Journal of Progressive Research in Mathematics
Vol 19,2 - 2022 — E-ISSN: 2395-0218
http://www.scitecresearch.com/journals/index.php/jprm

Traveling wave solutions and numerical solutions for a mBBM equation

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Received: May 10, 2022; Accepted: June 12, 2022; Published: July 4, 2022

Abstract

In this paper, some exact meromorphic solutions and generalized trigonometric solutions of the space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation are established by a new transformation and reliable methods. Moreover, some numerical solutions are obtained by using the optimal decomposition method (ODM), and their accuracy is shown in tables and images.

Keywords

mBBM equation; complex method; extended direct algebraic method; ODM.

1. Introduction

In 1972, Benjamin, Bona and Mahony constructed the long waves model in nonlinear dispersive system, that is, the Benjamin-Bona-Mahony(BBM) equation [1]

$$u_t + u_x + uu_x + u_{xxx} = 0.$$

It was originally a nonlinear partial differential equation used to simulate small amplitude long waves in hydrodynamics. For further study, the BBM equation is modified into the following form

$$u_t + u_x + au^2u_x + u_{xxx} = 0,$$

and it can not only describe the approximation of surface long waves in a nonlinear dispersive medium, but also characterize the hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals and acoustic-gravity waves in compressible fluids [1–3].

In the process of exploration, people find that many phenomena in nature can not be accurately described by integer order equations, so fractional order differential equations have entered the public field of vision

Cite this paper: Ni, W., Li, Y. (2022). Traveling wave solutions and numerical solutions for a mBBM equation. Journal of Progressive Research in Mathematics, 19(2), 1-14. Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2145

and gradually become a popular research topic. Naturally, the BBM equation is further rewritten into the following mBBM equation [4]

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}}+\frac{\partial^{\alpha} u}{\partial x^{\alpha}}-v u^{2} \frac{\partial^{\alpha} u}{\partial x^{\alpha}}+\frac{\partial^{3 \alpha} u}{\partial x^{3 \alpha}}=0, \quad \alpha \in(0,1). \tag{1}$$

However, it is not easy to solve the fractional differential equations accurately, so solving and verifying the accuracy of the solutions is a meaningful work. Therefore, the purpose of this paper is to find some analytical and numerical solutions of mBBM equation, and verify that our solutions are reliable. Eq.(1) was studied by some scholars in the past few years. They have adopted Exp-function method, extended (G'/G)-expansion method, the first integral method and extended tanh method, and achieved lots of results [5–11].

The structure of this paper is as follows: in Section 2, Eq.(1) is transformed into an ordinary differential equation(ODE) by using the definition and properties of truncated M-fractional derivative; in Section 3, the meromorphic solutions of Eq.(1) are obtained by complex method; the generalized trigonometric solutions of Eq.(1) are worked out by extended direct algebraic method in Section 4; in Section 5, ODM is used to obtain approximate solutions, and some images and tables show us the accuracy of them; the summary of our work is written in Section 6.

2. Simplification of mBBM equation

At the beginning, we will introduce the main tool used in simplification——truncated M-fractional derivative [12].

Definition 1. *Let $f:[0,\infty)\to\mathbb{R}$. For $\alpha\in(0,1)$, a truncated M-fractional derivative type of f of order α, denoted by ${}_{i}\mathbb{D}_{M}^{\alpha,\beta}$, is*

$${}_{i}\mathbb{D}_{M}^{\alpha,\beta} f(t):=\lim_{\varepsilon\to 0}\frac{f(t_{i}\mathbb{E}_{\beta}(\varepsilon t^{-\alpha}))-f(t)}{\varepsilon},$$

$\forall t>0$, and ${}_{i}\mathbb{E}_{\beta}(\cdot),\beta>0$ is a truncated Mittag-Leffler function of one parameter, defined as

$${}_{i}\mathbb{E}_{\beta}(z)=\sum_{k=0}^{i}\frac{z^{k}}{\Gamma(\beta k+1)}, z\in\mathbb{C}.$$

Theorem 1. *Let $0<\alpha\leq 1,\beta>0,a,b\in\mathbb{R}$ and f,g α-differentiable, at a point $t>0$. Then:*

(1) ${}_{i}\mathbb{D}_{M}^{\alpha,\beta}(f\circ g)(t)=f'(g(t))_{i}\mathbb{D}_{M}^{\alpha,\beta}g(t)$, for f differentiable at $g(t)$.

