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New Types of Pythagorean Fuzzy Modules and Applications in Medical Diagnosis

Areej Almuhaimeed

Department of Mathematics, College of Science,

Taibah University, Madinah, Saudi Arabia

Email: aamuhaimeed@taibahu.edu.sa

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Abstract

In this article, we discuss several distinct categories of pythagorean fuzzy modules, study pythagorean fuzzy relations, and provide applications in the field of medical diagnosis. The concept of pythagorean fuzzy prime modules, along with its characteristics, is presented. In addition,an investigation is conducted into a pythagorean fuzzy multiplication module. Moreover, pythagorean fuzzy relations and pythagorean fuzzy homomorphisms are introduced. By making use of pythagorean fuzzy sets and pythagorean fuzzy relations., we propose a novel approach to the medical diagnosis process. This approach is achieved by pointing the smallest distance between the symptoms of the patients and the symptoms related to diseases.

Mathematics Subject Classification:

03E72, 16P70.

Keywords

Pythagorean fuzzy set, pythagorean fuzzy module, homomorphism, medicaldiagnosis.

1. Introduction

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Zadeh [15] was the one who first proposed using a notation for fuzzy sets. A fuzzy set is a generalization of the classical set that can construct some information that was not discovered in the classical instance. The classical set provided the basis for the development of the fuzzy set notion, as a fuzzy set is a generalization of the classical set. In fuzzy settings, for each element, there exists a membership η and non-membership $\hat{\eta}$ in which $0 \leq \eta, \hat{\eta} \leq 1$ that presents the degree that this element belongs to the discourse set.

The formed results in fuzzy sets encouraged researchers to investigate several concepts and results in classical set theory and move them in fuzzy sitting. These lead to a huge number of new ideas regarding fuzzy sets which makes it a new area to investigate.

After that, new classes of fuzzy sets under some conditions are introduced. Intuitionistic fuzzy subset was found by Atanassov[1] and [2]. In this class, the membership η and non-membership $\hat{\eta}$ satisfied: $0 \leq \eta + \hat{\eta} \leq 1$. Many applications using Intuitionistic fuzzy subset applied on decision making. In [7] , a decision-making model is found for benchmark medical datasets. Xue and Deng [12] introduced a method for decision making that took use of intuitionistic fuzzy subset. This algorithm is able to tackle some problems that other models are unable to address.

A pythagorean fuzzy set is an example of one of the fundamental categories of fuzzy sets. Yager was the one who initiated this class. [13]. The membership η and non-membership $\hat{\eta}$ in this class satisfied: $0 \leq \eta^2 + \hat{\eta}^2 \leq 1$. As a result, one might consider a pythagorean fuzzy set to be an extension of an intuitionistic fuzzy set. Moreover, this generaliztion leads to increase the applications that can be found using pythagorean fuzzy set more than the application that solved by intuitionistic fuzzy set as the condition $0 \leq \eta^2 + \hat{\eta}^2 \leq 1$ provides more pairs $(\eta, \hat{\eta})$ than the condition $0 \leq \eta + \hat{\eta} \leq 1$. Thus, using pythagorean fuzzy sets can lead to solve more problems and provide efficient and accurate algorithms.

In order to address issues relating to decision-making, an algorithm for multiple group decision making is presented in [5].using linguistic characteristic and the proposed operators. Many other applications in decision making using pythagorean fuzzy set were presented, for example [10], [9], [14], [6], [4] and [3].

Turning into medical diagnosis, there are several approaches investigated in order to solve medical diagnosis problems.

Both intuitionistic fuzzy sets and pythagorean fuzzy sets are used in medical diagnosis. In [8], a new approach to medical diagnosis that makes use of intuitionistic fuzzy sets has been proposed. Divergence measure of pythagorean fuzzy sets is used to present a new approach in medical diagnosis [11]. In [16], an application in medical diagnosis is presented by considering a divergence measure of pythagorean fuzzy sets.

