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Idempotent Structures Of Semigroup Of Singular - Regular Matrices

Adeniji, Adenike Olusola

Department of Mathematics, Faculty of Science, University of Abuja, Abuja, Nigeria
Email: adenike.adeniji@uniabuja.edu.ng

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Abstract

Let S be a semigroup of singular matrices over a subset of the set of integers. The idempotent elements of S are classified into tridempotent, fractional idempotent and skew idempotent elements. Matrices are generally known to be non-commutative but an example of commutative matrices is established in this work. Matrix multiplication and its axioms are employed to establish the results. Regularity was first established and the condition for regularity of each idempotent structure obtained is discussed. Some of the structures obtained are used to establish bands of regular elements. The fractional component of S is obtained combinatorially by choosing a number with its sign for every row of each matrix. There are m values of matrices with first and second rows being equal, which are removed from the set since their multiplication gives zero. The satisfaction of the condition for commutativity implies that all the matrices have the same characteristic equation and all are of a specified order n. This is evident in the fact that the product of the principal diagonal, D_1 of matrix A is the same as the product of the principal diagonal D_2 of matrix B. Also, the product of the off- diagonal of matrices A and B are the same.

Keywords

Tridempotents, Idempotents, Normal Semigroup, Band, Principal Diagonal.

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1 Introduction

Let S be a non - empty set defined on

$$\overline{Z}_m = \{0, \pm 1, \pm 2, \ldots \pm m\},$$

we define a multiplication

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$$ab = a(a, b \in S)$$

and obtain a left zero semigroup. An element a of a semigroup S is called regular provided that there exists $b \in S$ such that $aba = a$. Elements $a, b \in S$ form a pair of inverse elements provided that $aba = a$ and $bab = b$. Lawson[6] explained the concepts of inverse semigroup. Standard definition of terms and notations on semigroup are contained in [5] and [7]. Studies like understanding idempotent depth, Idempotent rank and some other important results on idempotency of some semigroups were obtained by authors like [1], [3], [4], [8] and [9]. $E(S)$ is the usual notation for the set of idempotents of a semigroup S and $T_E(S)$ denotes tridempotents in this work. Tridempotent element a, is defined as having the property that $a^3 = a$. A square matrix has its usual meaning and a set of square matrices satisfying certain conditions, as discussed here, is a semigroup. Richardson[2] defined conjugate matrices as a set satisfying the following conditions:

(i) they are commutative;

(ii) their symmetric functions are scalar matrices which are equal to the corresponding symmetric functions of the scalar roots of the characteristic equation of one of the matrices;

(iii) all the matrices have the same characteristic equation;

(iv) all are of a specified order n.

Four different idempotent structures are obtainable from $\overline{Z}_m$, two of which form a semigroup, to which the identity and zero elements can be adjoined. The semigroup $R_B(\overline{Z}_m, 2)$ is a set of 2×2 singular matrices, taken here to be S and the set of matrices, $(\overline{Z}_m, 2)$ is denoted by $\overline{S}$. It should be noted that $S \subset \overline{S}$. The common feature among the matrices to be classified under various idempotent structures is that for $A \in S$,

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $ad = bc \neq 0$. Also, k is defined to be $\mid d \mid - \mid a \mid$.

The following terms are used to describe idempotents structures of $\overline{S}$:

* E denotes idempotent if $A^2 = A$ for $k = 1$;

* F for fractional idempotents, if $\frac{A^2}{k} = A, k > 1$;

* $\overline{S}_E$ is skew-idempotent matrix, if $A^2 = -kA, k \geq 1$;

* $T_E(S)$ is tridempotent, if $A^3 = A$ for $K = 1$ and multiple tridempotent if $k > 1$.

These terms are observed in the idempotents structures, which are classified into four components as:

The first idempotent structure has the following properties:

$a = -b$ and $c = -d; b, d \in N$;

$A^2 = A$ if $k = 1$;

$A^2 = kA$ if $k > 1$, which is multiple idempotency;

$\mid d \mid \neq \mid a \mid$ and $\mid d \mid > \mid a \mid$.

The second idempotent structure is classified with the following characteristics:

$b = -a$ and $d = -c; a, c \in N$;

if $\mid a \mid > \mid d \mid$ and $k \geq 1$, then $A^2 = kA$;

if $\mid a \mid < \mid d \mid$, then $A^2 = -kA, k \geq 1$.

Note that, if $A^2 = -A, A$ is skew - idempotent and if $A^2 = -kA, k > 1$ then A is multiple skew - idempotent.

The third idempotent structure satisfies the following:

$c = -a$ and $d = -b; a, b \in N$; $\mid a \mid > \mid d \mid$ and $A^2 = kA, k \geq 1$.

The fourth and the last idempotent structure satifies the following features:

$a = -c$ and $b = -d; c, d \in N$; $\mid a \mid < \mid d \mid$ and $A^2 = kA, k \geq 1$.

Remark: In the third and fourth idempotent structures, whenever $\frac{A^2}{k} = A, k > 1$, then A is fractional idempotent.

