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Hidden properties of the equations of mathematical physics. Evolutionary relation for the state functionals and its connection with the field-theory equations

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Abstract

It is shown that the equations of mathematical physics describing material systems (material media) such as the thermodynamic, gas-dynamic and cosmic systems as well as the systems of charged particles and others have double solutions, and this fact enables one to describe the processes of emergence of various structures and formations (waves, vortices and so on). This follows from the evolutionary relation in skew-symmetric differential forms for state functionals (such as the action functional, entropy, Pointing's vector, Einstein's tensor, wave function, and others). This relation arises when studying the integrability of the equations of mathematical physics.

The evolutionary relation has the properties of the field-theory equations. This fact discloses a connection of the field-theory equations with the equations of mathematical physics and enables one to understand the basic principles of the field theory and the properties of physical fields.

Keywords: Material media; nonidentity of the evolutionary relation; double solutions of the equations of mathematical physics; conservation laws; properties of the field theory.

1. Introduction

Functional properties of the equations of mathematical physics and specific features of their solutions were described by the author in a body of papers (see, [1,2]).

In present paper an attempt will be made to call once more an attention to the peculiarities of the equations of mathematical physics due to which these equations can describe not only the change of physical quantities (such as energy, pressure, density), but also possess unique possibilities in describing evolutionary processes and the processes of emergence various structures and formations such as waves, vortices, turbulent pulsations and others.

Such peculiarities of the equations of mathematical physics relate to the properties of these equations that are hidden ones since they do not directly follow from these equations. Such properties of the equations of mathematical physics are discoverd when studying the integrability of the equations of mathematical physics.

In the process of investigating the integrability of the equations of mathematical physics, which depends on the consistency of the conservation law equations for energy, linear momentum, angular momentum, and mass (which made up the set of equations of mathematical physics), one obtains the evolutionary relation in skew-symmetric differential forms for such functionals as the action functional, entropy, Pointing's vector, Einstein's tensor, wave function and others. As it was shown, such functionals describe the state of a material medium, that is, they are state functionals [3].

The evolutionary relation obtained possesses a peculiarity, namely, it appears to be nonidentical one. Such nonidentical relation discloses a potentiality of the mathematical physics equations in the description of the processes of origin of various structures and formations.

From the evolutionary relation it follows that the equations of mathematical physics have double solutions, namely, the solutions that are not functions (they depends not only on the variables, their derivatives do not made up a differential) and solutions that are discrete functions. The solutions of the first type are defined on a nonintegrable (tangent) manifold. Such solutions describe a the non-equilibrium state of material medium. And the solutions of the second type are defined on integrable structures and describe a locally-equilibrium state of a material medium.The transition from the solutions of the first type to ones of the second type describe the process of origin of physical structures and observable formations.

Peculiarities of the equations of mathematical physics, their hidden properties and unique possibilities were demonstrated for a gas-dynamic system in the paper [4] "Hidden properties of the Navier-Stokes equations. Double solutions. Origination of turbulence."

The evolutionary relation discloses one more property of the equations of mathematical physics, namely, a connection between the field-theory equations, which describe physical fields, and the equations of mathematical physics, which describe material media. This is justified by the fact that there is a correspondence between the field-theory equations and the evolutionary relation. A connection of the field-theory equations with the equations of mathematical physics and a correspondence between the field-theory equations and the evolutionary relation enables one to understand the field-theory foundations, inherent connections of the field-theory equations and the properties of physical fields.

It should be emphasized that it appears to be possible to disclose the properties of the equations of mathematical physics due to the apparatus of skew-symmetric differential forms, whose basis is nonintegrable, deforming, manifolds (in contrast to exterior skew-symmetric forms). Such skew-symmetric forms [5, 6] possess an unconventional mathematical apparatus that includes such concepts as degenerate transformations and nonidentical relations. They can generate closed exterior forms, which are invariant ones.

In addition, it should be emphasized that, when studying the integrability of the equations of mathematical physics the property of physical quantities of a material medium (which are commonly ignored when solving the equations of mathematical physics) were taken into account. Since the physical quantities (like temperature, energy, pressure or density) relates to a single material medium, a connection between them should exist. Such a connection is described by functionals, the examples of which are (above noted) the action functional, entropy, Pointing's vector, Einstein's tensor, wave function and others. The evolutionary relation, which is a relation for such functionals, just allows to show up hidden properties of the equations of mathematical physics.

