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On the computation of zeros of Bessel functions

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Abstract

The subject of zeros of some chosen Bessel functions of different orders is revised using the well-known bisection method , McMahon formula is also reviewed and the calculation of some zeros are carried out implementing a recent version of MATLAB software.

The obtained results are analyzed and discussed on the lights of previous calculations.

Keywords: Bessel functions, Zeros, Bisection Method, Formula.

1. Introduction

Bessel functions (BFs) are very important in many applied fields especially in solving boundary-value problems in physics and engineering such as problems in potential theory [1] [2] [3].

Many properties of BFs were studied during the last centuries [1] ; one of the important subjects related to them is the calculation of their zeros [4] [5] [6] ; it is an interesting topic which compelled us to revisit the subject and perform the computation for some chosen BFs zeros with help of a recent MATLAB software [7].

In the next section , we give a very short account on the different kinds of BFs ; in section 3 , we introduce the bisection method (BM) technique and its use in computing the zeros of BFs [8] ; the McMahon formula(MMF) is also introduced [9].Our obtained results will be shown and discussed in section 4, at the end of which we present an interesting application emphasizing the importance of the zeros of BFs.

;. Finally, we give our conclusions in section 5

2. Different Kinds ofBFs

Bessel's equation of order $\boldsymbol{n}$ is given by

$$x^2 \frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad (1)$$

$\boldsymbol{n}$ is a positive real number and the equation has a regular singular point at $x = 0$ [1]. Using Frobenius method , one gets BFs of order $\boldsymbol{n}$ and of the first kind , namely

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n} \qquad (2)$$

If $n \to -n$ (n is not an integer) , one gets the other independent solution of Equation (1) , which is $J_{-n}(x)$. Note that if n is an integer , then $J_{-n}(x) = (-1)^n J_n(x)$.

The other kinds of BFs are

a) Neumann Functions (NFs) defined as

$$Y_n(x) = \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi} \qquad (3)$$

And are also called BFs of the second kind [1].

b) Hankel Functions (HFs) and they are of two kinds ; HFof the first kind which is defined as

$$\mathrm{H}_\alpha^1(x) = J_\alpha(x) + i\,Y_\alpha(x) \qquad (4)$$

And that of the second kind given as

$$\mathrm{H}_\alpha^2(x) = J_\alpha(x) - i\,Y_\alpha(x) \qquad (5)$$

c) Modified Bessel Functions (MBFs) which are of three kinds

MBF of the first kind given by

$$\mathrm{I}_{\pm n}(x) = \sum_{r=0}^{\infty} \frac{\left({}^{x}/_{2}\right)^{2r\pm n}}{r!\,\Gamma(r \pm n + 1)} \;\; ; n \geq 0 \qquad (6)$$

MBF of the second kind given by

$$\mathrm{K}_n(x) = \frac{\pi}{2} \,.\frac{\mathrm{I}_{-n}(x) - \mathrm{I}_n(x)}{\sin n\pi} \qquad (7)$$

d) Spherical Bessel Functions (SBFs) : these are solutions to the radial part of Helmholtz equation in spherical coordinates and are given by [2]

SBF of the first kind defined as

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\,\mathrm{J}_{\ell+\frac{1}{2}}(x)\,; \;\ell \text{is a non-negative integer} \qquad (8)$$

SBF of the second kind defined as

$$n_\ell(x) = \sqrt{\frac{\pi}{2x}}\,\mathrm{Y}_{\ell+\frac{1}{2}}(x)\ ;\ \ell \text{ is a non-negative integer} \qquad (9)$$

SBF of the third kind given by

$$\mathrm{h}_0^1(x) = \frac{-ie^{ix}}{x} \quad ; \quad h_0^2(x) = \frac{ie^{-ix}}{x} \qquad (10)$$

3. The Bisection Method

Recapitulating the same procedure adopted in reference 4 , the positive roots (or zeros) of the BF $J_v(x)(v \geq 0)$are calculated using the BM as follows [4]

Assuming that $f(x)$ is a continuous function defined on the interval [a,b] where

$$f(a)f(b) < 0 \qquad (11)$$

Which means that the zero of $f(x) = 0$ lies between a and b . Computing $x_1 = \frac{a+b}{2}$

And if $f(x_1) \neq 0$, then one makes the following test

(i) if $f(x_1)f(b) < 0$, a is then replaced by x_1 ; and

(ii) if $f(x_1)f(a) < 0$, b is then replaced by x_1 ; steps (i) and (ii) continue to be repeated until a certain tolerance (∈) is achieved , i.e

$$|x_{i+1} - x_i| < \in (12)$$

∈is a desired small positive number [4] .

