Two Classes of Optimal Fourth-Order Iterative Methods Free from Second Derivative for Solving Nonlinear Equations

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Abstract

This work proposes new fourth-order iterative methods to solve non-linear equations $f(x) = 0$. The iterative methods proposed here are presented by modifications of a third-order iterative method to be two classes of optimal fourth order. Convergence analysis was done for the iterative methods proposed in this paper. Multiple numerical examples were taken to explain the accuracy and efficiency of the proposed iterative methods.

Keywords: Optimal Iterative Methods; Order of Convergence; Weight Function; Nonlinear Equations.

1 Introduction

One of the common subjects discussed in applied mathematics is the approximation of solutions to the nonlinear equation $f(x) = 0$, where $f: D \to \mathbb{R}$ is a function over the open interval D. Therefore, constructing iterative methods for solving $f(x) = 0$ operations important in numerical analysis. To improve the local ranking of the affinity index and efficiency, several third and fourth order modified methods have been presented in the literature. For details we refer to [2], [4-9] and the references therein.

2 Description of The Method

A two steps third order method for solving nonlinear equations developed by Noor et. al. [3], which used five functions evaluations. This method had modified by Mastoi *et. al.* [1] which has two functions and two derivatives evaluations:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

$$x_{n+1} = y_n - \frac{4(y_n - x_n)f(y_n)}{(y_n - x_n)[f'(x_n) + f'(y_n)] + 2[f(y_n) - f(x_n)]} \tag{2}$$

This method to be optimal, we need to reduce the number of functions evaluations to three, $n = 3$. One way to do this is to replace the derivative of y_n. Two approximations to $f'(y_n)$ has been developed by ABABNEH [2]:

$$f_1'(y_n) \approx \frac{f'(x_n)f(x_n)^2}{(f(x_n) + f(y_n))^2}, \tag{3}$$

$$f_2'(y_n) \approx \frac{f'(x_n)[f(x_n) + (\beta - 2)f(y_n)]}{f(x_n) + \beta f(y_n)}, \tag{4}$$

where $\beta\epsilon R$. If $f_1^{'}(y_n)$ or $f_2^{'}(y_n)$ have been used instead of $f'(y_n)$ in the method of Sehrish et. al., [1], will also got a third order method. To go throw this, the weighted function $\mathcal{H}(v), v = \frac{f(y_n)}{f(x_n)}$ will be used to increase the order from three to optimal forth.

3 Convergence Analysis

More, two algorithms will be haven as:

Algorithm 1:

$$y_n = x_n - \frac{f(x_n)}{f^{'}(x_n)}, \tag{5}$$

$$x_{n+1} = y_n - \frac{4(y_n - x_n)f(y_n)}{(y_n - x_n)[f^{'}(x_n) + f_1^{'}(y_n)] + 2[f(y_n) - f(x_n)]}\mathcal{H}(v), v = \frac{f(y_n)}{f(x_n)}. \tag{6}$$

Theorem 3.1: Let D be an open interval and $f : D \to \mathbb{R}$ be sufficiently differentiable function. Suppose that m is a simple zero of f and x_o is sufficiently close to m. Denote $e_n = x_n - m$ and $c_k = f^k\ (m)/k!$. Then the family defined by (6) is an optimal fourth-order convergence if $v = \frac{f(y_n)}{f(x_n)}$ and $\mathcal{H}(0) = \mathcal{H}^{'}(0) = 1$.

Proof: Let $e_n = x_n - m$ be the error of the approximation in the nth iterative step. Using Taylor expansion of $f(x_n)$ about m and considering that $f^{'}(x_n) \neq 0$, we have

$$f(x_n) = f^{'}(m)(e + c_2e^2 + c_3e^3 + c_4e^4) + O(e^5) \tag{7}$$

Furthermore, we have

$$f'(x_n) = f^{'}(m)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4) + O(e^5) \tag{8}$$

and

$$\frac{f(x_n)}{f^{'}(x_n)} = e - c_2e^2 + (2c_2^2 - 2c_3)e^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e^4 + O(e^5) \tag{9}$$

Substituting (9) in (5) yields

$$y(x_n) = m + c_2e^2 + (-2c_2^2 + 2c_3)e^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e^4 + O(e^5). \tag{10}$$

Expanding $f(y_n)$ about δ and using (10), we have

$$f(y_n) = f^{'}(m)(c_2e^2 + (-2c_2^2 + 2c_3)e^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e^4) + O(e^5). \tag{11}$$

