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On the Solutions of Systems of Rational Difference Equations

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Abstract

In this paper we study the form of the solutions of the following systems of difference equations

$$w_{n+1} = \frac{s_n(w_{n-3} + s_{n-4})}{s_{n-4} + w_{n-3} - s_n}, \quad s_{n+1} = \frac{w_{n-2}(w_{n-2} + s_{n-3})}{2w_{n-2} + s_{n-3}}.$$

$$w_{n+1} = \frac{(s_{n-4} - w_{n-3})s_n}{s_{n-4} - w_{n-3} + s_n}, \quad s_{n+1} = \frac{(s_{n-3} - w_{n-2})w_{n-2}}{s_{n-3}}.$$

With nonzero real numbers initial conditions.

Keywords

Solutions of difference equations, periodic solution, recursive sequences.

Mathematics Subject Classification: 39A10.

1 Introduction:

Recently, there has been great interest in studying difference equation systems. This is due,to the need for certain methods that can be used to analyze equations emerging in mathematical models depicting exact situation in the fields of population biology, economics, probability theory, genetics, psychology, and others. Difference equations naturally arise as discrete analogs and as numerical solutions to differential and delay differential equations having applications in biology, ecology, economy, physics, and other fields. See [1–10] and the references cited

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therein. There are many papers related to the difference equations system, for example, the periodicity of solutions of the system of rational difference equations

$$w_{n+1} = \frac{w_{n-1} + s_n}{w_{n-1}s_{n-1}}, \quad s_{n+1} = \frac{s_{n-1} + w_n}{s_{n-1}w_{n-1}}$$

was studied by Kurbanli et al. in [18].

Touafek et al.[21] studied the periodic nature and gave of the solutions of the systems of difference equations

$$w_{n+1} = \frac{w_{n-3}}{\pm 1 \pm w_{n-3}s_{n-1}}, \quad s_{n+1} = \frac{s_{n-3}}{\pm 1 \pm s_{n-3}w_{n-1}}.$$

Kurbanli et al. [19] studied the behavior of the positive solutions of the following system

$$w_{n+1} = \frac{w_{n-1}}{1 + w_{n-1}s_n}, \quad s_{n+1} = \frac{s_{n-1}}{1 + s_{n-1}w_n}.$$

Mansour et al. [20] investigated the behavior of solutions of the difference equations systems

$$w_{n+1} = \frac{w_{n-5}}{-1 + w_{n-5}s_{n-2}}, \quad s_{n+1} = \frac{s_{n-5}}{\pm 1 \pm s_{n-5}w_{n-2}}.$$

Zhang et al. [27] studied the dynamics of a system of the third-order difference equation

$$w_{n+1} = \frac{w_{n-2}}{B + s_n s_{n-1} s_{n-2}}, \quad s_{n+1} = \frac{s_{n-2}}{A + sw_{nn-1}w_{n-2}}.$$

Similarly, difference equations and nonlinear systems of the rational difference equations were investigated see [1]-[27].
Our aim in this paper is to consider the following systems of difference equations

$$w_{n+1} = \frac{s_n(w_{n-3} + s_{n-4})}{s_{n-4} + w_{n-3} - s_n}, \quad s_{n+1} = \frac{w_{n-2}(w_{n-2} + s_{n-3})}{2w_{n-2} + s_{n-3}}.$$

$$w_{n+1} = \frac{(s_{n-4} - w_{n-3})s_n}{s_{n-4} - w_{n-3} + s_n}, \quad s_{n+1} = \frac{(s_{n-3} - w_{n-2})w_{n-2}}{s_{n-3}}.$$

With nonzero real numbers initial conditions.