Now , referring to the recurrence relation for BFs

$$J_{v-2}(x) + J_v(x) = 2\ \frac{(v-1)}{x} J_{v-1}(x) \qquad (13)$$

And with a few manipulations , one gets [4]

$$\frac{1}{(v+2k)(v+2k+1)} J_{v+2k+2}(x) + \frac{2}{(v+2k-1)(v+2k+1)} J_{v+2k}(x) + \frac{1}{(v+2k)(v+2k-1)} J_{v+2k-2}(x) = \frac{4}{x^2} J_{v+2k}(x) (14)$$

With $v \geq 0$ and $k = 1,2,3, \dots \dots \dots$

Equation (14) represents an infinite triangular matrix [4]. This equation can be written as

$$M_j = \frac{4}{x^2}\vec{J} (15)$$

Where

$$\vec{J}^{\,T} = (J_{v+2}(x)\,, J_{v+4}(x)\,, J_{v+6}(x)\,, \dots \dots) \qquad (16)$$

And $M \equiv (m_{k,p})$ is a triangular matrix with elements

$$m_{k,k} = \frac{2}{(v+2k-1)(v+2k+1)},$$

$$m_{k+1,k} = \frac{1}{(v+2k+1)(v+2k+2)},$$

$$m_{k+1,k} = \frac{1}{(v+2k)(v+2k+1)} \; ; \; k = 1,2,3, \dots \dots \qquad (17)$$

Putting $\lambda = \frac{x^2}{4}$ and $D = (d_{kp})$ where$d_{kp} = 0$ for $k \neq p$ and $d_{kk} = k(v+2k)$; $k =$ constant $\neq 0$ and $k = 1,2,3, \dots ....$

Moreover , one constructs $A = DMD^{-1}$ and putting $f = f(\lambda) = D_j$, we get from Equation (15)

$$A\overrightarrow{f} = \frac{1}{\lambda}\overrightarrow{f} \qquad (18)$$

Where A is a symmetric matrix with no-zero elements defined as

$$a_{k,k} = \frac{2}{(v+2k-1)(v+2k+1)},$$

$$a_{k,k+1} = a_{k+1,k} = \frac{1}{(v+2k+1)\sqrt{(v+2k)(v+2k+2)}} \qquad (19)$$

Note that $k = 1,2,3, \dots \dots$ and $\overrightarrow{f}^T$ is the vector

$$\overrightarrow{f}^T = [d_{1,1} J_{v+2}(x), d_{2,2} J_{v+4}(x), d_{3,3} J_{v+6}(x), \dots \dots] \equiv (f_1, f_2, f_3, \dots \dots) \qquad (20)$$

From Equation (18) and noting that $\frac{1}{\lambda} = \frac{4}{x^2}$, where x is a zero of $J_v(x)$; and λ is an eigenvalue of A , this shows that the problem of evaluating the zeros of BFs is equivalent to getting the eigenvalues of the matrix A [4].

The algorithm is summarized in the following steps:

(i) For any $v \geq 0$ we need to get $x > 0$ such that $x_1 < x_2 < x_3 < \cdots$ and $J_v(x_k) = 0$, x_k is obtained from the relation $\frac{4}{x^2} = \lambda_k$, where λ_k is the eigenvalue of A (Equation (18)).

(ii) Taking $a_0 = 0$, $b_0 = 1$ and considering the necessary condition for the zero to be in (a_0, b_0) and if μ_k is assumed to be in (a_p, b_p) obtained via the BM , i.e.

$$\mu = \frac{1}{2}(a_{r-1} + b_{r-1}) \qquad (21)$$

And with Sturm series and the convergence criterion in mind , which includes choosing an appropriate tolerance value ϵ, one gets the zeros (for $k = 1,2,3, \dots$) of $J_v(x)$ with $v = 0,0.5,1,15,20$ as shown in Table 1.

Table 1. Zeros of $J_v(x)$ for $v = 0,0.5,1,15,20$ [4].

i/v					
	2.4048256				
					55.246576
					58.602022

McMahon formula (MMF)

Many authors represented $J_n(x)$ and $Y_n(x)$ in somewhat convergent series forms for large x with the aim of facilitating the calculation of their zeros [2][9] ; e.g.
For $\upsilon = 0$ and large xMcMahon got the formula for the zeros of $J_{\upsilon,k}(x)$ as

$$x = \left(k - \frac{1}{4}\right)\pi + \frac{1}{8\left(k - \frac{1}{4}\right)\pi} - \frac{31}{384\left(k - \frac{1}{4}\right)^3 \pi^3} + \cdots \qquad (22)$$

While $J_{\upsilon,k}$,for $k \to \infty$and a constant υ, are given as [4][9]

$$J_{\upsilon,k} = k\pi + \frac{\pi}{2}\left(\upsilon - \frac{1}{2}\right) - \frac{(4\upsilon^2 - 1)}{8\left[k\pi + \frac{\pi}{2}\left(\upsilon - \frac{1}{2}\right)\right]} - \frac{(4\upsilon^2 - 1)(28\upsilon^2 - 31)}{384\left[k\pi + \frac{\pi}{2}\left(\upsilon - \frac{1}{2}\right)\right]^3} - \cdots \qquad (23)$$

MMF , also leads to different formulae for $J_{\upsilon,k}$whenυ is large [4].