2. Studying the integrability of the equations of mathematical physics. Evolutionary relation

In the present paper the equations of mathematical physics, which describe material media such as thermodynamic, gas-dynamic, cosmic systems, the systems of charged particles and so on, are considered. They describe processes proceeded in material medium. (See Appendix 2.) It is known that such equations of mathematical physics involve the conservation law equations for energy, linear momentum, angular momentum, and mass which are conservation laws for material media [7, 8]. The integrability of such a set of differential equations depends, firstly, on the consistency of derivatives along different directions and, secondly, on the consistency of equations in the set of equations.

In the present paper the investigation of integrability of the equations of mathematical physics caused by the consistency of the conservation law equations will receive a primary attention. Such an investigation enables one not only to study the integrability of the equations of mathematical physics, but it also enables to understand the mechanism of evolutionary processes and an origin of various structures.

2.1 Analysis of consistency of the conservation law equations. Evolutionary relation for the state functionals

As it is shown below, the consistency of the conservation law equations is realized under correlation of the conservation law equation between themselves.

Let us analyze the correlation of the equations that describe the conservation laws for energy and linear momentum. We introduce two frames of reference: the first is an inertial one (this frame of reference is not connected with the material system), and the second is an accompanying one (this frame of reference is connected with the manifold built by the trajectories of the material system elements).

The energy equation in the inertial frame of reference can be reduced to the form:

$$\frac{D\psi}{Dt} = A_1$$

where D/Dt is the total derivative with respect to time, ψ is the functional of the state that specifies the material system, A_1 is the quantity that depends on specific features of material system and on external energy actions onto the system. [The action functional, entropy, wave function can be regarded as examples of the functional ψ. Thus, the equation for energy presented in terms of the action functional S has a similar form: $DS/Dt = L$, where $\psi = S$, $A_1 = L$ is the Lagrange function. In mechanics of continuous media the equation for energy of an ideal gas can be presented in the form [8]: $Ds/Dt = 0$, where s is the entropy.]

In the accompanying frame of reference the total derivative with respect to time is transformed into the derivative along the trajectory. Equation of energy is now written in the form

$$\frac{\partial \psi}{\partial \xi^1} = A_1 \tag{1}$$

Here ξ^1 is the coordinate along the trajectory.

In a similar manner, in the accompanying reference system the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial \psi}{\partial \xi^\nu} = A_\nu, \quad \nu = 2,... \tag{2}$$

where ξ^ν are the coordinates in the direction normal to the trajectory, A_ν are the quantities that depend on the specific features of material system and on external force actions. [In the case of the Euler and Navier-Stokes equations a particular form of relation (2) and its properties were considered in papers [4].]

Eqs. (1) and (2) can be convoluted into the relation

$$d\psi = A_\mu d\xi^\mu, \quad \mu = 1, \nu \tag{3}$$

where $d\psi$ is the differential expression $d\psi = (\partial\psi/\partial\xi^\mu)d\xi^\mu$. .

Relation (3) can be written as

$$d\psi = \omega \tag{4}$$

here $\omega = A_\mu d\xi^\mu$ is the skew-symmetrical differential form of the first degree. (A summing over repeated indices is carried out.)

Relation (4) has been obtained from the equation of the conservation laws for energy and linear momentum. In this relation the form ω is that of the first degree. If the equations of the conservation laws for angular momentum be added to the equations for energy and linear momentum, this form will be a form of the second degree. And in combination with the equation of the conservation law for mass this form will be a form of degree 3. In general case the evolutionary relation can be written as

$$d\psi = \omega^p \tag{5}$$

where the form degree p takes the values $p = 0,1,2,3$. [The relation for $p = 0$ is an analog to that in the differential forms, and it was obtained from the interaction of energy and time.]

[A concrete form of relation (4) and its properties in the case of the Euler and Navier-Stokes equations were considered in papers [5, 9]. In this case the functional ψ is the entropy s. A concrete form of relation (5) for $p = 2$ were considered for electromagnetic field in Appendix 3 of paper [6] and in paper http://arxiv.org/pdf/math-ph/0310050v1.pdf]

Since the conservation law equations are evolutionary ones, the relations obtained are also evolutionary relations, and the skew-symmetric forms ω and ω^p are evolutionary ones.

2.2 Properties of evolutionary relation for the state functional

Evolutionary relation obtained from the equations of the conservation laws possesses some peculiarity. This relation proves to be nonidentical and selfvarying.

2.2.1. Nonidentity of the evolutionary relation.

Evolutionary relation proves to be nonidentical since the differential form in the right-hand side of this relation is not a closed form, and, hence, this form cannot be a differential like the left-hand side.