2 The system $w_{n+1} = \frac{s_n(w_{n-3}+s_{n-4})}{s_{n-4}+w_{n-3}-s_n}, s_{n+1} = \frac{w_{n-2}(w_{n-2}+s_{n-3})}{2w_{n-2}+s_{n-3}}$

In this section, we study the solutions of the system of the difference equations

$$w_{n+1} = \frac{s_n(w_{n-3} + s_{n-4})}{s_{n-4} + w_{n-3} - s_n}, \quad s_{n+1} = \frac{w_{n-2}(w_{n-2} + s_{n-3})}{2w_{n-2} + s_{n-3}}. \tag{1}$$

The initial conditions of system (1) are arbitrary real numbers.
Theorem 1. Suppose that $\{w_n, s_n\}$ are solutions of system (1). Then solutions of $\{w_n\}$ are periodic with period four and given by the following formulas for n=1,2,... ,

$$w_{4n-1} = c, \quad w_{4n} = d, \qquad w_{4n+1} = \frac{g(f + a)}{f + a - g}, \quad w_{4n+2} = b,$$

$$s_{4n-1}=\frac{d(d\phi_{2n+1}+t\phi_{2n})}{d\phi_{2n+2}+t\phi_{2n+1}}, \qquad s_{4n}=\frac{g(a+f)((a+f)\phi_{2n}-g\phi_{2n-2})}{(f+a-g)((a+f)\phi_{2n+1}-g\phi_{2n-1})},$$

$$s_{4n+1}=\frac{b(b\phi_{2n+1}+h\phi_{2n})}{b\phi_{2n+2}+h\phi_{2n+1}}, \qquad s_{4n+2}=\frac{c(c\phi_{2n+1}+r\phi_{2n})}{c\phi_{2n+2}+r\phi_{2n+1}}.$$

Where $w_{-3}=a,\ w_{-2}=b,\ w_{-1}=c,\ w_0=d,\ s_{-4}=f,\ s_{-3}=h,\ s_{-2}=r,\ s_{-1}=t$**,** $s_0=g$ **and** $\{\phi\}_{m=0}^{\infty}=\{1,1,2,3,5,...\}$**,** $\phi_{m+2}=\phi_{m+1}+\phi_m$**.**

Proof: For n=0, the result holds. Now, suppose that $n>0$ **and that our assumption holds for** $n-1$ **and** $n-2$**. That is**

$$w_{4n-5}=c, \quad w_{4n-4}=d, \qquad w_{4n-3}=\frac{g(f+a)}{f+a-g}, \quad w_{4n-2}=b,$$

$$s_{4n-5}=\frac{d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}, \qquad s_{4n-4}=\frac{g(a+f)((a+f)\phi_{2n-2}-g\phi_{2n-4})}{(f+a-g)((a+f)\phi_{2n-1}-g\phi_{2n-3})},$$

$$s_{4n-3}=\frac{b(b\phi_{2n-1}+h\phi_{2n-2})}{b\phi_{2n}+h\phi_{2n-1}}, \qquad s_{4n-2}=\frac{c(c\phi_{2n-1}+r\phi_{2n-2})}{c\phi_{2n}+r\phi_{2n-1}},$$

$$w_{4n-9}=c, \quad w_{4n-8}=d, \qquad w_{4n-7}=\frac{g(f+a)}{f+a-g}, \quad w_{4n-6}=b,$$

$$s_{4n-9}=\frac{d(d\phi_{2n-3}+t\phi_{2n-4})}{d\phi_{2n-2}+t\phi_{2n-3}}, \qquad s_{4n-8}=\frac{g(a+f)((a+f)\phi_{2n-4}-g\phi_{2n-6})}{(f+a-g)((a+f)\phi_{2n-3}-g\phi_{2n-5})},$$

$$s_{4n-7}=\frac{b(b\phi_{2n-3}+h\phi_{2n-4})}{b\phi_{2n-2}+h\phi_{2n-3}}, \qquad s_{4n-6}=\frac{c(c\phi_{2n-3}+r\phi_{2n-4})}{c\phi_{2n-2}+r\phi_{2n-3}}.$$