This article presents the concept of pythagorean fuzzy submodule. After that, we investigate some types of pythagorean fuzzy submodules. Firsly, we introduce pythagorean fuzzy prime submodules. In addition, some resuts regarding this type are proved. Then the concept of pythagorean fuzzy multiplication submodule is introduced and invetigated. Moreover, homomorphism of pythagorean fuzzy multiplication submodule is presented. Finally, we found an algorithm depending on the relations of pythagorean fuzzy sets and applied this algorithm on medical diagnosis.

Throughout this paper, R denotes a commutative ring with unity, M an R-module, PFSS(X) a set of pythagorean fuzzy subsets of X, PFI(R) a set of pythagorean fuzzy ideals of R, PFSM(M) a set of pythagorean fuzzy submodules of M, PFP is a pythagorean fuzzy point, PFPM is a pythagorean fuzzy prime module, PFPSM is a pythagorean fuzzy prime submodule and PFMSM is a pythagorean fuzzy multiplication submodule, unless stated otherwise.

2 Preliminaries

This section concerns the fundamental notions regarding pythagorean fuzzy sets and operations.

Definition 1. *Let X be a universe of discourse. A pythagorean fuzzy set (PFS) P of X can be defined as following*

$$P = \{(a, \eta_P(a), \hat{\eta}_P(a)) : a \in X\},$$

where $\eta_P(a)$ and $\hat{\eta}_P(a)$ are the membership and non-membership values of a respectively in which

$$0 \leq \eta_P(a) \leq 1, \quad 0 \leq \hat{\eta}_P(a) \leq 1$$

and

$$0 \leq \eta_P(a)^2 + \hat{\eta}_P(a)^2 \leq 1,$$

for every $a \in X$.

We prsent some basic operations defined on PFSs.

Definition 2. *Let P, S be PFSs in a fixed universal set X. Then*

- *P is a subset of S if for all $a \in X$, we have*

$$\eta_P^2(a) \leq \eta_S^2(a) \quad \text{and} \quad \hat{\eta}_P^2(a) \geq \hat{\eta}_S^2(a)$$

.

- *The intersection $P \cap S$ is defined by:*

$$\eta^2_{P\cap S}(a) = \min\{\eta^2_P(a), \eta^2_S(a) : a \in X\}$$
$$\hat{\eta}^2_{P\cap S}(a) = \max\{\hat{\eta}^2_P(a), \hat{\eta}^2_S(a)\}.$$

- *The union $P \cup S$ can be defined by:*

$$\eta^2_{P\cup S}(a) = \max\{\eta^2_P(a), \eta^2_S(a)\}$$
$$\hat{\eta}^2_{P\cup S}(a) = \min\{\hat{\eta}^2_P(a), \hat{\eta}^2_S(a)\}.$$

- *The sum $P + S$ is defined as follows:*

$$\eta^2_{P+S}(a) = \eta^2_P(a) + \eta^2_S(a) - \eta^2_P(a)\eta^2_S(a)$$
$$\hat{\eta}^2_{P+S}(a) = \hat{\eta}^2_P(a)\hat{\eta}^2_S(a)$$

3 Pythagorean Fuzzy Module

This section concerns pythagorean fuzzy submodules.

Definition 3. *Let M be an R-module and P a PFSS of M. Suppose that*

(1) $\eta^2_P(0) = 1$ *and* $\hat{\eta}^2_P(1) = 0$.

(2) $\eta^2_P(a + b) \geq \min\{\eta^2_P(a), \eta^2_P(b)\}$ *for all* $a, b \in M$ *and* $\hat{\eta}^2_P(a + b) \leq \max\{\hat{\eta}^2_P(a), \hat{\eta}^2_P(b)\}$ *for all* $a, b \in M$.

(3) $\eta^2_P(ra) \geq \eta^2_P(a)$ *and* $\hat{\eta}^2_P(ra) \leq \hat{\eta}^2_P(a)$ *for all* $a \in M$ *and* $r \in R$, *then we say that* P *is a PFSM of* M, *and we write* $P \leq_{PF} M$.

For two pythagorean fuzzy submodules P, T of a module M, we say that P is a submodule of T if $P \subseteq T$.

