Equivalently, the projected system (2.1) also has a unique adapted solution $(\mathbf{u}^N,\mathbf{v}^N,Z^N)$ *with.*

$$
\begin{aligned}
&E(\sup_{t\in[0,T]}|\mathbf{u}^N(t,x)|^2)+E\int_0^T\|\mathbf{u}^N(t,x)\|^2dt\\
+&E(\sup_{t\in[0,T]}|\mathbf{v}^N(t,x)|^2)+E\int_0^T\|\mathbf{v}^N(t,x)\|^2dt\\
+&E\int_0^T\|Z^N(t,x)\|^2_{L_Q}dt\le K.
\end{aligned}
$$

Proof. Based on assumptions (A.1)-(A.3), the existence and uniqueness of the solution of system (2.2) is guaranteed by the main results in [7]. The regularity can be obtained using standard method. The equivalence of (2.1) and (2.2) yields the second half of the theorem. □

4 The Galerkin Approximation

Now we are ready to provide the main result of this paper.

Theorem 4.1. *Assume that (A.1)-(A.4) hold. System* (1.1) *has an adapted solution* $(\mathbf{u}, \mathbf{v}, Z)$ *in the space*

$$\begin{aligned}&\Big\{L^2_{\mathcal{F}}(\Omega; L^\infty([0,T]; L^2(G))) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T; H^1_0(G)))\Big\}\\ \times&\Big\{L^2_{\mathcal{F}}(\Omega; L^\infty([0,T]; L^2(G))) \cap L^2_{\mathcal{F}}(\Omega; L^2(0,T; H^1_0(G)))\Big\}\\ \times& L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_Q)).\end{aligned}$$

Proof. **Step 1:** It is shown in Theorem 3.1 that $\{\mathbf{u}^N\}_{N=1}^\infty$ and $\{\mathbf{v}^N\}_{N=1}^\infty$ are uniformly bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T; H^1_0(G)))$. Thus along a subsequence,

$$\mathbf{u}^N \xrightarrow{w} \mathbf{u} \text{ and } \mathbf{v}^N \xrightarrow{w} \mathbf{v}$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T; H^1_0(G)))$. Denote

$$S^N(t,x,u,v) = -a\mathcal{L}u + b^N(t,x,u,v)$$

and

$$T^N(t,x,u,v) = h\mathcal{L}v + k^N(t,x,u,v).$$

Under assumption (A.1)-(A.2), b and k are uniformly bounded, and $\mathcal{L}$ is linear, we know that P-almost surely,

$$S^N(t,x,\mathbf{u}^N,\mathbf{v}^N) \xrightarrow{w} S(t,x) \text{ and } T^N(t,x,\mathbf{u}^N,\mathbf{v}^N) \xrightarrow{w} T(t,x)$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T; H^1_0(G)))$ along a subsequence. Also it is clear that

$$Z^N \xrightarrow{w} Z \text{ and } \sigma^N(t,x,\mathbf{u}^N,\mathbf{v}^N) \xrightarrow{w} \sigma(t,x) \quad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_Q)),$$

P-almost surely. For every t, we define

$$\begin{aligned}L_t : L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_Q)) &\to L^2_{\mathcal{F}}(\Omega; L^2(0,T; H^{-1}))\\ L_t(M(\cdot)) &\to \int_t^T M(s)dW(s,x).\end{aligned}$$

Then by Burkholder-Davis-Gundy's inequality,

$$\begin{aligned}&E\int_0^T \|L_t(M(\cdot))\|^2_{H^{-1}}dt\\ \le& TE\sup_{0\le t\le T}|L_t(M(\cdot))|^2\end{aligned}$$

$$\begin{aligned}
&\le 2TE|\int_0^T M(s)dW(s,x)|^2 + 2TE\sup_{0\le t\le T}|\int_0^t M(s)dW(s,x)|^2\\
&\le 4TE\sup_{0\le t\le T}|\int_0^t M(s)dW(s,x)|^2\\
&\le 4TCE\int_0^T \|M(s)\|_{L_Q}^2 ds
\end{aligned}$$