Now, it follows from Eq.(1) that

$$w_{4n}=\frac{s_{4n-1}(w_{4n-4}+s_{4n-5})}{s_{4n-5}+w_{4n-4}-s_{4n-1}}$$

$$=\frac{\frac{d(d\phi_{2n+1}+t\phi_{2n})}{d\phi_{2n+2}+t\phi_{2n+1}}\left(d+\frac{d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}\right)}{\frac{d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}+d-\frac{d(d\phi_{2n+1}+t\phi_{2n})}{d\phi_{2n+2}+t\phi_{2n+1}}}$$

$$=\frac{\frac{d(d\phi_{2n+1}+t\phi_{2n})}{d\phi_{2n+2}+t\phi_{2n+1}}\left(\frac{d(d\phi_{2n}+t\phi_{2n-1})+d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}\right)}{\frac{d(d\phi_{2n-1}+t\phi_{2n-2})(d\phi_{2n+2}+t\phi_{2n+1})+d(d\phi_{2n}+t\phi_{2n-1})(d\phi_{2n+2}+t\phi_{2n+1})-d(d\phi_{2n+1}+t\phi_{2n})(d\phi_{2n}+t\phi_{2n-1})}{(d\phi_{2n}+t\phi_{2n-1})(d\phi_{2n+2}+t\phi_{2n+1})}}=d.$$

And

$$s_{4n}=\frac{w_{4n-3}(w_{4n-3}+s_{4n-4})}{2w_{4n-3}+s_{4n-4}}$$

$$=\frac{\frac{g(f+a)}{f+a-g}\left(\frac{g(f+a)}{f+a-g}+\frac{g(a+f)((a+f)\phi_{2n-2}-g\phi_{2n-4})}{(f+a-g)((a+f)\phi_{2n-1}-g\phi_{2n-3})}\right)}{\frac{2g(f+a)}{f+a-g}+\frac{g(a+f)((a+f)\phi_{2n-2}-g\phi_{2n-4})}{(f+a-g)((a+f)\phi_{2n-1}-g\phi_{2n-3})}}$$

$$=\frac{\frac{g(f+a)}{f+a-g}\left(\frac{((a+f)(\phi_{2n-1}+\phi_{2n-2})-g(\phi_{2n-3}+\phi_{2n-4})}{((a+f)\phi_{2n-1}-g\phi_{2n-3})}\right)}{\frac{2((a+f)\phi_{2n-1}-g\phi_{2n-3})+((a+f)\phi_{2n-2}-g\phi_{2n-4})}{((a+f)\phi_{2n-1}-g\phi_{2n-3})}}$$

$$= \frac{g(a+f)((a+f)\phi_{2n} - g\phi_{2n-2})}{(f+a-g)((a+f)\phi_{2n+1} - g\phi_{2n-1})}.$$

Also,

$$w_{4n-1} = \frac{s_{4n-2}(w_{4n-5} + s_{4n-6})}{s_{4n-6} + w_{4n-5} - s_{4n-2}}$$

$$= \frac{\frac{c(c\phi_{2n-1}+r\phi_{2n-2})}{c\phi_{2n}+r\phi_{2n-1}}\left(c + \frac{c(c\phi_{2n-3}+r\phi_{2n-4})}{c\phi_{2n-2}+r\phi_{2n-3}}\right)}{\frac{c(c\phi_{2n-3}+r\phi_{2n-4})}{c\phi_{2n-2}+r\phi_{2n-3}} + c - \frac{c(c\phi_{2n-1}+r\phi_{2n-2})}{c\phi_{2n}+r\phi_{2n-1}}}$$

$$= \frac{\frac{c(c\phi_{2n-1}+r\phi_{2n-2})}{c\phi_{2n}+r\phi_{2n-1}}\left(\frac{c(c\phi_{2n-2}+r\phi_{2n-3})+c(c\phi_{2n-3}+r\phi_{2n-4})}{c\phi_{2n-2}+r\phi_{2n-3}}\right)}{\frac{c(c\phi_{2n-3}+r\phi_{2n-4})(c\phi_{2n}+r\phi_{2n-1})+c(c\phi_{2n-2}+r\phi_{2n-3})(c\phi_{2n}+r\phi_{2n-1})-c(c\phi_{2n-1}+r\phi_{2n-2})(c\phi_{2n-2}+r\phi_{2n-3})}{(c\phi_{2n-2}+r\phi_{2n-3})(c\phi_{2n}+r\phi_{2n-1})}} = c.$$