To justify this we shall analyze relation (4). The form $\omega = A_\mu d\xi^\mu$ is not a close form since its differential is nonzero. The differential $d\omega$ can be written as $d\omega = K_{\alpha\beta} d\xi^\alpha d\xi^\beta$, where $K_{\alpha\beta} = \partial A_\beta / \partial \xi^\alpha - \partial A_\alpha / \partial \xi^\beta$ are the components of the differential form commutator built of the mixed derivatives (here the term connected with the nonintegrability of the manifold has not yet been taken into account). The coefficients A_μ of the form ω have been obtained either from the equation of the conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. The commutator of the form ω made up by derivatives of such coefficients is nonzero. This means that the differential of the form ω is nonzero as well. Thus, the form ω proves to be unclosed and cannot be a differential like the left-hand side. [The skew-symmetric form in the evolutionary relation is defined in the manifold made up by trajectories of the material system elements. Such a manifold is a deforming manifold. (About the properties of such skew-symmetric form one can read, for example, in papers [5, 6]. Some properties of such skew-symmetric differential forms are described in Appendix 1.). The commutator of skew-symmetric form defined on such manifold includes an additional term, namely, the metric form commutator, which is nonzero. And this fact ones more emphasize that the evolutionary form that enters into evolutionary relation cannot be closed.]

Hence, without the knowledge of a particular expression for the form ω, one can argue that for actual processes the evolutionary relation proves to be nonidentical.

In similar manner it can be shown that general relation (5) is also nonidentical. It should be noted that the relation obtained will remain to be nonidentical regardless of the accuracy with what the conservation law equations were written. And this fact has a physical meaning.

2.2.2. Selfvariation of the evolutionary relation.

The evolutionary relation possesses one more peculiarity, namely, this relation is a selfvarying relation.

The evolutionary nonidentical relation is a selfvarying one, because, firstly, it is a nonidentical, namely, it contains two objects one of which appears to be unmeasurable, and, secondly, it is an evolutionary relation, that is, the variation of any object of the relation in some process leads to a variation of another object; and, in turn, the variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot stop. Selfvariation of the evolutionary relation proceeds by exchange between the evolutionary form coefficients and manifold characteristic. The evolutionary form is defined on the deforming manifold made up by trajectories of the material system elements. This means that the evolutionary form basis varies. In turn, this leads to variation of the evolutionary form, and the process of interacting change of the evolutionary form and the basis is repeated.

3 Hidden properties and possibilities of the equations of mathematical physics

The evolutionary relation has an unique significance for the equations of mathematical physics. This relation discloses functional properties of the equations of mathematical physics (such as an existence of double solutions), a role of the equations of mathematical physics in description the processes of origin of various structures and a connection between the equations of mathematical physics and the field-theory equations. (Such possibilities of the equations of mathematical physics occurs to be hidden and does not follow directly from initial equations of mathematical physics.)

3.1 Double solutions of the equations of mathematical physics

From the evolutionary relation it follows that the equations of mathematical physics have double solutions, namely, the solutions, which are not functions (these solutions can be called inexact solutions), and generalized solutions, i.e. solutions that are discrete functions.

3.1.1. Inexact solutions to the mathematical physics equations.

The nonidentity of the evolutionary relation points to the fact that the conservation law equations appear to be inconsistent. This means that the initial set of equations of mathematical physics proves to be nonintegrable (it cannot be convoluted into identical relation for differentials and be integrated). That is, the solutions to the mathematical physics equations are not functions (they will depend on the commutator of the form ω^p). This also points to the fact that the tangent manifold, on which the solutions are defined, is not integrable.

3.1.2. Generalized solutions to the mathematical physics equations.

From the evolutionary relation it follows that the mathematical physics equations can have solutions which are functions. However, it is possible only in the case **when from the evolutionary skew-symmetric form in the right-hand side of nonidentical evolutionary relation a closed skew-symmetric form, which is a differential, is realized**. In this case the identical relation is obtained from the nonidentical relation, and this will point out to a consistency of the conservation law equations and an integrability of the mathematical physics equations. But here there is some delicate matter. From the evolutionary unclosed skew-symmetric form, which differential is nonzero, one can obtain a closed exterior form with a differential being equal to zero only under degenerate transformation, namely, under a transformation that does not conserve differential. (The Legendre transformation is an example of such a transformation.) Degenerate transformations can take place under additional conditions, which due degrees of freedom. The vanishing of such functional expressions as determinants, Jacobians, Poisson's brackets, residues, and others corresponds to these additional conditions. The conditions of degenerate transformation specify the integrable structures on which the solutions become discrete functions. These conditions can be realized under a change of nonidentical evolutionary relation, which, as it was noted, appears to be a selfvarying relation.