4. Results

In Table 1 we have shown the calculation results of BFs zeros for $\upsilon = 0,0.5,1,15,20$ [4]. In Table 2 , the zeros of $J_k$were calculated using Equation (22) and the obtained values are compared with old known values [9]. Thetable shows the precision of our calculations as we go higher in k and as expected from MMH.

Table 2. Zeros of $J_{ok}(x)$ compared with old data.

K	j_{0k} by MMH	Old real value	Absolute error
1	2.4154177	2.4048256	0.01
2	5.5210093	5.5200781	0.0009
3	8.6539736	8.6537279	0.0002
4	11.791632	11.7915344	0.00009
5	14.930965	14.9309177	0.00004
6	18.071091	18.0710640	0.00002
7	21.211653	21.2116366	0.00001
8	24.352482	24.3524715	0.00001
9	27.493486	27.4934791	0.000006
10	30.634612	30.6346065	0.000005

Now , using Equation (23) , $J_{v,k}(x)$ are computed for $v = 1,2,3,4,5,7,10,15,20$; and the results are shown in Table 3 .

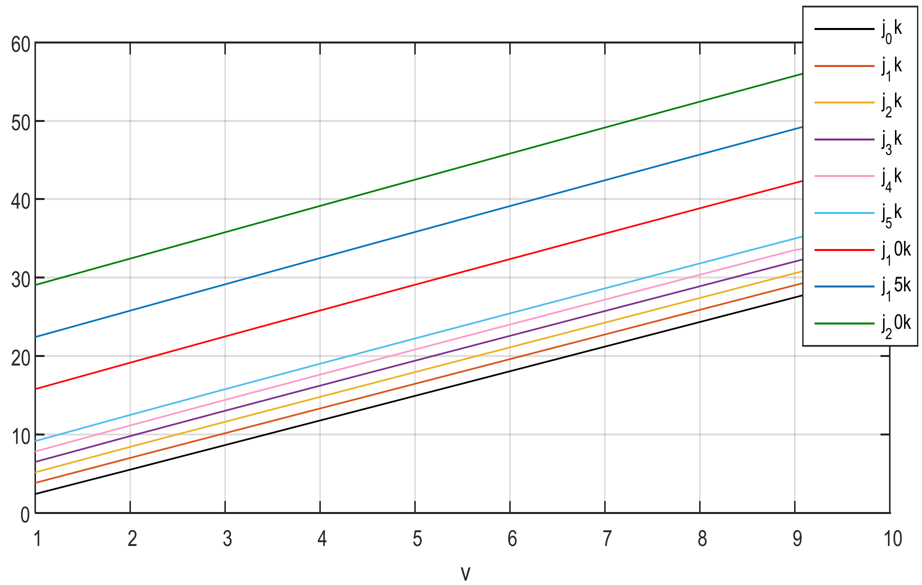
Table 3. Zeros of $J_{v,k}(x)$ for $v = 1,2,3,4,5,7,10,15,20$ using equation (23)

ν K	1	2	3	4	5	7	10	15	20
1	3.8311108	5.1757814	6.5066813	7.8339517	9.1601930	11.8122665	15.7912684	22.4253490	29.0611927
2	7.0154654	8.4272571	9.80060676	11.15436268	12.4973878	15.1672706	19.1555137	25.7907669	32.4236900
3	10.1734260	11.6237525	13.03255960	14.415428851	15.7815012	18.48350408	22.4982120	29.1511086	35.7890187
4	13.3236727	14.7978681	16.23259272	17.63981786	19.0274850	21.7639995	25.8130824	32.49580007	39.1468280
5	16.4706197	17.9608978	19.41479813	20.84156325	22.2478217	25.0167913	29.1023663	35.82199904	42.4925413
6	19.6158523	21.1176628	22.58616795	24.02863928	25.4505810	28.2486921	32.3700058	39.12995810	45.8244998
7	22.7600804	24.27055183	25.75049565	27.20575042	28.6409134	31.4647626	35.6197905	42.42109559	49.1425186
8	25.9036694	27.42087878	28.91000090	30.37581250	31.8221891	34.6686709	38.8549226	45.69717444	52.4471382
9	29.0468266	30.56942503	32.06606398	33.54071664	34.9966832	37.8630857	42.0779971	48.95995587	55.7392295
10	32.18967851	33.71668400	35.21958644	36.70173846	38.1659780	41.0499739	45.29108866	52.21106257	59.0197848
11	35.33230649	36.86298245	38.37118130	39.85976771	41.3312045	44.2308078	48.49585351	55.45193519	62.2898090
12	38.47476541	40.00854528	41.521279614	43.01544299	44.4931906	47.4067065	51.69361924	58.68382966	65.5502641
13	41.61709356	43.15353234	44.670193103	46.16923410	47.6525557	50.57853405	54.88545646	61.90783142	68.8020438
14	44.75931847	46.29806031	47.818152733	49.32149409	50.8097717	53.74696629	58.07223499	65.12487508 7	72.0459637
15	47.90146046	49.44221637	50.965333203	52.47249330	53.9652042	56.91253922	61.25466690	68.33576494	75.2827611
16	51.04353483	52.58606695	54.111869070	55.62244215	57.1191408	60.07568271	64.43333961	71.54119383	78.5130979
17	54.18555334	55.72966356	57.257865642	58.77150683	60.27181060	63.23674533	67.60874127	74.74175992	81.7375668
18	57.32752518	58.87304674	60.403406498	61.91982029	63.42339841	66.39601251	70.78128046	77.93798102	84.9566975