And

$$s_{4n-1} = \frac{w_{4n-4}(w_{4n-4} + s_{4n-5})}{2w_{4n-4} + s_{4n-5}}$$

$$= \frac{d\left(d + \frac{d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}\right)}{2d + \frac{d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}}$$

$$= \frac{d\left(\frac{d(d\phi_{2n}+t\phi_{2n-1})+d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}\right)}{\frac{2d(d\phi_{2n}+t\phi_{2n-1})+d(d\phi_{2n-1}+t\phi_{2n-2})}{d\phi_{2n}+t\phi_{2n-1}}}$$

$$= \frac{d(d\phi_{2n+1} + t\phi_{2n})}{d\phi_{2n+2} + t\phi_{2n+1}}.$$

Similarly, obtaining the other relations is very simple. Thus, the proof is completed.
Example 1. Figure 1 we consider the solution of Eq.(1) with the initial conditions $w_{-3} = 0.66,\ w_{-2} = 0.4,\ w_{-1} = 2.2,\ w_0 = 1.9,\ s_{-4} = 1.6,\ s_{-3} = 3.8,\ s_{-2} = 4.3,\ s_{-1} = 1.66$ **and** $s_0 = 3.3$**.**
Example 2. Figure 2 shows the solution of Eq.(1) when we assume that $w_{-3} = 1.5,\ w_{-2} = 0.4,\ w_{-1} = 2.2,\ w_0 = 3.9,\ s_{-4} = 0.5,\ s_{-3} = 1.8,\ s_{-2} = 0.3,\ s_{-1} = 3.66$ **and** $s_0 = 2.1$**.**

3 The system $w_{n+1} = \frac{(s_{n-4}-w_{n-3})s_n}{s_{n-4}-w_{n-3}+s_n}, s_{n+1} = \frac{(s_{n-3}-w_{n-2})w_{n-2}}{s_{n-3}}$

In this section, we study the solutions of the system of the difference equations

$$w_{n+1} = \frac{(s_{n-4} - w_{n-3})s_n}{s_{n-4} - w_{n-3} + s_n}, \quad s_{n+1} = \frac{(s_{n-3} - w_{n-2})w_{n-2}}{s_{n-3}}. \tag{2}$$

The initial conditions of system (2) are arbitrary real numbers.
Theorem 1. Suppose that $\{w_n, s_n\}$ **are solutions of system (2). Then for n=0,1,2, ... , we have**

$$w_{4n} = \frac{(t-d)d^2t}{(t\phi_{n-2} + d\phi_{n-3})(t\phi_{n-1} + d\phi_{n-2})(t\phi_n + d\phi_{n-1})},$$

$$w_{4n+1} = \frac{(f-a)^2g^2}{((f-a)\phi_{n+1} + g\phi_n)((f-a)\phi_n + g\phi_{n-1})((f-a)\phi_{n-1} + g\phi_{n-2})},$$

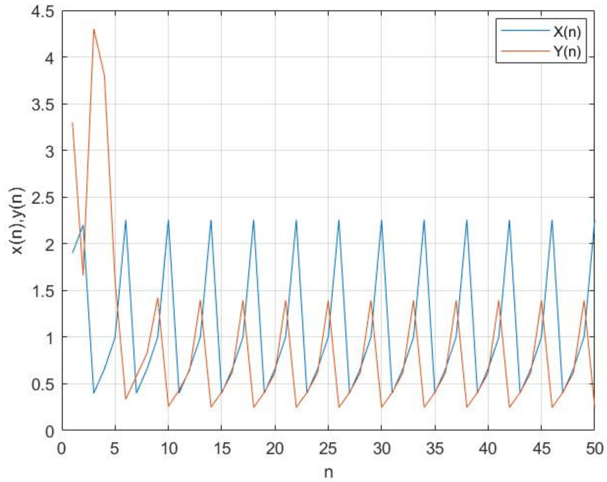

Figure 1. This figure shows the behavior of the solution of the system (1) with initial values as in example (1).