If the conditions of degenerate transformation are realized, from the unclosed evolutionary form ω^p (see evolutionary relation (5)) with non vanishing differential $d\omega^p \neq 0$, one can obtain a closed (only on some pseudostructure) exterior form with vanishing (interior) differential. That is, it is realized the transition

$$d\omega^p \neq 0 \rightarrow \text{degenerate transformation} \rightarrow d_\pi \omega^p = 0,\ d_\pi{}^* \omega^p = 0$$

The realization of the conditions $d_\pi{}^* \omega^p = 0$ and $d_\pi \omega^p = 0$ means that it is realized a closed dual form ${}^*\omega^p$, which describes some structure π (which is a pseudostructure with respect to its metric properties), and the closed exterior (inexact) form ω_π^p, which basis is a pseudostructure, is obtained.

The closed dual form and associated closed inexact exterior form made up a differential-geometrical structure that describes a pseudostructure with conservative quantity. (A closed dual form describes a pseudostructure. And a closed exterior form, as it is known, describes a conservative quantity, since the differential of closed form is equal to zero).

Such a differential-geometrical structure is an integrable structure, on which the solutions to the mathematical physics equations become functions. The structures like the characteristics, singular points, characteristic and potential surfaces, which are obtained when solving the mathematical physics equations, are such integrable structures. (Below it will be shown that the pseudostructures with conservative quantity are structures that have physical meaning.)

On an integrable structure, from evolutionary relation (5) it follows the relation

$$d\psi_\pi = \omega_\pi^p \tag{6}$$

which occurs to be an identical one, since the form ω_π^p is a differential.

The identity of the relation obtained from the evolutionary relation means that on the integrable structure the equations of conservation laws (of the equations of mathematical physics) become consistent. This points out to that the equations of mathematical physics become locally integrable (only on integrable structure).

The solutions to the mathematical physics equations on integrable structures are generalized solutions, which are discrete functions, since they are realized only under additional conditions (on the integrable structures). The solutions on characteristics or on potential surfaces are examples of such generalized solutions.

On integrable structures the desired quantities of the material system (such as the temperature, pressure, density) become functions of only independent variables and do not depend on the commutator (and on the path of integrating). Such functions may be found by means of integrating (on integrable structures) the equations of mathematical physics.

Since generalized solutions are defined only on realized integrable structures, they or their derivatives have discontinuities in the direction normal to integrable structure [10].

Thus, from the evolutionary relation it follows that the equations of mathematical physics have solutions of two types:

(1**) the inexact solutions** that are not functions, i.e., they depend not only on independent variables, and

(2) **the generalized solutions**, which are discrete functions.
The specific feature is the fact that these solutions are defined on different spatial objects. The solutions of the first type are inexact solutions defined on tangent nonintegrable manifold of original mathematical physics equations. And the solutions of the second type are generalized solutions defined on integrable structures that arise spontaneously under realization of additional conditions caused, for example, by degrees of freedom of the material medium described. Such properties of the solutions to the mathematical physics equations, as it was shown in papers [2, 11], lead to difficulties in modelling and numerical solving differential equations.

3.2 Physical meaning of double solutions to the equations of mathematical physics

An unique importance of nonidentical evolutionary relation consists in the fact that this relation discloses a mathematical and physical meaning of double solutions to the equations of mathematical physics and their role in description of the evolutionary procsses in material media as well the processes of origin of various structures.

3.2.1 Physical meaning of inexact solution. Description of non-equilibrium state of material medium.

From the evolutionary relation it follows that this relation can describe the material medium state. This relates to the fact that it includes the state functional, which specifies the material system state. But here there is some delicate matter. Although the evolutionary relation includes the state functional (which specifies the material medium state), but since this relation is nonidentical one, from this relation one cannot get the differential of the state functional ψ . This points out to the absence of the state function and means that the material medium is in the non-equilibrium state.

The non-equilibrium means that an internal force acts in material medium. It is evident that the internal force is described by the commutator of skew-symmetric form ω^p . (If the evolutionary form commutator be zero, the evolutionary relation would be identical, and this would point to the equilibrium state, i.e. the absence of internal forces). Everything that gives a contribution into the commutator of evolutionary form ω^p leads to emergence of internal forces that causes the non-equilibrium state of material medium (see [4]).

A non-equilibrium state of material medium is described by inexact solution to the mathematical physics equations. This follows from the fact that, in the case of nonidentity of the evolutionary relation (which points out to the non-equilibrium state of material medium) the inexact solution is a solution to the mathematical physics equations. (The nonidentity of evolutionary relation points also to the fact that the non-equilibrium state of material system is caused by the noncommutativity of the conservation laws that follows from the inconsistency of the conservation law equations.)