19	60.46945763	62.01624886	63.548558816	65.06749017	66.57405489	69.55372012	73.95130142	81.13030681	88.1709633
20	63.61135651	65.15929604	66.693377202	68.21460453	69.72390402	72.71006470	77.11909606	84.31912911	91.3807871

In Figure (1) ,weshow the behavior of the zeros of $J_{v,k}(x)$ for $k = 1,2,\dots,10$, using MMF.

Figure (1). The behavior of the zeros of some chosen$J_{v,k}(x)$, using MMF.



Moreover , we give in Table 4 a comparison of the absolute errors in the values of BFs zeros for $\upsilon = 15,20$ using MMF and the BM respectively.

Table 4. A comparison of the absolute errors in the values of zeros of $J_{v,k}$for $\upsilon = 15,20$

	Mac Mahon Method		Bisection method	
K				
1	22.4253490	29.0611927	19.994431	25.417141
2	25.7907669	32.4236900	24.269180	29.961604
3	29.1511086	35.7890187	28.102415	33.988703
4	32.49580007	39.1468280	31.733413	37.772858
5	35.82199904	42.4925413	35.247087	41.413065
6	39.12995810	45.8244998	38.684276	44.957677
7	42.42109559	49.1425186	42.067917	48.434239
8	45.69717444	52.4471382	45.412190	51.860020
9	48.95995587	55.7392295	48.726464	55.246576
10	52.21106257	59.0197848	52.017241	58.602022
11	55.45193519	62.2898090	55.289204	61.932273
12	58.68382966	65.5502641	58.545829	65.241766

13	61.90783142	68.8020438	61.789760	
14	65.124875087	72.0459637		
15	68.33576494	75.2827611		
16	71.54119383	78.5130979		
17	74.74175992	81.7375668		
18	77.93798102	84.9566975		
19	81.13030681	88.1709633		
20	84.31912911	91.3807871		

While in Figure (2), we present the behavior of the zeros of $J_\upsilon(x)$ for $\upsilon = 0,1,3,4,5,10,15,20$using MMF.

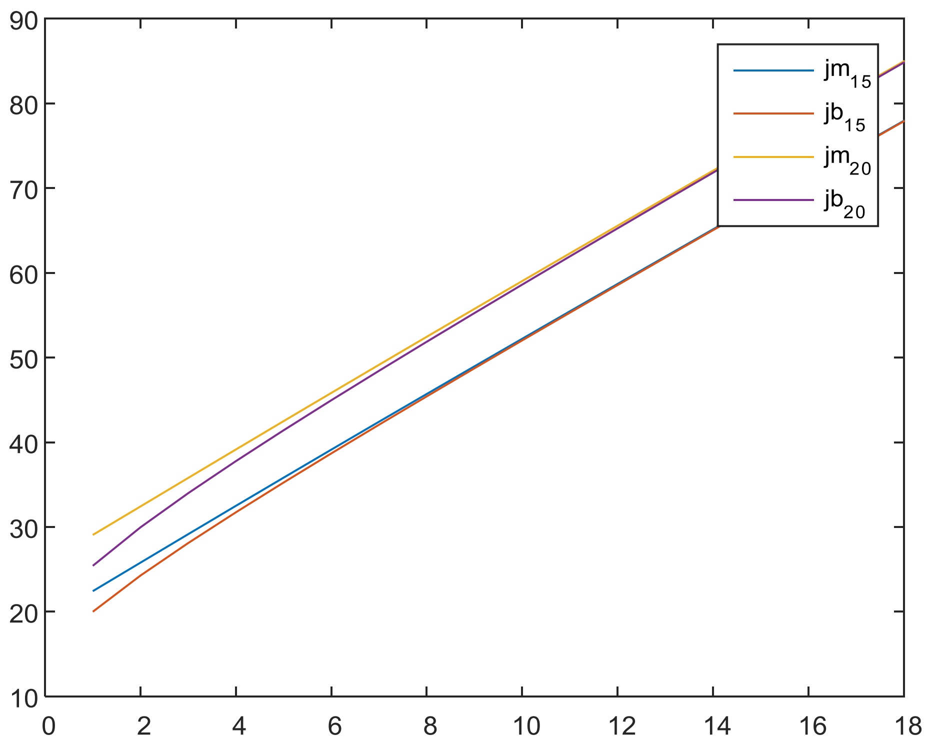


Figure (2). Comparison of $J_\upsilon(x)$ for $\upsilon = 15,20$ using BM and MMF.