Furthermore, we have

$$f_1{'}(y) = f^{'}(m)(1 + (-c_3 + 5c_2^2)e^2 + (-2c_4 + 18c_2c_3 - 20c_2^3)e^3 + (-3c_5 + 26c_2c_4 + 16c_3^2 - 103c_2^2c_3 + 80c_2^4)e^4) + O(e^5). \tag{12}$$

Using Taylor expansion of $\mathcal{H}(v), v = \frac{f(y_n)}{f(x_n)}$, we have

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(0) + \mathcal{H}^{'}(0)c_2e + (-3\mathcal{H}^{'}(0)c_2^2 + 2\mathcal{H}^{'}(0)c_3) + (8\mathcal{H}^{'}(0)c_2^3 - 10\mathcal{H}^{'}(0)c_2c_3 + 3\mathcal{H}^{'}(0)c_4)e^3 \\ &\quad + (-20\mathcal{H}^{'}(0)c_2^4 + 37\mathcal{H}^{'}(0)c_2^2c_3 - 14\mathcal{H}^{'}(0)c_2c_4 - 8\mathcal{H}^{'}(0)c_3^2 + 4\mathcal{H}^{'}(0)c_5)e^4 + O(e^5)\end{aligned} \tag{13}$$

Substituting (7-13) in (6) yields

$$\begin{aligned}x_{n+1} &= m + (-\mathcal{H}(0)c_2 + c_2)e^2 + (3\mathcal{H}(0)c_2^2 - \mathcal{H}'(0)c_2^2 - 2\mathcal{H}(0)c_3 - 2c_2^2 + 2c_3)e^3 \\ &\quad + \left(4c_2^3 - 7c_2c_3 + 3c_4 + 6c_2^3\mathcal{H}^{'}(0) - 4\mathcal{H}(0)c_2c_3 - \left(\frac{25}{4}\right)c_2^3\mathcal{H}(0) + 10c_2\mathcal{H}(0)c_3 \right. \\ &\quad \left. - 3\mathcal{H}(0)c_4\right)e^4 + O(e^5)\end{aligned}$$

Using the conditions $\mathcal{H}(0) = \mathcal{H}^{'}(0) = 1$, we got,

$$x_{n+1} = m + \left(\frac{15}{4}c_2^3 - c_2c_3\right)e^4 + O(e^5).$$

Then the algorithm 1, has order at least four, and the proof has been completed.

Algorithm 2:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{14}$$
$$x_{n+1} = y_n - \frac{4(y_n - x_n) f(y_n)}{(y_n - x_n)[f'(x_n) + f_2'(y_n)] + 2[f(y_n) - f(x_n)]} \mathcal{H}(v), v = \frac{f(y_n)}{f(x_n)}. \tag{15}$$

Theorem 3.2: Let D be an open interval and $f : D \rightarrow \mathbb{R}$ be sufficiently differentiable function. Suppose that m is a simple zero of f and x_o is sufficiently close to m. Denote $e_n = x_n - m$ and $c_k = f^k \ (m)/k!$. Then the family defined by (15) is a family optimal fourth-order convergence if $v = \frac{f(y_n)}{f(x_n)}$ and $\mathcal{H}(0) = \mathcal{H}'(0) = 1$.

Proof: using the equations (7) and (8) from the last proof, we got

$$f_2'(y) = f'(m)(1 + (-c_3 + 2\beta c_2^2 + 2c_2^2)e^2 + (-2c_4 - 8\beta c_2^3 + 8\beta c_2 c_3 - 4c_2^3 + 6c_2 c_3 - 2\beta^2 c_2^3)e^3 + (14\beta^2 c_2^4 + 2\beta^3 c_2^4 + 26\beta c_2^4 + 8\beta c_3^2 + 8c_2^4 - 3c_5 + 4c_3^2 + 8c_2 c_4 - 42\beta c_2^2 c_3 + 12\beta c_2 c_4 - 12\beta^2 c_3 c_2^2 - 16c_2^2 c_3)e^4 + O(e^5), \beta \in R. \tag{16}$$