1.1 Matrix Algebra of $R_B(\overline{Z}_m, 2)$

The following proposition shows the usual addition and multiplication of matrices on the elements of $\overline{S}$.

Proposition 1.1. *Let $A, B \in \overline{S}$, then*

- $A^T \cdot A^T = A^T$
- $(A^T)^T \cdot (A^T)^T = A$
- $A \cdot (A^T)^T = -A$
- $A \cdot A^T \neq A^T \cdot A$
- $A \cdot -A = -A \cdot A$
- $-A \cdot A = -A$
- $-A \cdot -A = A$
- $-A + A = I$

Proposition 1.2. *If $A \in \overline{S}$, then $(nA)^x = n^x A$.*

Proof. Let $n, x \in N$. The proof follows from the fact that $A^x = A$.

□

2 Main Results

The matrices with $k = 1$ from the first two idempotent structures and $| a | > | d |$ in the second structure form the elements considered as a semigroup, S in this study. The results obtained for S and $\overline{S}$ are stated and proved in this section.

Theorem 1. *The cardinality of S is $2(m-1)$.*

Proof. This is simply established from choosing possible values of two consecutive numbers occupying the two rows of a matrix alternately, applying $k = 1$. If the identity element is adjoined, as made possible in the theory of semigroup, then the cardinality will be $2m - 1$.

□

Theorem 2. *Let $A, B \in \overline{S}$, a set of 2×2 matrices. If $B = -A$, then $AB = BA$.*

Proof. The computation of eigenvalues relation of matrix A given as
$AX = \lambda X$
and the eigenvalues relation of matrix B as
$BX = \beta X$
show that
$\lambda_1 = \beta_1$ and $\lambda_2 = \beta_2$
$\forall \lambda_1, \lambda_2 \in \lambda$ and $\beta_1, \beta_2 \in \beta$.
Then,
$\lambda = \beta$.
$\Rightarrow D_1 = D_2$
where the product of the principal diagonal, D_1 of matrix A is the same as the product of the principal diagonal, D_2 of matrix B.
Also, the product of the off- diagonal of matrices A and B are the same.
Hence $AB = BA$. □

Theorem 3. *If $A^2 = -A$, then A is tridempotent.*

Proof. The statement of this theorem satisfies the condition for commutativity in theorem (2). The second structure of idempotency on $\overline{Z}_m$, where $k = -1$, and skew idempotency shows that, $A^2 = -A$
$\Rightarrow A^3 = -A \cdot A = A \cdot -A = A$.

□

Theorem 4. *Let $A, B \in S$ and $k = 1$, then S is a regular semigroup.*

Proof. One of the characteristics of S is that $AB = A$ and $BA = B$.
$\Rightarrow ABA = (AB)A = AA = A$ and $A(BA) = AB = A$
also
$BAB = B(AB) = BA = B$ and $(BA)B = BB = B$,
since $AB \neq BA$.
Commutativity can only be satisfied if $B = -A$ as in theorem (2).

□

Theorem 5. *The semigroup S is not inverse and its idempotents do not commute.*

Proof. Theorem (4) has shown that S is regular. Let $A \in S$ and $B, C \in S$ be inverses of A.
Since $ABA = A$ and $ACA = A$, then
$(ABA)(ACA) = ABAACA = ABACA = ABA = A$ and
$(ACA)(ABA) = ACAABA = ACABA = ACA = A$.
Infact, $ACA = ABA(C)ABA = AB(ACA)BA = ABABA = ABA = A$
and $CAC = C$.
$\Rightarrow AC \neq CA$, hence S is not inverse.
If S is inverse, that is
$AC = A$ and $CA = C$,
$\Rightarrow (AC)(AC) = A(CA)C = AC^2 = AC = A \qquad (i)$
$(CA)(CA) = C(AC)A = CA^2 = CA = C \qquad (ii)$.
From (i) and (ii)
S does not commute.

□

Theorem 6. *$R_B(\overline{Z}_m, 2)$ is a normal semigroup.*

Proof. Let $A, B, C, D \in S$. The regular idempotent elements of S satisfy
$AB = A \forall A, B \in S$.
$ABCD = A(BC)D = A(BD) = AB = A$ and
$ACBD = (AC)(BD) = AB = A$.
Hence, $ABCD = ACBD$

□

2.1 Transformation on S

Taking any two elements $\alpha, \beta \in S$, the transformation
$\gamma : S \mapsto S$ defined by $\alpha_{ij} \mapsto \beta_{ij}, i, j \in \{1, 2\}$ is obtained as follows:
Let $\alpha =$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

and
$\beta =$

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$$

then

$\gamma = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{21} & \alpha_{22} \\ \beta_{11} & \beta_{12} & \beta_{21} & \beta_{22} \end{bmatrix}$.

The following four possible cases are considered for obtaining γ; $j \in \{1, 2\}$:

$\beta_{2j} = -\alpha_{2j}, \;\; \beta_{1j} \neq \alpha_{1j}$.

$\gamma^n \neq \phi, n \in N$. Most importantly, $\gamma^2 = \gamma^4$. (i)

$\beta_{1j} = -\alpha_{1j}, \;\; \beta_{2j} \neq \alpha_{2j}$.