From the obtained results in Table 1 , Table 2 , Table 3 , Table 4 , and from Figure (1) and Figure (2) , it is clear that the BM is more precise than MMF especially when υ becomes large.

Note that Olver [10] introduced a form for evaluating $J_{\upsilon,k}$ when υ is large ; for instance and according to olver$J_{\upsilon,3}$ and $J_{\upsilon,5}$ are given as

$$J_{v,3} = v + 4.3816712v^{\frac{1}{3}} + 5.7597129v^{-\frac{1}{3}} - 0.22608v^{-1} - 2.80395v^{-\frac{5}{3}} + 3.9760v^{-\frac{7}{3}} + \cdots \quad (24)$$

And

$$J_{v,5} = v + 6.305263v^{\frac{1}{3}} + 11.9269025v^{-\frac{1}{3}} - 0.701926v^{-1} - 12.01933v^{-\frac{5}{3}} + 24.8020v^{-\frac{7}{3}} + \cdots \quad (25)$$

Accordingly and using Olver formulae (OF) for $J_{v,k}$, we present in Table 5 the zeros of $J_{v,k}$ for $v = 1,2,3,4,5,15,20$. These showed that OF gave good approximate values for $v = 1$; $k = 1,2$. but some discrepancies arose when $v = 1$ and $k > 3$. In general the precision of OF improves as v increases , while it deviates from real values of the zeros if k increases ..

Table 5. zeros of $J_{v,k}$ for $v = 1,2,3,4,5,15,20$ according to OF.

K V							
	3.8375114	5.1361439909	6.38028	7.5883828	8.5206434	19.9534	25.3729159
	7.33984	8.458684	9.772977	11.069547	12.34099	24.2534627	29.9449630
	12.08735	11.884756	13.094644	14.405699	15.7168	28.0935029	33.9790442
	19.50516222	15.694791	16.500806	17.7339	19.04021	31.7287143	37.7667263
	31.312902	20.1950477	20.114092	21.131026	22.374419	35.2472598	41.4099046
	49.440163	25.726508	24.0583125	24.6612	25.762191	38.6918668	44.958156
	76.0885043	32.691647	28.4847705	28.4046006	29.256283	42.087237	48.439939
	1.1356105	41.5064717	33.541583	32.431359	32.896472	45.4494598	51.8734070
	1.6433348	52.6258006	39.394357	36.822279	36.729602	48.7900215	55.2710612
	2.310175	66.532957	46.2197956	41.663379	40.380511	52.117700	58.6420518

An interesting Application

In this application, we present the vibrating circular membrane problem and shed light on the usefulness of th zeros of BFs in determining the frequencies and modes of the related vibrations [11]. Consider the following wave equation in polar coordinates,

$$\frac{\partial^2 \mathrm{u}}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) \quad (26)$$

With

$$u = u(r,t) \ , t \geq 0 \ \ , 0 < r < a \ \ , u(a,t) = 0 \ \ , \text{and} u(r,0) = f(r) \ \ , \frac{\partial u(r,0)}{\partial r} = g(r) \quad (27)$$

Solving the above equation and using the above initial and boundary condition with the use of separation of variables, we obtain

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r) \quad (28)$$

‘

$$R(a) = c_1 J_0(\lambda a) = 0 \; \rightarrow \; J_0(\lambda a) = 0 \quad (29)$$

Hence λa are the zeros of BF of the zeroth order and

$$\lambda a = \alpha_{0n} \quad , n = 1,2, \ldots \quad (30)$$

While the eigenfunctions are given by

$$R(r) = J_0\left(\frac{\alpha_{0n} r}{a}\right) \quad (31)$$

The solution for the vibrating membrane is as follows

$$u_{0n}(\mathrm{r,t}) = (A_n \cos(c\lambda_{0n} t) + B_n sinc(\lambda_{0n} t) J_0(\lambda_{0n} r) \quad (32)$$