Another property of the nonidentical evolutionary relation, namely, its self variation, points out to the fact that the non-equilibrium state of material medium turns out to be selfvarying. State of material medium changes but in this case remains to be non-equilibrium during this process, since the evolutionary relation remains to be nonidentical durining the process of self variation.

3.2.2 Physical meaning of generalized solution. Transition of a material medium to a locally equilibrium state.

The generalized solution arises under realization of identical relation. From identical relation one can define the differential of the state functional, and this points out to a presence of the state function and the transition of material medium from non-equilibrium state into equilibrium one. However, such a state of material medium turns out to be realized only locally due to the fact that differential of the state functional obtained is an differential interior (only on pseudostructure). And yet the total state of material medium remains to be non-equilibrium state because the evolutionary relation, which describes the material medium state, remains nonidentical one. (That is, there exists a duality. Nonidentical evolutionary relation goes on to act simultaneously with identical relation.)

The transition from non-equilibrium state to locally equilibrium state means that unmeasured quantity, which is described by the commutator and act as internal force, converts into a measured quantity of material medium. This reveals in emergence of some observed formations in material medium. Waves, vortices, fluctuations, turbulent pulsations and so on are examples of such formations. The intensity of such formations is controlled by a quantity accumulated by the evolutionary form commutator. (This discloses a mechanism of such processes like an origin of vortices and turbulence. In papers [4, 9] the process of production of vorticity and turbulence was described.) Such emerged formations is described by generalized solutions to the equations of mathematical physics. The functions that correspond to generalized solutions, as it is known, are discrete functions. Such functions or its derivative have a discontinuity in the direction normal to integrable structure. Realization of integrable structures with such discontinuous functions and the transition from nonintegrable tangent manifold to integrable structure just describes the emergence of some such formations.

Thus, from the evolutionary relation it follows that the equations of mathematical physics have solutions of two types, one of which is defined on nonintegrable original manifold and is not a function, and the other is realized on integrable structures and is a discrete function. The first solution describes a non-equilibrium state of material medium, and the second solution describes a locally equilibrium state of a medium. The transition from the solution of the first type to the generalized solution describes the transition of the material medium state from non-equilibrium state to locally equilibrium state and is accompanied by an emergence in material medium of some observed formation, which is described by the generalized solution.

It should be emphasized that such properties are inherent to the solutions of the mathematical physics equations, which describe actual processes in material media and on which any additional conditions (integrability conditions and other requirements) are not imposed.

4. Connection between the equations of mathematical physics and the field-theory equations

The evolutionary relation discloses one more unique property of the mathematical physics equations, namely, a connection between the mathematical physics equations, which describe material media, and the field-theory equations, which describe physical fields. This is justified by the fact that there is a correspondence between the field-theory equations and the evolutionary relation.

4.1. Correspondence between the field-theory equations and the evolutionary relation

The field-theory equations, which describe physical fields, are equations for functionals such as wave function, the action functional, Pointing's vector, Einstein's tensor, and others. The nonidentical evolutionary relations derived from the equations of mathematical physics, which describe material media, are relations for all these functionals.

From the evolutionary relation, as it was shown, closed inexact exterior forms, which degree p changes from 0 to 3, follow. Such closed inexact exterior or dual forms are solutions of the field-theory equations:

- closed exterior forms of zero degree correspond to quantum mechanics;

- the Hamilton formalism bases on the properties of closed exterior and dual forms of first degree;

- the properties of closed exterior and dual forms of second degree are at the basis of the equations of electromagnetic field;

- the closure conditions of exterior and dual forms of third degree made up the basis of equations for gravitational field.

To the connection between the field-theory equations and the equations of mathematical physics it also points out to the fact that, all equations of field theories, as well as the evolutionary relation, are nonidentical relations in differential form or in the forms of their tensor or differential (i.e. expressed in terms of derivatives) analogs. For example,

- the Einstein equation is a relation in differential forms;

- the Dirac equation relates Dirac's *bra*- and *cket*- vectors, which made up a differential form of zero degree;

- the Maxwell equations have the form of tensor relations;

- the Schrodinger's equations have the form of relations expressed in terms of derivatives and their analogs.

[The field-theory equations are those whose solutions must be not functions but differentials (closed exterior forms). The equations whose solutions are differentials are obtained from usual differential equations under the integrability condition. Equations that obey the integrability conditions turn out to be nonidentical relations between differential skew-symmetric forms or its analogs. The field-theory equations appears to be such nonidentical relations.]