for some constant C. This shows that L_t is a bounded linear operator. Hence L_t maps weakly convergent sequence $\{Z^N\}_{N=1}^\infty$ to a weakly convergent sequence $\{\int_t^T Z^N(s)dW^N(s,x)\}_{N=1}^\infty$ in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}))$ with the limit $\int_t^T Z(s)dW(s,x)$, P-a.s. Here we have used the fact that

$$\int_t^T Z^N(s)dW(s,x) = \int_t^T Z^N(s,x)dW^N(s)$$

by letting $Z^N(t)(\mathbf{e}_i)$=0 for $i > N$. Similarly, we can show that

$$\int_0^t \sigma^N(s,x,\mathbf{u}^N,\mathbf{v}^N)dW^N(s,x) \xrightarrow{w} \int_0^t \sigma(s,x)dW(s,x)$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}))$, P-.a.s.,

$$\int_0^t S^N(s,x,\mathbf{u}^N,\mathbf{v}^N)ds \xrightarrow{w} \int_0^t S(s,x)ds$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}))$, P-a.s., and

$$\int_0^t T^N(s,x,\mathbf{u}^N,\mathbf{v}^N)ds \xrightarrow{w} \int_0^t T(s,x)ds$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}))$, P-a.s. It is clear that by assumption (A.2), for any $\mathbf{w} = \sum_{i=1}^\infty w_i\mathbf{e}_i \in L^2(G)$ and $M \in \mathbb{N}$,

$$\begin{aligned}
&\langle g^N(\mathbf{u}^N(T,x)) - g(\mathbf{u}(T,x)), \mathbf{w}\rangle\\
=&\langle g^N(\mathbf{u}^N(T,x)) - g(\mathbf{u}(T,x)), \sum_{i=1}^M w_i\mathbf{e}_i\rangle\\
&+ \langle g^N(\mathbf{u}^N(T,x)) - g(\mathbf{u}(T,x)), \sum_{i=M+1}^\infty w_i\mathbf{e}_i\rangle\\
\le&\langle g^N(\mathbf{u}^N(T,x)) - g(\mathbf{u}(T,x)), \sum_{i=1}^M w_i\mathbf{e}_i\rangle\\
&+ \Big(|g\big(\mathbf{u}^N(T,x) - \mathbf{u}(T,x)\big)| + |g(\mathbf{u}(T,x))|\Big)|\sum_{i=M+1}^\infty w_i\mathbf{e}_i|\\
\le&\langle g^N(\mathbf{u}^N(T,x)) - g(\mathbf{u}(T,x)), \sum_{i=1}^M w_i\mathbf{e}_i\rangle\\
&+ \Big(|g(0)| + c_1|\mathbf{u}^N(T,x) - \mathbf{u}(T,x)| + |g(\mathbf{u}(T,x))|\Big)|\sum_{i=M+1}^\infty w_i\mathbf{e}_i|.
\end{aligned}$$

By the linearity of g and the regularity of $\mathbf{u}^N$, we see that $g^N(\mathbf{u}^N(T,x))$ converges weakly to $g(\mathbf{u}(T,x))$ in $L^2_{\mathcal{F}}(\Omega;L^2(0,T;L^2(G)))$. Thus one has

$$\begin{aligned}\mathbf{u}^N(t,x)&=\mathbf{u}_0^N(x)+\int_0^t S^N(s,x,\mathbf{u}^N,\mathbf{v}^N)ds+\int_0^t\sigma^N(s,x,\mathbf{u}^N,\mathbf{v}^N)dW^N(s,x)\\ &\xrightarrow{w}\mathbf{u}_0(x)+\int_0^t S(s,x)ds+\int_0^t\sigma(s,x)dW(s,x)\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\mathbf{v}^N(t,x)&=g^N(\mathbf{u}^N(T,x))-\int_t^T T^N(s,x,\mathbf{u}^N,\mathbf{v}^N)ds+\int_t^T Z^N(s,x)dW^N(s,x)\\ &\xrightarrow{w} g(\mathbf{u}(T,x))-\int_t^T T(s,x)ds+\int_t^T Z(s,x)dW(s,x)\end{aligned}\tag{4.2}$$

in $L^2_{\mathcal{F}}(\Omega;L^2(0,T;H^{-1}(G)))$, P-a.s. This also shows that

$$\mathbf{u}(t,x)=\mathbf{u}_0(x)+\int_0^t S(s,x)ds+\int_0^t\sigma(s,x)dW(s,x),\quad \text{P-a.s.}\tag{4.3}$$

and

$$\mathbf{v}(t,x)=g(\mathbf{u}(T,x))-\int_t^T T(s,x)ds+\int_t^T Z(s,x)dW(s,x),\quad \text{P-a.s.}\tag{4.4}$$