With

$$\lambda_{0n} = \frac{\alpha_{0n}}{a} n = 1,2, \ldots \quad ,$$

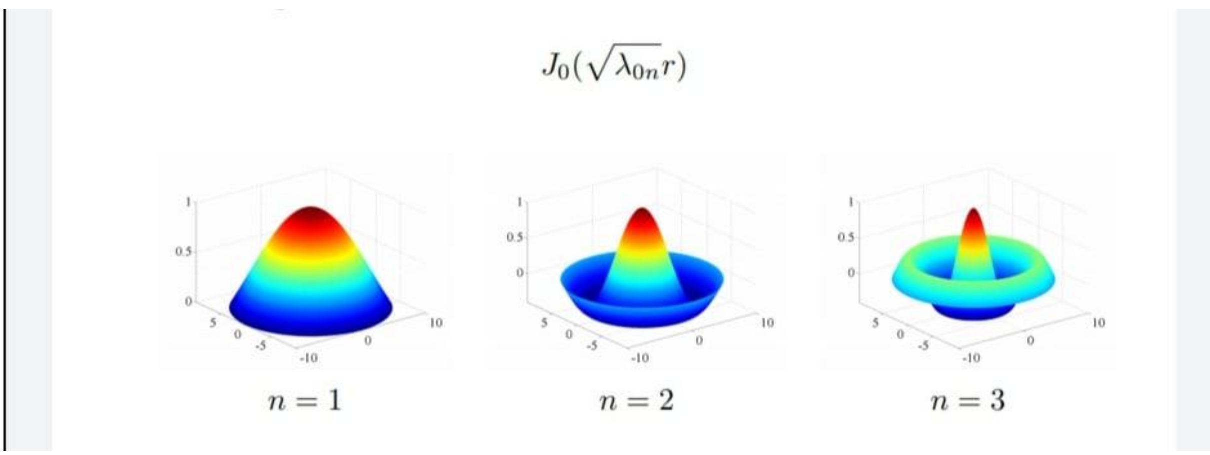
And the coefficients are given by

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\lambda_n r) r dr \quad (33)$$

,

$$B_n = \frac{2}{ac\alpha_n J_1^2(\alpha_n)} \int_0^a g(r) J_0(\lambda_n r) r dr \quad (34)$$

With appropriate choice of $f(r)$ and $g(r)$,one obtains modes of vibration for the circular membrane as shown in Figure (3) [11]



Figure(3).Three different modes of vibrations for the circular membrane.

Now if we take $a = 1$, $c = 1$, $f(r) = J_0(\alpha_{03} r)$, $g(r) = 1 - r^2$,then we can calculate the coefficients $A's$ and $B's$ with the help of the zeros of BF of zeroth order and these are $A_n = \frac{2}{J_1^2(\alpha_{03})} \int_0^1 J_0^2(\alpha_{03} r) r dr = 1$ and $B_n = \frac{8}{\alpha_{0n}^4 J_1(\alpha_n)}$; the solution is then by $u(r,t) = J_0(\alpha_{03} r)\cos(\alpha_{03} t) + 8\sum_{n=1}^{\infty} \frac{J_0(\alpha_{0n} r)}{\alpha_{0n}^4 J_1(\alpha_n)} \sin(\alpha_{0n} t)$ [11].

Using the first five zeros of BF of the zeroth order given in Table 6, we get the solution as [11]

$$u(r,t) = J_0(\alpha_{03}r)\cos(\alpha_{03}t) + 0.46081J_0(2.4048r)\sin(2.4048t) - 0.025318J_0(5.5201r)\sin(5.5201t) + 0.005256J_0(8.6537r)\sin(8.6537t) - 0.001779J_0(11.7915r)\sin(11.7915r) + 0.0007795J_0(14.9309r)\sin(14.9309t)$$

Table 6.The first five zeros of BF .

N	1	2	3	4	5
	2.4048	5.5201	8.6537	11.7915	14.9309
	0.5191	-0.3403	0.2714	-0.2325	0.2065
	0.46081	-0.0253	0.0052	-0.0017	0.00077

In general ,when there is no symmetry, the wave equation will take the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \quad (35)$$

With conditions as$u(r,\theta,0) = f(r,\theta)$, $\frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta)$,and $0 < r < a$,$0 < \theta < 2\pi$,$t > 0$.

Again, with the use of separation of variables we obtain

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta))\cos(c\lambda_{mn}t) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(a_{mn}^{*}\cos(m\theta) + b_{mn}^{*}\sin(m\theta)\sin(c\lambda_{mn}t)) \quad (36)$$

With the various coefficients given as

$$a_{0n} = \frac{1}{\pi a^2 J_1^2((a_{0n})}\int_0^a\int_0^{2\pi} f(r,\theta)J_0(\lambda_{0n}r)rd\theta dr \quad (37)$$

,

$$a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2((a_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\cos(m\theta)J_m(\lambda_{mn}r)rd\theta dr \quad (38)$$

,

$$b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2((a_{mn})}\int_0^a\int_0^{2\pi} f(r,\theta)\sin(m\theta)J_m(\lambda_{mn}r)rd\theta dr \quad (39)$$

$$a_{0n}^{*} = \frac{1}{\pi c a \alpha_{0n} J_1^2(a_{0n})}\int_0^a\int_0^{2\pi} g(r,\theta)J_0(\lambda_{0n}r)rd\theta dr \quad (40)$$