From the field-theory equations, as well as from the nonidentical evolutionary relation, the identical relation, which contains the closed exterior form, is obtained. As one can see, from the field-theory equations it follows such identical relation as

- the Poincare invariant, which connects closed exterior forms of first degree;

- the relations $d\theta^2=0$, $d\,^*\theta^2=0$ are those for closed exterior forms of second degree obtained from Maxwell equations;

- the Bianchi identity for gravitational field.

Thus, one can see that there exists a correspondence between the field-theory equations, which describe physical fields, and the evolutionary relation obtained from the equations of mathematical physics for a material medium.

Such a correspondence enables one to understand the basic principles of field theory and the properties of physical fields.

From the correspondence between the field-theory equations and the evolutionary relation it follows that the functionals of field-theory equations (such as wave function, the action functional, the Pointing vector, Einstein tensor, and others) are functionals that specify the state of relevant material medium (material system), that is, they are state functionals.

Moreover, one can see that the field theory equations (as well as the evolutionary relation) are connected with closed exterior forms of a certain degree. This points out to the fact that there exists an internal connection between field theories, which describe physical fields of various types. It is evident that the degree of closed exterior forms is a parameter that integrates field theories into unified field theory.

The correspondence between the field-theory equations and the evolutionary relation points out to the fact that the field-theory equations, as well as the evolutionary relation, are a consequence of the mathematical physics equations on which the integrability conditions are imposed. This discloses the basic principles of the field theory that are common for all field-theory equations. This fact has to be taken into account when building the general field theory. The evolutionary relation obtained from the equations of mathematical physics, which discloses common properties and peculiarities of existing equations of field theory, can play a role of the equation of general field theory.

The connection between the field-theory equations, which describe physical fields, and the equations of mathematical physics, which describe material media, points out to the fact that it has to exist a connection between physical fields and material media.

4.2. Connection of physical fields with material media

The evolutionary relation possesses one more unique properties. The evolutionary relation discloses a mechanism of generation of physical structures, from which physical fields are formatted. This follows from the properties of conservation laws.

4.2.1. Conservation laws.

The field-theory equations, as well as the equations of mathematical physics, are connected with the properties of conservation laws.

The equations of mathematical physics are connected with the conservation laws for energy, linear momentum, angular momentum, and mass. These are conservation laws for material media. They establish a balance between the change of physical quantities and the external action. Such conservation laws are described by differential (or integral) equations.

In field theory "the conservation laws" are those that claim an existence of conservative quantities or objects. The conservation laws for physical fields, which can be named exact ones, are such conservation laws. They are described by closed exterior skew-symmetric forms. (The Noether theorem is an example.)

From the evolutionary relation it follows that there exists a connection between the balance and exact conservation laws.

4.2.2. Fulfillment of the exact conservation law.

As it follows from the nonidentity of evolutionary relation, the equations of the balance conservation laws (conservation laws for energy, linear momentum, angular momentum, and mass) are inconsistent. (This points out to a noncommutativity of the balance conservation laws). Under realization of additional conditions the identical relation is realized from nonidentical evolutionary relation. That is, the equations of the balance conservation laws become a locally consistent. (This means that the conservation laws become locally commutative).

As it was shown, the realization of identical relation is connected with the realization of closed dual form and closed inexact exterior skew-symmetric forms that describe a pseudostructure with conservative quantity, i.e. a conservative object. The realization of the conservative object means that the exact conservation law is fulfilled.

One can see that the realization of consistence of the equations of balance conservation laws relates to a fulfillment of exact conservation laws.

Such a process discloses the mechanism of emerging physical structures from which physical fields are formatted.

4.2.3. Physical structures that format physical fields and manifolds.

As it was noted before, closed inexact exterior and dual forms are differential-geometrical structures. Such a differential-geometrical structure describes a pseudostructure with conservative quantity, i.e. a structure on which the exact conservation law fulfills. Physical structures from which physical fields and relevant manifolds are formatted just are such structures. (Massless particles, structures made up by eikonal surfaces and wave fronts, and so on are examples of such physical structures.)

It should be noted that the differential-geometrical structure possesses a duality. On the one hand, as it was shown in Subsection 3.1.2, it describes the integrable structure a, namely, a pseudostructure with generalized solution, and, on the other hand, it describes a physical structure, i.e. a pseudostructure with conservative quantity on which the exact conservation law is fulfilled.

The realization of integrable structures, i.e. pseudostructures with generalized solution, describes the transition of a material system from non-equilibrium state into a locally equilibrium state, which is accompanied by an emergence of observable formations in material medium.