$$w_{4n+2} = \frac{(h-b)b^2h}{(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})(h\phi_{n+1}+b\phi_n)},$$

$$w_{4n+3} = \frac{(r-c)c^2r}{(r\phi_{n-1}+c\phi_{n-2})(r\phi_n+c\phi_{n-1})(r\phi_{n+1}+c\phi_n)},$$

$$s_{4n} = \frac{(f-a)^2g^2}{((f-a)\phi_{n-1}+g\phi_{n-2})((f-a)\phi_n+g\phi_{n-1})^2},$$

$$s_{4n+1} = \frac{(h-b)b^2h}{(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})^2},$$

$$s_{4n+2} = \frac{(r-c)c^2r}{(r\phi_{n-1}+c\phi_{n-2})(r\phi_n+c\phi_{n-1})^2},$$

$$s_{4n+3} = \frac{(t-d)d^2t}{(t\phi_{n-1}+d\phi_{n-2})(t\phi_n+d\phi_{n-1})^2}.$$

Where $\{\phi\}_{m=-3}^{\infty} = \{-1,1,0,1,1,2,3,...\},\ \phi_{m+2} = \phi_{m+1}+\phi_m$**.**
Proof: For n=0, the result holds. Now, suppose that $n > 0$ **and that our assumption holds for** $n-1$ **and** $n-2$**. That is**

$$w_{4n-4} = \frac{(t-d)d^2t}{(t\phi_{n-3}+d\phi_{n-4})(t\phi_{n-2}+d\phi_{n-3})(t\phi_{n-1}+d\phi_{n-2})},$$

$$w_{4n-3} = \frac{(f-a)^2g^2}{((f-a)\phi_n+g\phi_{n-1})((f-a)\phi_{n-1}+g\phi_{n-2})((f-a)\phi_{n-2}+g\phi_{n-3})},$$

$$w_{4n-2} = \frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})},$$

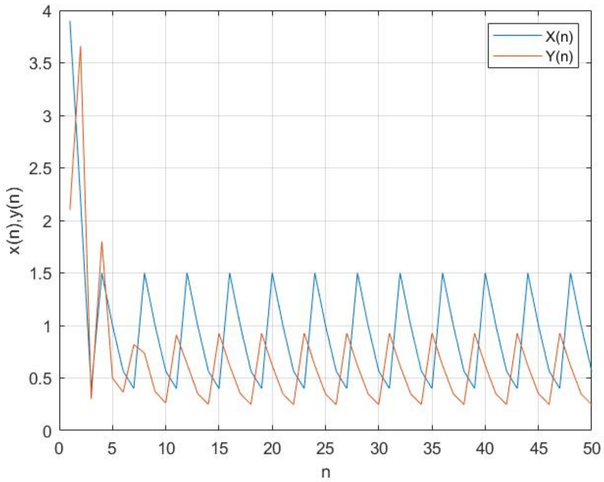

Figure 2. This figure shows the behavior of the solution of the system (1) with initial values as in example (2).

$$w_{4n-1} = \frac{(r-c)c^2r}{(r\phi_{n-2}+c\phi_{n-3})(r\phi_{n-1}+c\phi_{n-2})(r\phi_n+c\phi_{n-1})},$$

$$s_{4n-4} = \frac{(f-a)^2g^2}{((f-a)\phi_{n-2}+g\phi_{n-3})((f-a)\phi_{n-1}+g\phi_{n-2})^2},$$

$$s_{4n-3} = \frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})^2},$$

$$s_{4n-2} = \frac{(r-c)c^2r}{(r\phi_{n-2}+c\phi_{n-3})(r\phi_{n-1}+c\phi_{n-2})^2},$$

$$s_{4n-1} = \frac{(t-d)d^2t}{(t\phi_{n-2}+d\phi_{n-3})(t\phi_{n-1}+d\phi_{n-2})^2}.$$

$$s_{4n-5} = \frac{(t-d)d^2t}{(t\phi_{n-3}+d\phi_{n-4})(t\phi_{n-2}+d\phi_{n-3})^2}.$$