Substituting (7-11), (13) and (16) in (6) yields

$$x_{n+1} = m + (-\mathcal{H}(0)c_2 + c_2)e^2 + (3\mathcal{H}(0)c_2^2 - \mathcal{H}'(0)c_2^2 - 2\mathcal{H}(0)c_3 - 2c_2^2 + 2c_3)e^3 + \left(4c_2^3 - 7c_2c_3 + 3c_4 + 6\mathcal{H}'(0)c_2^3 - 4\mathcal{H}'(0)c_2c_3 - 7\mathcal{H}(0)c_2^3 + 10\mathcal{H}(0)c_2c_3 - 3\mathcal{H}(0)c_4 + \frac{1}{2}\beta\mathcal{H}(0)c_2^3\right)e^4 + O(e^5)$$

Using the conditions $\mathcal{H}(0) = \mathcal{H}'(0) = 1$, we got,

$$x_{n+1} = m + \left(3c_2^3 - c_2c_3 + \frac{1}{2}\beta c_2^3\right)e^4 + O(e^5), \beta \in R.$$

Then the algorithm 2, has order at least four, and the proof has been completed.

4 Numerical examples

In this section, nonlinear numerical examples are presented to illustrate the effectiveness of the proposed methods in this paper.

The method (LM11) defined by using theorem 3.1the weighted with function

$$\mathcal{H}(v) = \cos(x) + \sin(x)\ .$$

The method (LM12) defined by using theorem 3.1the weighted with function

$$\mathcal{H}(v) = 1 + v + B\, v^a \ \ , \ B = -2, a = 2\ .$$

The method (LM21) defined by using theorem 3.2 the weighted 1 with function

$$\mathcal{H}(v) = \cos(x) + \sin(x)\ .$$

The method (LM22) defined by using theorem 3.2 the weighted with function

$$\mathcal{H}(v) = 1 + v + B\, v^a \ \ , \ B = -2, a = 2\ .$$

They were compared with some fourth-order methods.

The ABABNEH method [2] (AM) defined by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{4(y_n - x_n)f(y_n)}{(y_n - x_n)[f'(x_n) + f'(y_n)] + 2[f(y_n) - f(x_n)]},$$

the King method [7] (KM) defined by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \ \ \gamma = 1$$

The Traub method [8] (TM) defined by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{f^2(x_n)}{[f^2(x_n) - 2f(x_n)f(y_n)]} \frac{f(x_n)}{f'(x_n)},$$

The Hani I. Siyyam and I. A. Al-Subaihi method [9] (SSM) defined by:

$$G(t) = 1 + 2t + 4t^2 + t^3, \ \ t = \frac{y_n}{f(x_n)},$$
$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - \frac{2f(x_n)G(u_n) + (y_n - x_n)f'(x_n)[1 + G(u_n)]}{2f'(x_n)}.$$

All numerical calculations were performed using MATLAB R2018a software package of 1000-number floating point arithmetic. The following discontinuation criteria for computer programs are used in the calculations:

$$|x_n - x_{n-1}| < \varepsilon \quad and \quad |f(x_n)| < \varepsilon.$$

When the stopping criterion are met x_n is taken over the calculated exact root α. For numerical illustrations, ε is taken to be $\varepsilon = 10^{-30}$. As a convergence criterion, the distance of two successive approximations $m = |x_n - x_{n-1}|$ is shown in following Table. The number of iterations of the root approximation (IT) is also shown in the Table. Referring to [10], the arithmetic order of convergence (COC) is given by

$$COC = \frac{ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}{ln|(x_{n-1} - \alpha)/(x_{n-2} - \alpha)|}$$

We get,

Methods	IT	$\|f_n(x)\|$	$\|x_n - x_{n-1}\|$	COC
Example	$f_1(x) = 3x + \sin(x) - \exp(x)$			
AM	3	3.78688e-101	4.91489e-34	3
KM	NC			
TM	NC			
SSM	4	1.88768e-250	4.12256e-63	4
LM11	4	1.61418e-267	2.48142e-67	3.69249
LM12	4	4.68619e-242	4.92598e-61	4
LM21	NC			
LM22	NC			
Example	$f_2(x) = \cos(x) - x\exp(x)$			
AM	4	6.27261e-148	6.65019e-50	3
KM	4	1.83126e-431	1.41233e-108	4
TM	3	6.40732e-122	4.99293e-31	4
SSM	4	3.40014e-414	2.6141e-104	4
LM11	4	6.67121e-439	2.05929e-110	4
LM12	4	3.7127e-404	7.90383e-102	4
LM21	4	1.34614e-416	6.66179e-105	4
LM22	4	1.35731e-418	2.14695e-105	4
Example	$f_3(x) = 3x + \cos(x) - 1$			
AM	5	3.32987e-256	1.58685e-85	3
KM	4	1.50828e-229	1.37935e-57	4
TM	NC			
SSM	4	8.66945e-211	6.10278e-53	4
LM11	4	4.75062e-234	1.08152e-58	4
LM12	4	4.44857e-200	2.73194e-50	4
LM21	4	1.45715e-213	1.25346e-53	4
LM22	4	8.4001e-213	1.97192e-53	4
Example	$f_4(x) = lnx + \sqrt{x} - 5$			
AM	4	2.63012e-130	7.92492e-43	3
KM	4	1.16536e-401	7.94308e-100	4
TM	4	3.46512e-426	6.87106e-106	4
SSM	4	4.78943e-373	9.34162e-93	4
LM11	4	5.24948e-417	1.30553e-103	4
LM12	4	2.81124e-359	2.35818e-89	4
LM21	4	3.2836e-245	1.34159e-61	4
LM22	4	2.54369e-245	1.27301e-61	4
Example	$f_5(x) = \cos(x) - x$			