(2) If f is differentiable, then ${}_{i}\mathbb{D}_{M}^{\alpha,\beta}f(t)=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{df(t)}{dt}$.

From what is known above, we construct a transformation

$$u(x,t)=W(z), z=\frac{kx^{\alpha}\Gamma(\beta+1)}{\alpha}+\frac{lt^{\alpha}\Gamma(\beta+1)}{\alpha}.$$

Then one can calculate

$$\begin{aligned}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}&={}_{i}\,\mathbb{D}_{M,t}^{\alpha,\beta}u(x,t)={}_{i}\,\mathbb{D}_{M,t}^{\alpha,\beta}W[z(x,t)]=W'_{i}\mathbb{D}_{M,t}^{\alpha,\beta}z(x,t)\\&=W'\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{dz(x,t)}{dt}=W'\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{l\alpha t^{\alpha-1}\Gamma(\beta+1)}{\alpha}\\&=lW'.\end{aligned}$$

Similarly, $\frac{\partial^{\alpha}u}{\partial x^{\alpha}}=kW'$, $\frac{\partial^{3\alpha}u}{\partial x^{3\alpha}}=k^{3}W'''$.

So, Eq.(1) becomes to

$$(k+l)W' - kvW^2W' + k^3W''' = 0.$$

Integrating the above formula to obtain

$$(k+l)W - \frac{1}{3}kvW^3 + k^3W'' + b = 0. \tag{2}$$

After that, we only need to find the solution $W(z)$ of Eq.(2), then bring the transformation into it, and we'll get the solution $W(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha}) = u(x,t)$ for Eq.(1).

3. Meromorphic solutions with complex method

Considering Eq.(2) in the complex plane [13–17], its solutions are included in class W (a meromorphic function f belongs to the class W if f is a rational function of z, or a rational function of $e^{\gamma z}, \gamma \in \mathbb{C}$, or an elliptic function). And it is found by calculation that the Eq.(2) satisfies the weak ¡2,1¿ condition. This gives us a guide for assuming the form of the solutions [18, 19].

Remark 1. *The weak¡p,q¿ condition has the following definition.*

Let $p, q \in \mathbb{N}$. Suppose that the meromorphic solution $W(z)$ of Eq.(2) has at least one pole, we say that Eq.(2) satisfies the weak ¡p,q¿ condition if substituting Laurent series

$$W = \sum_{k=-q}^{\infty} c_k z^k, q > 0, c_{-q} \neq 0$$

into Eq.(2) and then we can determine p distinct Laurent singular parts $\sum\limits_{k=-q}^{-1} c_k z^k$.

According to the definition in Remark 1 and the homogeneous balance method, we can get $q = 1, p = 2, c_{-1} = \pm\sqrt{\frac{6k^2}{v}}$.

3.1. Rational solutions

Suppose the form of rational solutions of Eq.(2) with $z = 0$ as the pole is

$$W(z) = \frac{\varrho_{11}}{z-\tau} + \frac{\varrho_{21}}{z} + \varrho_{10}, \tag{3}$$

where $\varrho_{11}, \varrho_{21}, \varrho_{10}$ are all undetermined coefficients, and τ is any complex number. Substitute Eq.(3) into Eq.(2) and we get

$$\sum_{i=0}^{6} \frac{A_i z^i}{3(z-\tau)^3} = 0, \tag{4}$$

where

$$A_0 = \varrho_{21}^3 k\tau^3 v - 6\varrho_{21}k^3\tau^3,$$

$$A_1 = 18\varrho_{21}k^3\tau^2 + 3\varrho_{10}\varrho_{21}^2 k\tau^3 v - 3\varrho_{21}^3 k\tau^2 v - 3\varrho_{11}\varrho_{21}^2 k\tau^2 v,$$