Now, we define some important PFSs:

Definition 4. *Let T be a PFSS of a module M. Then*

(1) If $r, s \in [0, 1]$ *such that* $r^2 + s^2 \leq 1$, *then* $T_{[r,s]} = \{a \in M : \eta_T(a) \geq r$ *and* $\hat{\eta}_T(a) \leq s\}$ *is called the* $[r, s]$*-cut set of* M *with respect to* T.

(2) $P^\star = \eta^\star_P \cap \hat{\eta}^\star_P$, *where*

$$\eta^\star_T = \{a \in M : \eta_T(a) > 0\}$$
$$\hat{\eta}^\star_T = \{a \in M : \hat{\eta}(a) < 1\}$$

(2) $T_\star = \eta_{\star T} \cap \hat{\eta}_{\star T}$, *where*

$$\eta_{\star T} = \{a \in M : \eta_T(a) = 1\}$$
$$\hat{\eta}_{\star T} = \{a \in M : \hat{\eta}(a) = 0\}$$

Theorem 1. *Suppose that T is a PFSS of a module M. Then the (r,s)-cut of T is an R-module of M, for all $r, s \in (0,1]$ in which $r^2 + s^2 \leq 1$ if and only if T is a PFSM of M.*

Proof. Assume that the (r,s)-cut of T is an R-module of M. Then

$$\eta_T^2(0) \geq r \quad \text{and} \quad \hat{\eta}^2(1) \leq s.$$

Set $r = 1$ and $s = 0$, then $\eta_T^2(0) = 1$ and $\hat{\eta}^2(1) = 0$.. In order to see the second and third conditions, suppose that $a, b \in M$ and

$$r^2 = \min\{\eta_T^2(c), \eta_T^2(d)\},$$
$$s^2 = \max\{\hat{\eta}_T^2(c), \hat{\eta}_T^2(d)\}.$$

This implies that $c, d \in T_{[r,s]}$. That $T_{[r,s]}$ is a submodule of M, implies $ac + bd \in T_{[r,s]}$ for all $a, b \in R$. Thus

$$\eta_T^2(ac + bd) \geq r^2 = \min\{\eta_T^2(c), \eta_T^2(d)\},$$
$$\hat{\eta}_T^2(ac + bd) \leq s^2 = \max\{\hat{\eta}_T^2(c), \hat{\eta}_T^2(d)\}.$$

Hence T is a PFSS of M.

Conversly, assume that T is a PFSM of M. Suppose that $c, d \in T_{[r,s]}$ which means that $\eta_T(c) \geq r$ and $\eta_T(d) \geq r$. Since T is a PFSM of M, we obtain

$$\eta_T^2(c + d) \geq \min\{\eta_T^2(c), \eta_T^2(d)\} \geq r^2,$$
$$\hat{\eta}_T^2(ac + bd) \leq \max\{\hat{\eta}_T^2(c), \hat{\eta}_T^2(d)\} \leq s^2.$$

Hence $c + d \in T_{[r,s]}$. That T is a PFSM of M, implies

$$\eta_T^2(ac) \geq \eta_T^2(c) \geq r^2, \quad \forall a \in R,$$
$$\hat{\eta}_T^2(ac) \leq \hat{\eta}_T^2(c) \leq s^2, \quad \forall a \in R.$$

Thus $ac \in T_{[r,s]}$ and therefore, $T_{[r,s]}$ is an R-module of M. □

4 Pythagorean fuzzy prime module

In this section, we are interested in pythagorean fuzzy prime modules. We first define pythagorean fuzzy ideals, pythagorean fuzzy points and some related definitions after which the definition of pythagorean fuzzy prime module is presented.

Definition 5. *Suppose that S is a PFSS of a ring R. Then S is a PFI of R if it satisfies the following two conditions: for all $a, b \in R$,*

(1) $\eta_S^2(a-b) \geq \min\{\eta_S^2(a), \eta_S^2(b)\}$ *and* $\hat{\eta}_S^2(a-b) \leq \max\{\hat{\eta}_S^2(a), \hat{\eta}_S^2(b)\}$.