For notational simplicity, we denoted the index of the subsequences by N again.

Step 2: In the monotonicity condition (A.5), for any $\mathbf{u}'$, $\mathbf{v}'\in H_0^1(G)$, one has

$$\begin{aligned}&2E\int_0^T\langle S^N(s,x,\mathbf{u}^N,\mathbf{v}^N)-S^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{u}^N-\mathbf{u}'\rangle ds\\ &+E\int_0^T\|\sigma^N(s,x,\mathbf{u}^N,\mathbf{v}^N)-\sigma^N(s,x,\mathbf{u}',\mathbf{v}')\|^2_{L_Q}ds\\ \leq& E\int_0^T\delta|\mathbf{u}^N-\mathbf{u}'|^2ds+E\int_0^T\eta|\mathbf{v}^N-\mathbf{v}'|^2ds.\end{aligned}$$

Let us first discuss the situation when $\delta=0$. Rearranging the terms, one gets

$$\begin{aligned}&2E\int_0^T\langle S^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\mathbf{u}^N\rangle ds-E\int_0^T\eta|\mathbf{v}^N|^2ds\\ &+E\int_0^T\|\sigma^N(s,x,\mathbf{u}^N,\mathbf{v}^N)\|^2_{L_Q}ds\\ \leq&2E\int_0^T\Big\{\langle S^N(s,x,\mathbf{u}^N,\mathbf{v}^N)-S^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{u}'\rangle+\langle S^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{u}^N\rangle\Big\}ds\\ &+E\int_0^T\Big\{-\|\sigma^N(s,x,\mathbf{u}',\mathbf{v}')\|^2_{L_Q}+2\langle\sigma^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\sigma^N(s,x,\mathbf{u}',\mathbf{v}')\rangle_{L_Q}\Big\}ds\\ &+\eta E\int_0^T\{|\mathbf{v}'|^2-2\langle\mathbf{v}^N,\mathbf{v}'\rangle\}ds.\end{aligned}\tag{4.5}$$

Applying the Itô formula to equation (4.1) to get

$$
\begin{aligned}
&E|\mathbf{u}^N(T,x)|^2 - E|\mathbf{u}_0^N(x)|^2\\
=&2E\int_0^T \langle S^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\mathbf{u}^N\rangle ds + E\int_0^T \|\sigma^N(s,x,\mathbf{u}^N,\mathbf{v}^N)\|_{L_Q}^2 ds.
\end{aligned}
$$

Thus (4.5) becomes

$$
\begin{aligned}
&E|\mathbf{u}^N(T,x)|^2 - E|\mathbf{u}_0(x)|^2 - E\int_0^T \eta|\mathbf{v}^N|^2 ds\\
\leq&2E\int_0^T \Big\{\langle S^N(s,x,\mathbf{u}^N,\mathbf{v}^N) - S^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{u}'\rangle + \langle S^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{u}^N\rangle\Big\} ds\\
&+ E\int_0^T \Big\{-\|\sigma^N(s,x,\mathbf{u}',\mathbf{v}')\|_{L_Q}^2 + 2\langle\sigma^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\sigma^N(s,x,\mathbf{u}',\mathbf{v}')\rangle_{L_Q}\Big\} ds\\
&+ \eta E\int_0^T \{|\mathbf{v}'|^2 - 2\langle\mathbf{v}^N,\mathbf{v}'\rangle\} ds. \qquad (4.6)
\end{aligned}
$$