Now, it follows from Eq.(2) that

$$w_{4n+1} = \frac{(s_{4n-4}-w_{4n-3})s_{4n}}{s_{4n-4}-w_{4n-3}+s_{4n}}$$

$$= \frac{\rho(a,f,g)}{\kappa(a,f,g)}$$

$$\rho(a,f,g) = (\frac{(f-a)^2g^2}{((f-a)\phi_{n-2}+g\phi_{n-3})((f-a)\phi_{n-1}+g\phi_{n-2})^2}$$

$$-\frac{(f-a)^2g^2}{((f-a)\phi_n+g\phi_{n-1})((f-a)\phi_{n-1}+g\phi_{n-2})((f-a)\phi_{n-2}+g\phi_{n-3})})$$

$$\times\frac{(f-a)^2g^2}{((f-a)\phi_{n-1}+g\phi_{n-2})((f-a)\phi_n+g\phi_{n-1})^2}$$

$$\kappa(a,f,g)=\frac{(f-a)^2g^2}{((f-a)\phi_{n-2}+g\phi_{n-3})((f-a)\phi_{n-1}+g\phi_{n-2})^2}$$

$$-\frac{(f-a)^2g^2}{((f-a)\phi_n+g\phi_{n-1})((f-a)\phi_{n-1}+g\phi_{n-2})((f-a)\phi_{n-2}+g\phi_{n-3})}$$

$$+\frac{(f-a)^2g^2}{((f-a)\phi_{n-1}+g\phi_{n-2})((f-a)\phi_n+g\phi_{n-1})^2}$$

After some calculations we get

$$w_{4n+1}=\frac{\frac{((f-a)\phi_{n-2}+g\phi_{n-3})g^2(f-a)^2}{((f-a)\phi_n+g\phi_{n-1})((f-a)\phi_{n-1}+g\phi_{n-2})}}{((f-a)\phi_n+g\phi_{n-1})((f-a)\phi_{n-2}+g\phi_{n-3})+((f-a)\phi_{n-2}+g\phi_{n-3})((f-a)\phi_{n-1}+g\phi_{n-2})}$$

Hence, we have

$$w_{4n+1}=\frac{(f-a)^2g^2}{((f-a)\phi_{n+1}+g\phi_n)((f-a)\phi_n+g\phi_{n-1})((f-a)\phi_{n-1}+g\phi_{n-2})},$$

And

$$s_{4n+1}=\frac{(s_{4n-3}-w_{4n-2})w_{4n-2}}{s_{4n-3}}$$

$$s_{4n+1}=\frac{\left(\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})^2}-\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})}\right)}{\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})^2}}$$

$$\times\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})}$$

$$=\frac{(h\phi_n+b\phi_{n-1}-h\phi_{n-1}-b\phi_{n-2})(h-b)b^2h}{(h\phi_{n-1}+b\phi_{n-2})(h\phi_{n-2}+b\phi_{n-3})(h\phi_n+b\phi_{n-1})^2}$$

So, we have

$$s_{4n+1}=\frac{(h-b)b^2h}{(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})^2}.$$

Also,

$$w_{4n+2}=\frac{(s_{4n-3}-w_{4n-2})s_{4n+1}}{s_{4n-3}-w_{4n-2}+s_{4n+1}}$$

$$=\frac{\rho(b,h)}{\kappa(b,h)}$$

$$\rho(b,h)=\left(\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})^2}-\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})}\right)$$

$$\times\frac{(h-b)b^2h}{(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})^2}$$

$$\kappa(b,h)=\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})^2}$$
$$-\frac{(h-b)b^2h}{(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})}$$
$$+\frac{(h-b)b^2h}{(h\phi_{n-1}+b\phi_{n-2})(h\phi_n+b\phi_{n-1})^2}$$

After some calculations we get

$$w_{4n+2}=\frac{\frac{(h\phi_{n-2}+b\phi_{n-3})(h-b)b^2h}{(h\phi_n+b\phi_{n-1})(h\phi_{n-1}+b\phi_{n-2})}}{(h\phi_n+b\phi_{n-1})(h\phi_{n-2}+b\phi_{n-3})+(h\phi_{n-2}+b\phi_{n-3})(h\phi_{n-1}+b\phi_{n-2})}$$

Hence, we have

$$w_{4n+2}=\frac{(h-b)b^2h}{(h\phi_n+b\phi_{n-1})(h\phi_{n-1}+b\phi_{n-2})(h\phi_{n+1}+b\phi_n)}.$$