(2) $\eta_S^2(ab) \geq \max\{\eta_S^2(a), \eta_S^2(b)\}$ *and* $\hat{\eta}_S^2(ab) \leq \min\{\hat{\eta}_S^2(a), \hat{\eta}_S^2(b)\}$.

Definition 6. *Let X be a non-empty set and $r, s \in (0,1]$ such that $r^2 + s^2 \leq 1$. A PFSS $a_{(r,s)}$ of X which defined by:*

$$a_{(r,s)}(b) = \begin{cases} (r,s) & \text{if } \ a = b \\ (0,1) & \text{otherwise} \end{cases}$$

is called a pythagorean fuzzy point.

Definition 7. *A non constant PFI S of a ring R is prime if for any two PFPs $a_{(r,s)}, b_{(t,u)}$ of R, the following statment holds:*

$$\text{If } a_{(r,s)}, b_{(t,u)} \subseteq S, \text{ then } \ a_{(r,s)} \subseteq S \ \text{ or } \ b_{(t,u)} \subseteq S.$$

Recall that a proper submodule T of a module M is called prime if the following statment holds:

$$\textbf{If } am \in T, \ a \in R, \ m \in M \ \textbf{ then } \ aM \subseteq T \ \textbf{ or } \ m \in T.$$

Definition 8. *A non constant PFSM S of M is prime if for a PFI A of R and a PFSM B of (M) in which $A.B \subseteq S$, then $B \subseteq S$ or $A \subseteq (S : \chi_M)$.*

Now, we are able to present the following theorem.

Theorem 2. *Let T be a PFPSM of M. If $T_{[s,r]} \neq M_{[s,r]}$, where $s, r \in [0,1]$ in which $s^2 + r^2 \leq 1$, then $T_{[s,r]}$ is a prime submodule of $M_{[s,r]}$.*

Proof. Suppose that $am \in T_{[r,s]}$ for $a \in R$ and $m \in M$. Then

$$\eta_T^2(am) \geq r^2 \ \text{ and } \ \hat{\eta}_T^2(am) \leq s^2.$$

Thus $am_{(r,s)} = a_{(r,s)} \cdot m_{(r,s)} \subseteq T$. But T is PFPSM of M by assumpation, which means that $a_{(r,s)}M \subseteq T$ or $m_{(r,s)} \subseteq T$. Now, consider these cases:

Case(1) If $m_{(r,s)} \subseteq T$, then

$$\eta_T^2(m) \geq r^2 \quad \text{and} \quad \hat{\eta}_T^2(m) \leq s^2,$$

which implies that $m \in T_{[r,s]}$, and thus $T_{[r,s]}$ is a prime submodule of $M_{[s,r]}$.

Case(2) Assume that $a_{(r,s)}M \subseteq T$. Suppose that $d \in aM_{[s,r]}$, that is $d = ah$ for some $h \in M_{[s,r]}$. Then

$$\eta_M^2(h) \geq r^2 \quad \text{and} \quad \hat{\eta}_M^2(h) \leq s^2.$$

Thus

$$\begin{aligned} r^2 &= \min\{r^2, \eta_M^2(h)\} \\ &\leq \max\{\min\{r^2, \eta_M^2(m) : d = am\}\} \\ &= \eta^2_{a_{(r,s)}M}(d) \\ &\leq \eta_T^2(d) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} s^2 &= \max\{s^2, \hat{\eta}_M^2(h)\} \\ &\geq \min\{\max\{s^2, \hat{\eta}_M^2(m) : d = am\}\} \\ &= \hat{\eta}^2_{a_{(r,s)}M}(d) \\ &\geq \hat{\eta}_T^2(d) \end{aligned}$$

Hence $d \in T_{[r,s]}$, so that $aM_{[s,r]} \subseteq T_{[s,r]}$. Therefore, $T_{[r,s]}$ is a prime submodule of $M_{[s,r]}$.