On the other hand, the realization of pseudostructures with conservative quantity on which the exact conservation law is fulfilled points out to origination of physical structures, from which physical fields are formatted.

Such a duality of differential-geometrical structures discloses the process of generation of physical structures and a connection between physical fields and material media.

One can see that the transition of material media from non-equilibrium state to locally-equilibrium one is accompanied by the emergence of observable formations in material media and origination of physical structures that made up physical fields.

Physical structures and observed formations are a manifestation of the same phenomenon. (Light is an example of manifestation of such a duality, namely, as a massless particle (photon) and as a wave.) However, physical structures and observed formations are not identical objects. Whereas the wave is an observable formation, the element of wave front made up the physical structure in the process of its motion.

The mechanism of origination the physical structures elucidates the mechanism of forming the physical fields and manifolds [12]. (The dual forms, which are metric forms of the manifold, describe pseudostructures, from which pseudometric and metric manifolds are formatted). To every physical field it is assigned its own material medium (a material system). As examples of material systems it may be cosmic systems, systems of elementary particles and others.

5. Conclusion

It has been shown that the equations of mathematical physics, which describe actual processes in material media and on which any additional conditions (integrability conditions and other requirements) are not imposed, have double solutions. Due to this fact the equations of mathematical physics can describe the processes of emergence of various structures and formations.

[The Navier-Stokes equations (which describe a flow of a viscous gas or fluid) are example of the mathematical physics equations. The problem of solving these equations was claimed as the problem of the millennium. As the main problem of the Navier-Stokes equations it was claimed a proof of existence and smoothness of solutions of the Navier-Stokes equations. As it follows from the Ref.[4] and present paper, the smooth analytical solution does not exist. The Navier-Stokes equations, as well corresponding mathematical physics equations, have double solutions, namely, the solution on integral structures and the solution on a tangent (nonintegrable) manifold. The solution to the Navier-Stokes equations on a tangent manifold is a solution that is not a function (i.e., depends not only on the variables), since the tangent manifold of the Navier-Stokes equations (as well as the tangent manifold of corresponding mathematical physics equations) is nonintegrable (because these equations describe actual, nonpotential, processes). The analytical solution, i.e. a solution that is a function, is realized on integral structures, that is, this solution is discrete one. Only under realization of additional conditions and realization of integral structures (that can be caused by any degrees of freedom and proceed only discretely), the solutions that are discrete functions (namely, functions defined only on integrate structures) are obtained. The process of realization of the solutions that are discrete functions describes the process of emergence of turbulent pulsations. Such a process relates to the transition from tangent manifold to integral structures and to the transition from the solutions that are not functions to solutions that are discrete functions. One can see that only with the help of double solutions the development of turbulence can be described. (If there were existed only smooth solution, the description of the turbulence origination would be impossible.)]

The equations of mathematical physics possess one more unique property. The evolutionary relation, which follows from the equations of mathematical physics, corresponds to the field-theory equations, and this fact enables one to understand an internal connection of the field-theory equations and its foundations.

[As it was shown, a non-measurable quantity that is accumulated in the evolutionary form commutator partly converts into conservative quantity of physical (observed) structures under realization of any degrees of freedom (to which a degenerate transformation is assigned). A non-measurable quantity that does not convert into physical structure turns out to be a non-observed and non-measurable quantity. Dark matter and dark energy are such an essence which reveals as the result of the noncommutativity of conservation laws of relevant material media produced by various nonpotential actions. The deformation of manifolds, which is made up by the trajectories of the material system elements, arisen under the action of external forces (due to nonpotential actions) relates to this fact.]

1) Appendix 1

Some properties of skew-symmetric forms corresponding to the conservation laws

a) Closed inexact exterior forms: differential-geometrical structures.

The exterial differential form of the degree p (p - form) can be written as [5, 13]:

$$\theta^p = \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \quad 0 \le p \le n \tag{1}$$

where

$$\begin{aligned} dx^i \wedge dx^i &= 0 \\ dx^i \wedge dx^j &= -dx^j \wedge dx^i \end{aligned} \tag{2}$$

The exterior form differential θ^p is expressed by the formula

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \tag{3}$$

The form called as a closed one if its differential equals to zero:

$$d\theta^p = 0 \tag{4}$$

From condition (4) one can see that the closed form is a conservative quantity. This means that such a form can correspond to the conservation law (for physical fields), i.e. a conservative quantity.

If the form be closed only on pseudostructure, i.e. this form is a closed inexact one, the closure condition can be written as

$$d_\pi \theta^p = 0 \tag{5}$$

In this case the pseudostructure π obeys the condition

$$d_\pi {}^*\theta^p = 0 \tag{6}$$

here ${}^*\theta^p$ is the dual form.