And

$$s_{4n+2}=\frac{(s_{4n-2}-w_{4n-1})w_{4n-1}}{s_{4n-2}}$$

$$s_{4n+2}=\frac{\left(\frac{(r-c)c^2r}{(r\phi_{n-2}+c\phi_{n-3})(r\phi_{n-1}+c\phi_{n-2})^2}-\frac{(r-c)c^2r}{(r\phi_{n-2}+c\phi_{n-3})(r\phi_{n-1}+c\phi_{n-2})(r\phi_n+c\phi_{n-1})}\right)}{\frac{(r-c)c^2r}{(r\phi_{n-2}+c\phi_{n-3})(r\phi_{n-1}+c\phi_{n-2})^2}}$$
$$\times\frac{(r-c)c^2r}{(r\phi_{n-2}+c\phi_{n-3})(r\phi_{n-1}+c\phi_{n-2})(r\phi_n+c\phi_{n-1})}$$
$$=\frac{(r\phi_n+c\phi_{n-1}-r\phi_{n-1}-c\phi_{n-2})(c-r)c^2r}{(r\phi_{n-1}+c\phi_{n-2})(r\phi_{n-2}+c\phi_{n-3})(r\phi_n+c\phi_{n-1})^2}$$

So, we have

$$s_{4n+2}=\frac{(r-c)c^2r}{(r\phi_{n-1}+c\phi_{n-2})(r\phi_n+c\phi_{n-1})^2}.$$

Similarly, obtaining the other relations is very simple. Thus, the proof is completed.
Example 3. Consider the difference system equation (2) with the initial conditions $w_{-3}=0.5,\ w_{-2}=0.4,\ w_{-1}=0.2,\ w_0=0.9,\ s_{-4}=1.6,\ s_{-3}=0.8,\ s_{-2}=0.3,\ s_{-1}=1.6$ **and** $s_0=0.3$**. (See Fig.3) .**
Example 4. We assume that the initial conditions for the difference system equation (2) $w_{-3}=0.2,\ w_{-2}=1.2,\ w_{-1}=0.3,\ w_0=1.9,\ s_{-4}=0.3,\ s_{-3}=0.28,\ s_{-2}=0.5,\ s_{-1}=0.1$ **and** $s_0=0.13$**. (See Fig.4) .**

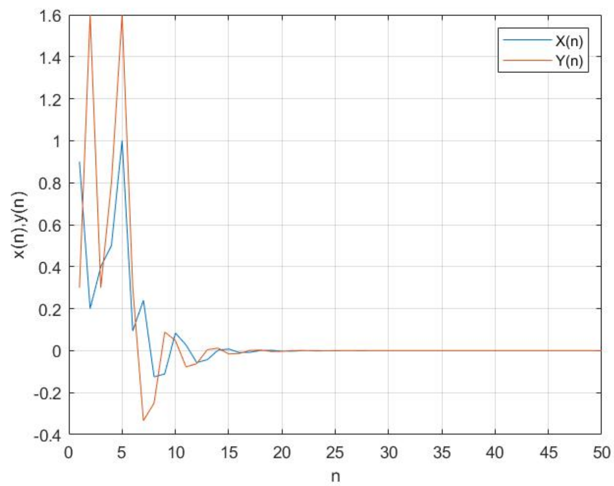

Figure 3. This figure shows the behavior of the solution of the system (2) with initial values as in example (3).

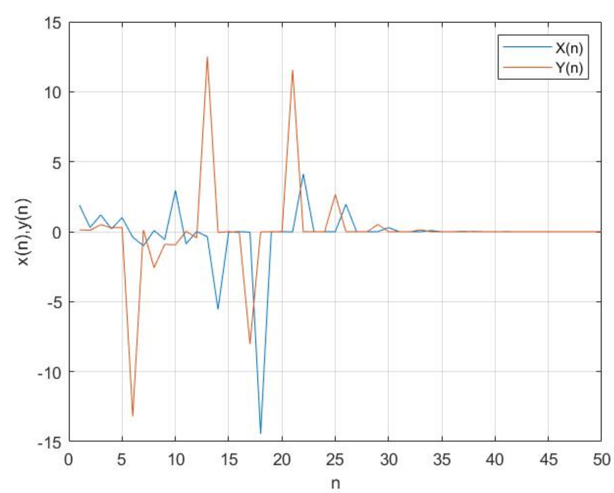

Figure 4. This figure shows the behavior of the solution of the system (2) with initial values as in example (4).

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