Since $\eta \leq 0$, taking the limit inferior on both sides of (4.6) yields

$$
\begin{aligned}
&E|\mathbf{u}(T,x)|^2 - E|\mathbf{u}_0(x)|^2 - E\int_0^T \eta|\mathbf{v}|^2 ds\\
\leq&2E\int_0^T \Big\{\langle S(s,x) - (-a\mathcal{L}\mathbf{u}' + b(s,x,\mathbf{u}',\mathbf{v}')),\mathbf{u}'\rangle\\
&\qquad + \langle -a\mathcal{L}\mathbf{u}' + b(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{u}\rangle\Big\} ds\\
&+ E\int_0^T \Big\{-\|\sigma(s,x,\mathbf{u}',\mathbf{v}')\|_{L_Q}^2 + 2\langle\sigma(s,x),\sigma(s,x,\mathbf{u}',\mathbf{v}')\rangle_{L_Q}\Big\} ds\\
&+ \eta E\int_0^T \{|\mathbf{v}'|^2 - 2\langle\mathbf{v},\mathbf{v}'\rangle\} ds. \qquad (4.7)
\end{aligned}
$$

An application of the Itô formula to (4.3) yields

$$
\begin{aligned}
&E|\mathbf{u}(T,x)|^2 - E|\mathbf{u}_0(x)|^2\\
=&2E\int_0^T \langle S(s,x),\mathbf{u}\rangle ds + E\int_0^T \|\sigma(s,x)\|_{L_Q}^2 ds.
\end{aligned}
$$

Replacing the left hand side of (4.7) using the above equality, one obtains

$$
\begin{aligned}
&2E\int_0^T \langle S(s,x),\mathbf{u}\rangle ds + E\int_0^T \|\sigma(s,x)\|_{L_Q}^2 ds - E\int_0^T \eta|\mathbf{v}|^2 ds\\
\leq&2E\int_0^T \Big\{\langle S(s,x) - (-a\mathcal{L}\mathbf{u}' + b(s,x,\mathbf{u}',\mathbf{v}')),\mathbf{u}'\rangle\\
&\qquad + \langle -a\mathcal{L}\mathbf{u}' + b^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{u}\rangle\Big\} ds\\
&+ E\int_0^T \Big\{-\|\sigma(s,x,\mathbf{u}',\mathbf{v}')\|_{L_Q}^2 + 2\langle\sigma(s,x),\sigma(s,x,\mathbf{u}',\mathbf{v}')\rangle_{L_Q}\Big\} ds\\
&+ \eta E\int_0^T \{|\mathbf{v}'|^2 - 2\langle\mathbf{v},\mathbf{v}'\rangle\} ds. \qquad (4.8)
\end{aligned}
$$

Rearranging the terms, one gets

$$\begin{aligned}&2E\int_0^T \langle S(s,x)-(-a\mathcal{L}\mathbf{u}'+b(s,x,\mathbf{u}',\mathbf{v}')),\mathbf{u}-\mathbf{u}'\rangle ds\\&+E\int_0^T \|\sigma(s,x)-\sigma(s,x,\mathbf{u}',\mathbf{v}')\|_{L_Q}^2 ds\\ \leq &E\int_0^T \eta|\mathbf{v}-\mathbf{v}'|^2 ds.\end{aligned}$$

Taking $\mathbf{u}'=\mathbf{u}$ and $\mathbf{v}'=\mathbf{v}$, the above inequality becomes

$$E\int_0^T \|\sigma(s,x)-\sigma(s,x,\mathbf{u},\mathbf{v})\|_{L_Q}^2 ds\leq 0,$$

which implies

$$\sigma(s,x)=\sigma(s,x,\mathbf{u},\mathbf{v}),\quad \text{P-a.s.}$$

Taking $\mathbf{u}'=\mathbf{u}-\varepsilon\mathbf{w}$ and $\mathbf{v}'=\mathbf{v}$, where $\mathbf{w}\in L^\infty([0,T]\times\Omega;H_0^1(G))$ and $\varepsilon>0$, one gets

$$E\int_0^T \langle S(s,x)-(-a\mathcal{L}(\mathbf{u}-\varepsilon\mathbf{w})+b(s,x,\mathbf{u}-\varepsilon\mathbf{w},\mathbf{v})),\mathbf{w}\rangle ds\leq 0.$$