□

Corollary 1. *If T is PFPSM of M. Then T_* is a prime submodule of M.*

Proof. Clearly, set $r = 1, s = 0$ in the previous theorem. □

Definition 9. *Let P, T be two PFSs of a module M and S a PFS of R. The residual quotients $(P : T)$ and $(P : S)$ as follows:*

$$(P : T) = \cup\{N : N \text{ is a PFSs of } R \text{ s.t. } N.T \subseteq P\},$$

$$(P : S) = \cup\{K : K \text{ is a PFSs of } M \text{ s.t. } S.K \subseteq P\}$$

Theorem 3. *Let P, T be two PFSs of a module M. Then*

$$(P : T) \cdot T \subseteq P.$$

Proof. The membership

$$\eta^2_{(P:T)\cdot T}(a) = \begin{cases} \max\{\min\{\eta^2_{(P:T)}(r), \eta^2_T(b)\}\} & \text{if } a = rb, r \in R, b \in M \\ 0 & \text{otherwise} \end{cases}$$

If $a \neq rb$ for some $r \in R, b \in M$, then $\eta^2_{(P:T)\cdot T}(a) = 0 \leq \eta^2_P$. Now, assume that $a = rb$ for some $r \in R, b \in M$. Then

$$\begin{aligned} \eta^2_{(P:T)\cdot T}(a) &= \max\{\min\{\eta^2_{(P:T)}(r), \eta^2_T(b)\}\} \\ &= \max\{\min\{\max\{\eta^2_N(r) : N \text{ is a PFSS of } R, N.T \subseteq P\}, \eta^2_T(b)\}\} \\ &= \max\{\min\{\eta^2_N(r), \eta^2_T(b) : N \text{ is a PFSS of } R, N.T \subseteq P\}\} \\ &\leq \max\{\eta^2_{N\cdot T}(rb) : N \text{ is a PFSS of } R, N.T \subseteq P\} \\ &\leq \max\{\eta^2_P(rb)\} \\ &= \eta^2_P(a) \end{aligned}$$

□

For a module M, a PFSS χ^{PF}_M is defined as follows:

$$\chi^{PF}_M = (\chi_M, \chi^c_M)$$

in which

$$\chi_M(x) = \begin{cases} 1 & \textbf{if } x \in M \\ 0 & \textbf{otherwise} \end{cases}$$

and

$$\chi^c_M(x) = \begin{cases} 0 & \textbf{if } x \in M \\ 1 & \textbf{otherwise} \end{cases}$$

Recall that a PFSM of M generated by a PFSS P of M is defined as

$$\langle P \rangle = \cap\{T : P \subseteq T; T \textbf{ is a PFSM of } M\}.$$

Now, we are able to prove the following theorem:

Theorem 4. *Let M be an R-module and P a proper PFSM of M. Then P is PFPSM of M if and only if $(P : T) = (P : \chi_M)$, for all PFSM T such that $P \subsetneq T \subseteq M$.*

Proof. Assume that P is PFPSMe of M and T is aPFSMs such that $P \subsetneq T \subseteq M$. Then $(P : \chi_M) \subseteq (P : T)$. Since $P \subsetneq T$, then there exists $m \in M$ such that $m_{(s,r)} \in T - P$, where $s, r \in [0, 1]$ and $s^2 + r^2 \leq 1$. Let $a_{(u,v)}$

be a PFPof R such that $a_{(u,v)} \in (P : T)$. By definition, $a_{(u,v)}T \subseteq P$ which implies that $a_{(u,v)}m_{(s,r)} \subseteq P$. Thus, by assumption, $a_{(u,v)} \in (P : \chi_M)$. Hence $(P : T) = (P : \chi_M)$.