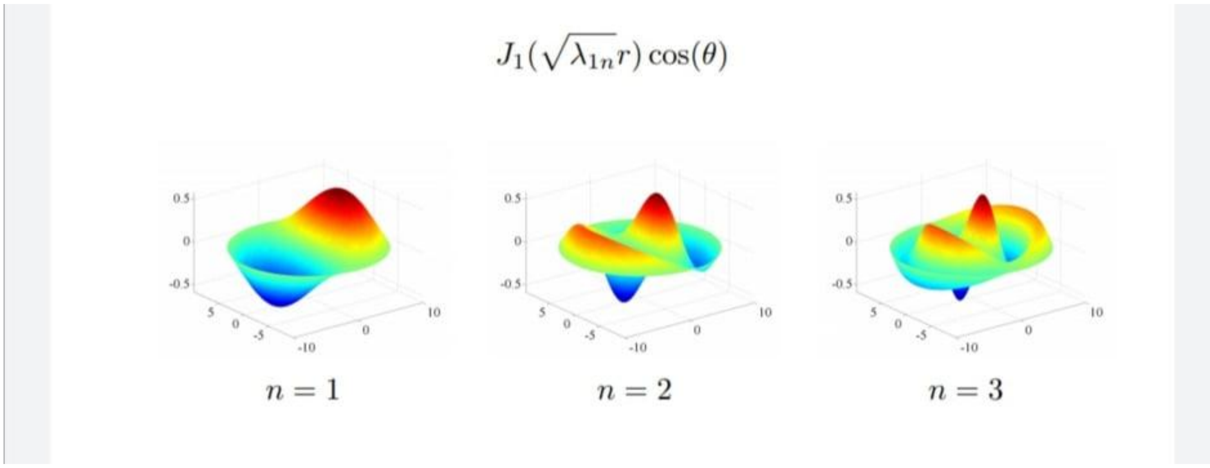
,

$$a_{mn}^{*} = \frac{2}{\pi c a \alpha_{mn} J_{m+1}^2(a_{mn})}\int_0^a\int_0^{2\pi} g(r,\theta)\cos(m\theta)J_m(\lambda_{mn}r)rd\theta dr \quad (41)$$

,

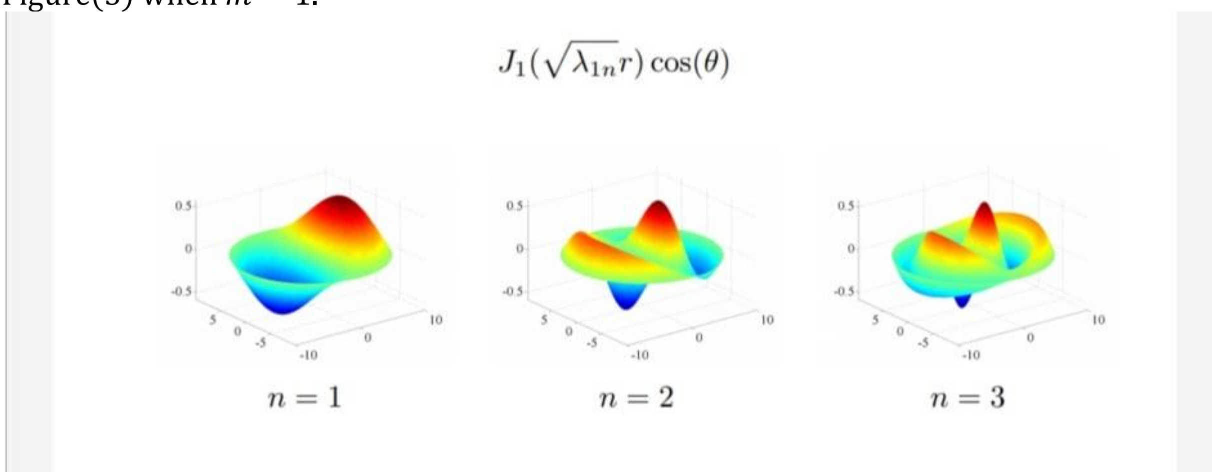
$$b_{mn}^{*} = \frac{2}{\pi c a \alpha_{mn} J_1^2(a_{mn})}\int_0^a\int_0^{2\pi} g(r,\theta)\sin(m\theta)J_m(\lambda_{mn}r)rd\theta dr \quad (42)$$

Where $m, n = 1,2, \ldots; \lambda_{mn} = \frac{\alpha_{mn}}{a}$ and α_{mn} are the zeros of BFs[11].
In Figure(4) ,we show the vibration modes for the circular membrane for the case of taking BF of the first order.