$\gamma^n \neq \phi, n \in N$. Also, $\gamma^2 = \gamma^4$. (ii)

$\beta_{1j} = -\alpha_{2j}, \;\; \beta_{2j} = -\alpha_{1j}$.

$\gamma^n \neq \phi, n \in N$. Here, $\gamma = \gamma^3$. (iii)

$\beta_{1j} = \alpha_{2j}, \;\; \beta_{2j} \neq \alpha_{1j}$.

$\gamma^2 \neq \phi$ but $\gamma^3 = \phi$. (iv)

For example,

if

$\alpha =$

$$\begin{bmatrix} 4 & -4 \\ 3 & -3 \end{bmatrix}$$

and

$\beta =$

$$\begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}$$

then

$\gamma = \begin{bmatrix} 4 & -4 & 3 & -3 \\ -2 & 2 & -3 & 3 \end{bmatrix}$,

$\Rightarrow \gamma^2 = \gamma^4$.

It should be noted that the transformation defined on S is not equivalent to being called a transformation semigroup.

2.2 On Fractional Idempotent Matrices

Let $\overline{Z}_m = \{\pm 1, \pm 2, \ldots \pm m\}$ with 2×2 matrices, denoted by $(\overline{Z}_m, 2)$, possessing the following properties:

If A =

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a, b, c, d \in N$;

i. a = -b and c = -d;

ii. $ad = bc \neq 0$;

iii. $\mid b \mid \neq \mid d \mid$ and $\mid d \mid > \mid b \mid$.

The matrices formed, emanate from one of the idempotents structures mentioned in the introduction. The matrices are split into two forms as:

(i) idempotents, $E(\overline{S})$, where $k =\mid d \mid - \mid a \mid= 1$;

(ii) fractional idempotents, $F(\overline{S})$, with $k =\mid d \mid - \mid a \mid> 1$.

The following algebraic result explains the relationship between idempotents and fractional idempotent matrices.

Theorem 7. *Let $A, B \in \overline{S}$, then for $k \in N$:*

(i) if $A, B \in E, AB = A$, hence $E(\overline{S})$ is a regular semigroup;

(ii) if $A \in E$ and $B \in F, AB = kA$;

(iii) if $A \in F$ and $B \in E$, $AB = A$;

(iv) if $A, B \in F, AB = kA$

Proof. (i) The characteristics of a regular semigroup, S as in theorem (1) is that $ABA = A, \forall A, B \in S$ which $E(\overline{S})$ satisfies, hence it is a regular semigroup.

Taking $k =\mid d \mid - \mid a \mid$ of matrix B, which is 1 in E. Thus $AB = kA = A$.

Also, since

$ABA = A$, assume that $AB \neq A$, say $AB = C \in E$ then $CA = A$.

Let $BA \neq B$, say $BA = D \in E$, then $AD = A$.

$\Rightarrow ACA = A$ and $ADA = A$

$\Rightarrow AC = A, CA = C$ and $AD = A, DA = D$ which is a contradiction.

Hence, for any $A, B \in E, AB = A$ and $BA = B$.

(ii) It is observed that $k > 1$ in $B \subset F$.

Taking $A \in E$ and $B \in F$ results in multiples of A.

Hence $AB = kA$.

(iii) Since k is determined by B and here, $k = 1$

following the proof of (i), then $AB = A$.

(iv) The proof of this follows the argument of (ii).

□

Theorem 8. *Let $A \in F$ and $k \in N, A^n = k^{n-1}A$.*

Proof. The proof follows theorem 7(ii). □

The following theorem explains the cardinalities of E, F and $\mid \overline{S} \mid$.

Theorem 9. *$\overline{S}$ is a set of matrices and* $\mid \overline{S} \mid = \frac{m(m-1)}{2}$

Proof. The elements of $\overline{S}$ are basically idempotents in nature having two components. In the first component, $k = 1$ and $\mid d \mid > \mid a \mid$ for consecutive numbers a and d. The cardinality of the first component, for each $m \in N$ is

$\mid E(\overline{S}) \mid = m - 1 \qquad\qquad (i)$

The fractional component, $\mid F(\overline{S}) \mid$ is obtained combinatorially by choosing a number with its sign for every row of each matrix, meaning choose two out of $m + 1$ for any m. Note that in fractional idempotent matrices, $\mid d \mid > \mid a \mid$. There are m values of matrices with first and second rows being equal, that is $\mid d \mid = \mid a \mid$, which are removed from the set since their multiplication gives zero. So,

$\mid F(\overline{Z}_m, 2) \mid = \binom{m+1}{2} - 2m + 1 \qquad\qquad (ii)$

Combining (i) and (ii), $\mid (\overline{S}) \mid = \binom{m+1}{2} - m$.

□

3 Conclusions

The set of matrices, $\overline{S}$ is categorised into four different idempotent structures. The first two structures with their identified conditions, are further studied to be regular band semigroup of matrices, $R_B(\overline{Z}_m, 2)$. Each of the structures is regular only if $k = 1$. This semigroup satisfies the properties of conjugate matrices according to Richardson[2].

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