Conversely, assume that $(P : T) = (P : \chi_M)$, for all PFSMs T such that $P \subsetneq T \subseteq M$. Suppose that $a_{(u,v)}m_{(s,r)} \subseteq P$ for some PFPs $a_{(u,v)}$ of R and $m_{(s,r)}$ of M and $m_{(s,r)} \subsetneq P$. By assumption, $(P : \chi_M) = (P : P + \langle m_{(s,r)} \rangle)$. That $a_{(u,v)} \subseteq (P : P + \langle m_{(s,r)} \rangle)$, implies that $a_{(u,v)} \subseteq (P : \chi_M)$ and therefore, P is a PFPSM of M. □

5 Pythagorean fuzzy Multiplication Module

Consider a module M. Then M is called a pythagorean fuzzy multiplication module if for every PFSM P of M, there exists a PFI S of R in which

(a) $\eta_S^2(0) = 1$.

(b) $\hat{\eta}^2(0) = 0$.

(c) $P = S \cdot \chi_M$.

By the above definition, note that $\eta_S^2(0) = 1, \hat{\eta}^2(0) = 0$ and $P = S \cdot \chi_M$. This implies that $P = (P : \chi_M) \cdot \chi_M$.

Theorem 5. *Any PFSM of a PFMM is multiplication.*

Proof. Clear. □

The theorem presented above does not necessarily imply that its inverse is correct.. That is, if M is a pythagorean fuzzy module that contains a pythagorean fuzzy multiplication submodule, then M need not be a pythagorean fuzzy multiplication module. For example, the zero submodule is a a pythagorean fuzzy multiplication submodule of any a pythagorean fuzzy module M.

Theorem 6. *If M is a pythagorean fuzzy module in which all its pythagorean fuzzy submodules are multiplication, then M is a pythagorean fuzzy multiplication module.*

Proof. Clear. □

Turning into homomorphisms. Let P, S be two pythagorean fuzzy R-modules and L a PFSM of P. Consider an R-homomorphism

$$\vartheta : P \longrightarrow S$$

For $s \in S$, we define:

$$\eta_{\vartheta(L)}(s) = \begin{cases} \max\{\eta_L(p) : s = \psi(p)\} & \textbf{if } s \in \mathrm{Im}(\vartheta) \\ 0 & \textbf{otherwise} \end{cases}$$

and

$$\hat{\eta}_{\vartheta(L)}(s) = \begin{cases} \min\{\eta_L(p) : s = \vartheta(p)\} & \textbf{if } s \in \mathrm{Im}(\vartheta) \\ 1 & \textbf{otherwise} \end{cases}$$

We are ready to establish the following theorem:

Theorem 7. *Assume that $\vartheta : P \longrightarrow S$ is a homomorphism of modules. If T is a PFMSM of P, then $\vartheta(T)$ is a PFMSM of S.*

Proof. Let N be a PFSM of T and let $s \in \vartheta(T)$ with $\eta_{\vartheta(N)}(s) = a$ and $\hat{\eta}_{\vartheta(N)}(s) = b$, for some $a, b \in [0, 1]$, $a^2 + b^2 \leq 1$. We need to show that

$$a \leq \eta_{(\vartheta(N):\chi_{\vartheta(T)})\cdot\chi_{\vartheta(T)}}(s), \quad b \geq \hat{\eta}_{(\vartheta(N):\chi_T)\cdot\chi_T}(s). \tag{1}$$

By definition,

$$\eta_{\vartheta(N)}(s) = \begin{cases} \max\{\eta_N(p) : s = \vartheta(p)\} & \text{if } s \in \mathrm{Im}(\vartheta) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\eta}_{\vartheta(N)}(s) = \begin{cases} \min\{\eta_N(p) : s = \vartheta(p)\} & \text{if } s \in \mathrm{Im}(\vartheta) \\ 1 & \text{otherwise} \end{cases}$$