AM	4	7.14748e-99	4.44112e-33	3
KM	4	1.17685e-256	1.96559e-64	3.906
TM	4	1.43971e-285	1.35423e-71	3.37438
SSM	4	3.73236e-243	4.33318e-61	4
LM11	4	2.54724e-260	2.45756e-65	3.83197
LM12	4	1.12657e-234	5.44263e-59	4
LM21	4	5.04373e-248	1.00698e-62	4
LM22	4	3.78596e-252	9.52131e-64	3.99972
Example	$f_6(x) = sin^2(x) - x^2 + 1$			
AM	4	1.16326e-93	9.13857e-32	3
KM	NC			
TM	NC			
SSM	4	3.9355e-246	2.94896e-62	4
LM11	NC			
LM12	4	1.0214e-237	3.51425e-60	4
LM21	4	2.47151e-376	4.32813e-94	4
LM22	4	6.58525e-375	9.58128e-94	4
Example	$f_7(x) = x^2 - e^x - 3x + 2$			
AM	4	6.43344e-139	2.69133e-46	3
KM	4	3.6704e-369	2.40901e-92	4
TM	4	1.01902e-360	2.77977e-90	4
SSM	4	4.33819e-378	1.62172e-94	4
LM11	4	6.6594e-367	8.55363e-92	4
LM12	4	1.59989e-390	1.60404e-97	4
LM21	4	1.24752e-419	1.27923e-104	4
LM22	4	4.09056e-411	1.38317e-102	4
Example	$f_8(x) = \sqrt{x} - \cos(x)$			
AM	4	1.43847e-145	1.92217e-48	3
KM	4	2.85e-397	2.95764e-99	4
TM	4	1.13654e-384	3.35802e-96	4
SSM	4	9.0189e-429	8.34616e-107	4
LM11	4	2.11355e-393	2.54628e-98	4
LM12	4	3.68971e-406	1.80912e-101	4
LM21	4	2.24135e-365	6.02801e-91	4
LM22	4	2.63833e-364	1.10294e-90	4
Example	$f_9(x) = 2x - \ln(x) - 7$			
AM	4	6.85985e-135	1.69968e-44	3
KM	4	4.71979e-359	2.17106e-89	4
TM	4	6.29542e-351	2.17611e-87	4
SSM	4	8.28065e-367	2.67711e-91	4
LM11	4	5.44735e-357	6.97982e-89	4
LM12	4	6.37494e-375	2.7081e-93	4
LM21	3	7.3799e-162	8.81847e-41	4
LM22	4	1.84337e-308	9.96293e-78	4
Example	$f_{10}(x) = \exp(x) - 5x$			
AM	4	6.01609e-226	1.74441e-75	3
KM	3	6.47341e-164	2.85633e-41	4

TM	3	1.17001e-168	2.12293e-42	4
SSM	3	1.69273e-161	1.0745e-40	4
LM11	3	7.3799e-162	8.81847e-41	4
LM12	3	7.00429e-160	2.60642e-40	4
LM21	3	6.74858e-165	1.66725e-41	4
LM22	3	3.04575e-162	7.14177e-41	4

5 Conclusion

In this work new optimal fourth order algorithms for solving nonlinear equations have been proposed. The given table shows the best performance of the new methods in terms of number of iterations, and order of convergence as compared to other well-known existing methods. We conclude that the given methods are compatible with the existing iterative methods.

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