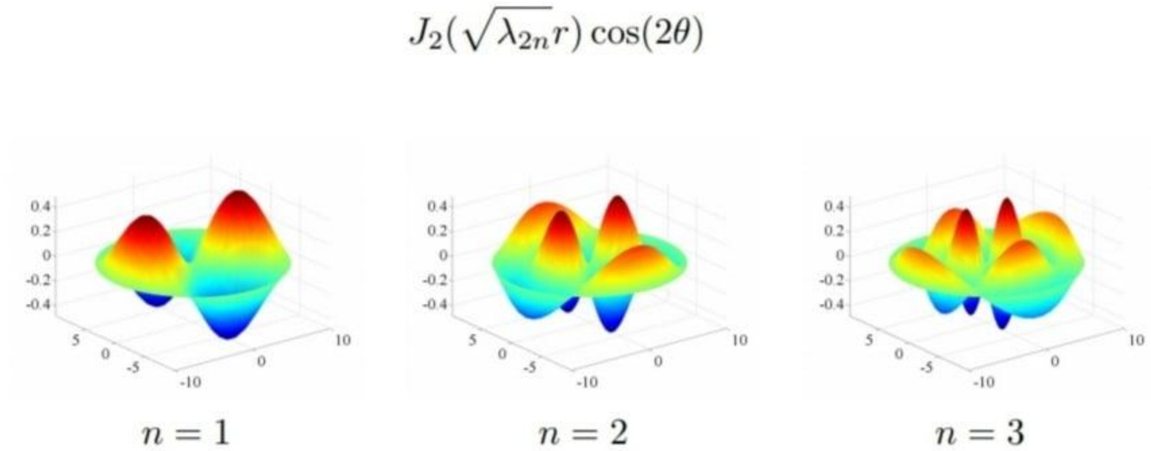
Figure(4).The vibration modes of the circular membrane in case of 1st order BF and $m = 1$.

The vibration modes for the circular membrane in the case of BF of the 2nd order is shown in Figure(5) when $m = 1$.



Figure(5). Vibration modes for the membrane in case of Bf of the 3rd order.

When $m = 2$ and the BF is of the 2nd order, the vibration modes will be as in Figure(6) shown below.



Figure(6). Vibration modes for the circular membrane when $m = 2$ and the BF is of the 2^{nd} order.

Finally, we show in Figure(7) the vibration modes for $m = 3$ and BF of the 3^{rd} order.

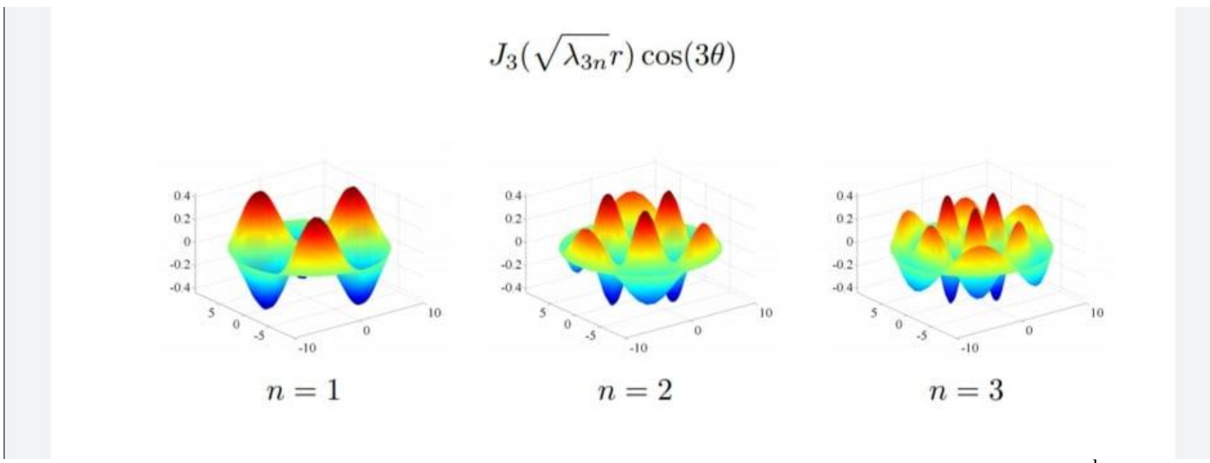


Figure (7).Vibration modes for the circular membrane when $m = 3$ and BF of the 3^{rd} order.

5. Concluding discussion

As we have seen from the outcome of this study ; the BM , though it is simple , proved to be a precise method in computing zeros of BFs in comparison with old methods such as MMF [9]. Note that the importance of computing zeros of BFs started long time ago when Stokes and Poisson used convergent series forms or representations for $J_v(x)$ [9] ; however , the subject is still of interest and a good number of research articles were writtenon them , i.e. on the zeros of $J_v(x)$, their computation , and their properties including monotonicity , convexity , and concavity[12][13][14][15].

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