Now, if $s \notin \mathrm{Im}(\vartheta)$, then 1 hold. Thus assume that $s \notin \mathrm{Im}(\vartheta)$. Then

$$a = \eta_{\vartheta(N)}(s) = \eta_N(p), \; b = \hat{\eta}_{\vartheta(N)}(s) = \hat{\eta}_N(p),$$

for some p, where $s = \vartheta(p)$. But T is a PFMSM of P. Thus

$$\begin{aligned} a =& \eta_{\vartheta(N)}(s) \\ =& \eta_N(p) \\ =& \eta_{(N:\chi_T)\cdot\chi_T}(p) \\ \leq& \eta_{(\psi(N):\chi_{\vartheta(T)})\cdot\chi_{\vartheta(T)}}(s) \end{aligned}$$

and

$$\begin{aligned} b =& \hat{\eta}_{\vartheta(N)}(s) \\ =& \hat{\eta}_N(p) \\ =& \hat{\eta}_{(N:\chi_T)\cdot\chi_T}(p) \\ \geq& \hat{\eta}_{(\vartheta(N):\chi_{\psi(T)})\cdot\chi_{\vartheta(T)}}(s) \end{aligned}$$

Therefore, $\vartheta(T)$ is aPFMSM of S. □

The theorem presented above does not necessarily imply that its inverse is correct. In general, if $\vartheta : P \longrightarrow S$ is a homomorphism of modules, T is a PFSM of P and $\vartheta(T)$ is a PFMSM of S, then it is not necessarily that T is a PFMSM of P. For example, consider the zero homomorphism.

6 Application in medical diagnosis

This section introduce an algorithm designed for medical diagnosis using PFS and relations. This algorithm considers the distances for PFSs by taking into account the three parameters: the membership, the non-membership functions and the hesitation margin.

Consider the set $P = \{p_1, p_2, \cdots p_r\}$ of r patients and $D = \{d_1, d_2, \cdots d_s\}$ of s diseases. Suppose that $X = \{x_1, x_2, \cdots, x_m\}$ is a set of symptoms regarding these diseases and patients. Define the pythagorean fuzzy relations $R_1 : P \to X$ and $R_2 : X \to D$ as follows:

$$R_1 = \{((p_i, x_j), \eta_{R_1}(p_i, x_j), \hat{\eta}_{R_1}(p_i, x_j)) : (p_i, x_j) \in P \times X\},$$

$$R_2 = \{((x_j, d_k), \eta_{R_2}(x_j, d_k), \hat{\eta}_{R_2}(x_j, d_k)) : (x_j, d_k) \in X \times D\},$$

Now, we present an algorithm aims to recognize which diseases affect patients.

Algorithm

Step 1: **Input the pythagorean fuzzy relations $(R_1, \eta_{R_1}, \hat{\eta}_{R_1})$ and $(R_1, \eta_{R_2}, \hat{\eta}_{R_2})$.**

Step 2: **Find the distance $|\eta_{R_1}(p_i, x_j) - \eta_{R_2}(x_j, d_k)|$ for $1 \leq i \leq r$, $1 \leq j \leq m$ and $1 \leq k \leq s$.**

Step 3: **Find $\pi_{R_1}(p_i, x_j) = \sqrt{1 - (\eta^2_{R_1}(p_i, x_j) + \hat{\eta}^2_{R_1}(p_i, x_j))}$, for every pair $(p_i, x_j) \in P \times X$ and $\pi_{R_2}(x_j, d_k) = \sqrt{1 - (\eta^2_{R_2}(x_j, d_k) + \hat{\eta}^2_{R_2}(x_j, d_k))}$, for every pair $(x_j, d_k) \in X \times D$.**

Step 4: **Compute**

$$h_{(p_i, d_k)} = \frac{1}{m}\left((\sum_{j=1}^{m} |\eta_{R_1}(p_i, x_j) - \eta_{R_2}(x_j, d_k)| + |\hat{\eta}_{R_1}(p_i, x_j)\pi_{R_1}(p_i, x_j) - \hat{\eta}_{R_2}(x_j, d_k)\pi_{R_2}(x_j, d_k)| \right).$$

Step 5: Find $h = \min\{h_{(p_i,d_1)}, h_{(p_i,d\triangleright 2)}, \cdots, h_{(p_i,d_s)},\}$ **which is the choice for** $p_i \in P$**.**

We prsent an example of using the above algorithm.