Letting $\varepsilon\downarrow 0$, since b is a smooth operator, one has

$$E\int_0^T \langle S(s,x)-(-a\mathcal{L}\mathbf{u}+b(s,x,\mathbf{u},\mathbf{v})),\mathbf{w}\rangle ds\leq 0$$

for all $\mathbf{w}\in L^\infty([0,T]\times\Omega;H_0^1(G))$. This means

$$S(s,x)=-a\mathcal{L}\mathbf{u}+b(s,x,\mathbf{u},\mathbf{v}),\quad \text{P-a.s.}$$

Combining with (4.3), we have

$$\begin{aligned}\mathbf{u}(t,x)=&\mathbf{u}_0(x)+\int_0^t\Big(-a\mathcal{L}\mathbf{u}+b(s,x,\mathbf{u},\mathbf{v})\Big)ds\\&+\int_0^t\sigma(s,x,\mathbf{u},\mathbf{v})dW(s,x),\quad \text{P-a.s.}\end{aligned}\tag{4.9}$$

For the case when δ is not equal to 0, simply replace S^N in (4.5) by $S^N-\delta$. The same result can be obtained similarly.

Step 3: Now let us consider the backward component of the system. For any $\mathbf{u}'$, $\mathbf{v}'\in H_0^1(G)$, one has

$$\begin{aligned}&E\int_0^T \langle -T^N(s,x,\mathbf{u}^N,\mathbf{v}^N)+T^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{v}^N-\mathbf{v}'\rangle ds\\ \leq &E\int_0^T \alpha|\mathbf{u}^N-\mathbf{u}'|^2 ds+E\int_0^T\beta|\mathbf{v}^N-\mathbf{v}'|^2 ds.\end{aligned}$$

Similarly, we only discuss the case when $\beta=0$. Rearranging the terms, one gets

$$2E\int_0^T \langle -T^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\mathbf{v}^N\rangle ds-2E\int_0^T\alpha|\mathbf{u}^N|^2 ds$$

$$
\begin{aligned}
\leq & 2E\int_0^T \Big\{\langle T^N(s,x,\mathbf{u}',\mathbf{v}') - T^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\mathbf{v}'\rangle - \langle T^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{v}^N\rangle\Big\}ds \\
& + 2\alpha E\int_0^T \{|\mathbf{u}'|^2 - 2\langle \mathbf{u}^N,\mathbf{u}'\rangle\}ds. \qquad (4.10)
\end{aligned}
$$

Applying the Itô formula to equation (4.2) to get

$$
\begin{aligned}
& -E|\mathbf{v}^N(T,x)|^2 + E|\mathbf{v}^N(0,x)|^2 + E\int_0^T \|Z^N(s,x)\|_{L_Q}^2 ds \\
= & -2E\int_0^T \langle T^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\mathbf{v}^N\rangle ds.
\end{aligned}
$$

Thus (4.10) becomes

$$
\begin{aligned}
& -E|g^N(\mathbf{u}^N(T,x))|^2 + E|\mathbf{v}^N(0,x)|^2 + E\int_0^T \|Z^N(s,x)\|_{L_Q}^2 ds \\
& -2E\int_0^T \alpha|\mathbf{u}^N|^2 ds \\
\leq & 2E\int_0^T \Big\{\langle T^N(s,x,\mathbf{u}',\mathbf{v}') - T^N(s,x,\mathbf{u}^N,\mathbf{v}^N),\mathbf{v}'\rangle - \langle T^N(s,x,\mathbf{u}',\mathbf{v}'),\mathbf{v}^N\rangle\Big\}ds \\
& + 2\alpha E\int_0^T \{|\mathbf{u}'|^2 - 2\langle \mathbf{u}^N,\mathbf{u}'\rangle\}ds. \qquad (4.11)
\end{aligned}
$$