From conditions (5) and (6) one can see that the dual form (pseudostructure) and closed inexact form (conservative quantity) describe a conservative object that can also correspond to some conservation law. (It appears that the closed inexact exterior and dual forms describe a structure with conservative quantity. Such structures made up physical fields and pseudometric and metric manifolds.)

It turns out that the closed inexact exterior forms are obtained from the skew-symmetric differential forms, whose basis is nonintegrable manifolds (in contrast to exterior skew-symmetric forms). Such forms, which possess the evolutionary properties, are obtained from the equations which describe any processes.

b) Distinction of evolutionary forms from exterior forms [14].

The evolutionary form can be written in a manner similar for exterior differential form [5]. However, in distinction from the exterior form differential, an additional term will appear in the evolutionary form differential. This is due to the fact that the evolutionary form basis changes since such a form is defined on nonintegrable manifold.

The evolutionary form differential takes the form

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} + \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} d(dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}) \qquad (7)$$

where the second term is connected with the basis differential being is nonzero: $d(dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}) \neq 0$ (for the exterior form defined on integrable manifold one has $d(dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}) = 0$.)

The peculiarity of skew-symmetric forms defined on nonintegrable manifold can be demonstrated by the example of a skew-symmetric form of first-degree.

Let us consider the first-degree form $\omega = a_\alpha dx^\alpha$. The differential of this form can be written as $d\omega = K_{\alpha\beta} dx^\alpha dx^\beta$, where $K_{\alpha\beta} = a_{\beta;\alpha} - a_{\alpha;\beta}$ are components of the commutator of the form ω, and $a_{\beta;\alpha}$, $a_{\alpha;\beta}$ are covariant derivatives. If we express the covariant derivatives in terms of connectedness (if it is possible), they can be written as $a_{\beta;\alpha} = \partial a_\beta / \partial x^\alpha + \Gamma^\sigma_{\beta\alpha}$, where the first term results from differentiating the form coefficients, and the second term results from differentiating the basis. If we substitute the expressions for covariant derivatives into the formula for commutator components, we obtain the following expression for commutator components of the form ω:

$$K_{\alpha\beta} = \left(\frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta} \right) + (\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta}) a_\sigma \qquad (8)$$

Here the expressions $(\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})$ entered into the second term are just components of the commutator of the first-degree metric form that specifies the manifold deformation and hence is nonzero. (It is well-known that the metric form commutators of the first-, second- and third degrees specifies, respectively, torsion, rotation and curvature.)

(In the commutator of exterior form, which is defined on integrable manifold, the second term absents: the connectednesses are symmetric, that is, the expression $(\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})$ vanishes).

Since the commutator, and hence the differential, of skew-symmetric form defined on nonintegrable manifold are nonzero, this means that such a form cannot be closed one.

The skew-symmetric form, which is obtained from the conservation law equations for material media, is just such evolutionary form. The basis of this form is an accompanying manifold, namely, the manifold built by the trajectories of the material system elements. Such a manifold is a deforming nonintegrable manifold. For this reason the basis differential of skew-symmetric form defined on such manifold is nonzero. Such evolutionary form is unclosed one.

The evolutionary form enters into the evolutionary relation (5). Since the evolutionary form is unclosed and is not a differential, the evolutionary relation turns out to be non identical. Such a property of the evolutionary relation just discloses the specific features of the solutions to the mathematical physics equations (which describe material systems), the mechanisms of origination of various structures and the connection of the field-theory equations with the equations of mathematical physics.

2) Appendix 2

The equations of mathematical physics describing material media.

In the paper, the equations of mathematical physics, which describe material media, are investigated.

Such equations are presented for example in papers

1. Smirnov V.I., "A course of higher mathematics", -Moscow, Tech.Theor.Lit. 1957, V.4 (in Russian)

(points 164,165 and 167);

2. R.Courant, "Partial Differential Eguations", New York.London, 1962 (Chapter 6, paragraph 3a),

3. W. Pauli "Theory of Relatiyity, Pergamon Press, 1958 (paragraphs 30, 37).

The equations of gas-dynamics, which describe the flow of ideal gas, are the example of such equations.

Such equations can be written in the form [8]:

The conservation law equations for energy (see 5.3.21)

$$\rho \frac{D}{Dt}(e + \frac{1}{2}u^2) = f_1$$

The conservation law equations linear momentum (see 5.3.15):

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k}) = f_j$$

The conservation law equations for mass (see 5.3.12)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$

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