Example 1. *[8], [11], [16] Let* $P = \{p_1, p_2, p_3, p_4\}$ *be a set of patients,*

$$X = \{x_1 := Temperature, x_2 := Headache, x_3 := Stomach\ pain, x_4 := Cough, x_5 := Chest\ pain\}$$

be a set of observed symptoms and

$$D = \{d_1 := Viral\ fever, d_2 := Malaria, d_3 := Typhoid, d_4 := Stomach\ problems, d_4 := Chest\ problems\}$$

be a set of diagnoses. Consider two relations $R_1 : P \to X$ *and* $R_2 : X \to D$ *represnted by table 1 and table 2 respectively:*

R_1	x_1	x_2	x_3	x_4	x_5
P_1	$(0.9, 0.1)$	$(0.7, 0.2)$	$(0.2, 0.8)$	$(0.7, 0.2)$	$(0.2, 0.7)$
P_2	$(0.0, 0.7)$	$(0.4, 0.5)$	$(0.6, 0.2)$	$(0.2, 0.7)$	$(0.1, 0.2)$
P_3	$(0.7, 0.1)$	$(0.7, 0.1)$	$(0.0, 0.5)$	$(0.1, 0.7)$	$(0.0, 0.6)$
P_4	$(0.5, 0.1)$	$(0.4, 0.3)$	$(0.4, 0.5)$	$(0.8, 0.2)$	$(0.3, 0.4)$

Table 1: $R_1 : P \longrightarrow X$ in Example 1

R_2	d_1	d_2	d_3	d_4	d_5
x_1	$(0.3, 0.0)$	$(0.0, 0.6)$	$(0.2, 0.2)$	$(0.2, 0.8)$	$(0.2, 0.8)$
x_2	$(0.3, 0.5)$	$(0.2, 0.6)$	$(0.5, 0.2)$	$(0.1, 0.5)$	$(0.0, 0.7)$
x_3	$(0.2, 0.8)$	$(0.0, 0.8)$	$(0.1, 0.7)$	$(0.7, 0.0)$	$(0.2, 0.8)$
x_4	$(0.7, 0.3)$	$(0.5, 0.0)$	$(0.2, 0.6)$	$(0.1, 0.7)$	$(0.1, 0.8)$
x_5	$(0.2, 0.6)$	$(0.1, 0.8)$	$(0.2, 0.8)$	$(0.2, 0.7)$	$(0.8, 0.1)$

Table 2: $R_2 : X \longrightarrow D$ in Example 1

Applying the above algorithm, we obtain:

	D_1	D_2	D_3	D_4	D_5
P_1	0.2768	0.5673	0.4156	0.7827	0.8260
P_2	0.6170	0.4568	0.4073	0.2697	0.4806
P_3	0.5083	0.61042	0.2877	0.6349	0.6946
P_4	0.2401	0.5070	0.4332	0.6273	0.6845

Table 3: The relation betwwen patients and diagnoses in Example 1

	P_1	P_2	P_3	P_4
Ngan et al. [8]	Viral fever	Stomach problems	Typhoid	Viral fever
Xiao and Ding [11]	Viral fever	Stomach problems	Typhoid	Viral fever
Zhou et al. [16]	Viral fever	Stomach problems	Typhoid	Viral fever
The new method	Viral fever	Stomach problems	Typhoid	Viral fever

Table 4: The results generated by several methods in Example 1

The results of the patients marked red in the above table. Considering other methods solving this problem, we have:

7 Conclusion

In this article, we discuss several distinct categories of pythagorean fuzzy modules, study pythagorean fuzzy relations, and provide applications in the field of medical diagnosis. The concept of pythagorean fuzzy prime modules, along with its characteristics, is presented. In addition,an investigation is conducted into a pythagorean fuzzy multiplication module. Moreover, pythagorean fuzzy relations and pythagorean fuzzy homomorphisms are introduced. By making use of pythagorean fuzzy sets and pythagorean fuzzy relations., we propose a novel approach to the medical diagnosis process. This approach is achieved by pointing the smallest distance between the symptoms of the patients and the symptoms related to diseases. This algorithm can be also used in decision making for several fields.

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