Since $\alpha \leq 0$, taking the limit inferior on both sides of (4.11) yields

$$
\begin{aligned}
& -E|g(\mathbf{u}(T,x))|^2 + E|\mathbf{v}(0,x)|^2 + E\int_0^T \|Z(s,x)\|_{L_Q}^2 ds \\
& -2E\int_0^T \alpha|\mathbf{u}|^2 ds \\
\leq & 2E\int_0^T \Big\{\langle h\mathcal{L}\mathbf{v}' + k(t,x,\mathbf{u}',\mathbf{v}') - T(s,x),\mathbf{v}'\rangle - \langle h\mathcal{L}\mathbf{v}' + k(t,x,\mathbf{u}',\mathbf{v}'),\mathbf{v}\rangle\Big\}ds \\
& + 2\alpha E\int_0^T \{|\mathbf{u}'|^2 - 2\langle \mathbf{u},\mathbf{u}'\rangle\}ds. \qquad (4.12)
\end{aligned}
$$

An application of the Itô formula to (4.4) yields

$$
\begin{aligned}
& -E|g(\mathbf{u}(T,x))|^2 + E|\mathbf{v}(0,x)|^2 + E\int_0^T \|Z(s,x)\|_{L_Q}^2 ds \\
= & -2E\int_0^T \langle T(s,x),\mathbf{u}\rangle ds.
\end{aligned}
$$

Replacing the left hand side of (4.12) using the above equality, one obtains

$$
\begin{aligned}
& -2E\int_0^T \langle T(s,x),\mathbf{u}\rangle ds - 2E\int_0^T \alpha|\mathbf{u}|^2 ds \\
\leq & 2E\int_0^T \Big\{\langle h\mathcal{L}\mathbf{v}' + k(t,x,\mathbf{u}',\mathbf{v}') - T(s,x),\mathbf{v}'\rangle - \langle h\mathcal{L}\mathbf{v}' + k(t,x,\mathbf{u}',\mathbf{v}'),\mathbf{v}\rangle\Big\}ds \\
& + 2\alpha E\int_0^T \{|\mathbf{u}'|^2 - 2\langle \mathbf{u},\mathbf{u}'\rangle\}ds. \qquad (4.13)
\end{aligned}
$$

Rearranging the terms, one gets

$$\begin{aligned}&E\int_0^T \langle -T(s,x)+h\mathcal{L}\mathbf{v}'+k(t,x,\mathbf{u}',\mathbf{v}'),\mathbf{v}-\mathbf{v}'\rangle ds\\ \le &E\int_0^T \alpha|\mathbf{u}-\mathbf{u}'|^2 ds.\end{aligned}$$

Taking $\mathbf{u}'=\mathbf{u}$ and $\mathbf{v}'=\mathbf{v}-\varepsilon\mathbf{w}$, where $\mathbf{w}\in L^\infty([0,T]\times\Omega;H_0^1(G))$ and $\varepsilon>0$, one gets

$$E\int_0^T \langle -T(s,x)+h\mathcal{L}(\mathbf{v}-\varepsilon\mathbf{w})+k(t,x,\mathbf{u},\mathbf{v}-\varepsilon\mathbf{w}),\mathbf{w}\rangle ds\le 0.$$

Letting $\varepsilon\downarrow 0$, since k is a smooth operator, one has

$$E\int_0^T \langle -T(s,x)+h\mathcal{L}\mathbf{v}+k(t,x,\mathbf{u},\mathbf{v}),\mathbf{w}\rangle ds\le 0.$$

for all $\mathbf{w}\in L^\infty([0,T]\times\Omega;H_0^1(G))$. This means

$$T(s,x)=h\mathcal{L}\mathbf{v}+k(t,x,\mathbf{u},\mathbf{v}),\quad \text{P-a.s.}$$

Combining with (4.4), we have

$$\begin{aligned}\mathbf{v}(t,x)=&\,g(\mathbf{u}(T,x))-\int_t^T(h\mathcal{L}\mathbf{v}+k(t,x,\mathbf{u},\mathbf{v}))ds\\ &+\int_t^T Z(s,x)dW(s,x),\quad \text{P-a.s.}\end{aligned}\tag{4.14}$$

Together with (4.9), we have showed that $(\mathbf{u},\mathbf{v},Z)$ is an adapted solution to system (1.1).

□

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