$$\begin{aligned} A_2 =& -18\varrho_{21}k^3\tau - 3\varrho_{21}k\tau^3 + 3\varrho_{10}^2\varrho_{21}k\tau^3 v - 9\varrho_{10}\varrho_{21}^2 k\tau^2 v - 6\varrho_{10}\varrho_{11}\varrho_{21}k\tau^2 v + 3\varrho_{21}^3 k\tau v + 6\varrho_{11}\varrho_{21}^2 k\tau v \\ &+ 3\varrho_{11}^2\varrho_{21}k\tau v - 3\varrho_{21}l\tau^3, \end{aligned}$$

$$
\begin{aligned}
A_3 =& -3b\tau^3+6\varrho_{11}k^3+6\varrho_{21}k^3-3\varrho_{10}k\tau^3+3\varrho_{11}k\tau^2+9\varrho_{21}k\tau^2+\varrho_{10}^3k\tau^3v-3\varrho_{10}^2\varrho_{11}k\tau^2v-9\varrho_{10}^2\varrho_{21}k\tau^2v\\
&+3\varrho_{10}\varrho_{11}^2k\tau v+9\varrho_{10}\varrho_{21}^2k\tau v+12\varrho_{10}\varrho_{11}\varrho_{21}k\tau v-\varrho_{11}^3kv-\varrho_{21}^3kv-3\varrho_{11}\varrho_{21}^2kv-3\varrho_{11}^2\varrho_{21}kv\\
&-3\varrho_{10}l\tau^3+3\varrho_{11}l\tau^2+9\varrho_{21}l\tau^2,\\
A_4 =& 9b\tau^2+9\varrho_{10}k\tau^2-6\varrho_{11}k\tau-9\varrho_{21}k\tau-3\varrho_{10}^3k\tau^2v+6\varrho_{11}\varrho_{10}^2k\tau v+9\varrho_{21}\varrho_{10}^2k\tau v-3\varrho_{11}^2\varrho_{10}kv-3\varrho_{21}^2\varrho_{10}kv\\
&-6\varrho_{11}\varrho_{21}\varrho_{10}kv+9\varrho_{10}l\tau^2-6\varrho_{11}l\tau-9\varrho_{21}l\tau,\\
A_5 =& -9b\tau-9\varrho_{10}k\tau+3\varrho_{11}k+3\varrho_{21}k+3\varrho_{10}^3k\tau v-3\varrho_{11}\varrho_{10}^2kv-3\varrho_{21}\varrho_{10}^2kv-9\varrho_{10}l\tau+3\varrho_{11}l+3\varrho_{21}l,\\
A_6 =& 3b+3\varrho_{10}k-k\varrho_{10}^3v+3\varrho_{10}l.
\end{aligned}
$$

Eq.(4) holds, which implies all $A_i=0$. By computing the system of equations we get $\varrho_{21}=\pm\sqrt{\frac{6k^2}{v}}$, $\varrho_{11}=\mp\frac{(k+l)\tau^2}{\sqrt{6k^4v}}$, $\varrho_{10}=\mp\frac{(k+l)\tau}{\sqrt{6k^4v}}$, $b=\pm2\frac{(k+l)}{\tau}\sqrt{\frac{2k^2}{3v}}$, $l=\frac{6k^3-k\tau^2}{\tau^2}$, and $\varrho_{21}=\pm\sqrt{\frac{6k^2}{v}}$, $\varrho_{11}=0, \varrho_{10}=0, b=0, l=-k$.

So all the rational solutions of Eq.(2) are

$$
W_1(z)=\mp\frac{\frac{(k+l)\tau^2}{\sqrt{6k^4v}}}{z-\tau-z_0}\pm\frac{\sqrt{\frac{6k^2}{v}}}{z-z_0}\mp\frac{(k+l)\tau}{\sqrt{6k^4v}},\quad z_0\in\mathbb{C},
$$

and

$$
W_2(z)=\frac{\pm\sqrt{\frac{6k^2}{v}}}{z-z_0},\quad z_0\in\mathbb{C}.
$$

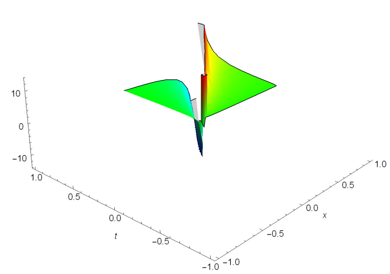
Naturally, the solutions to Eq.(1) are

$$
\begin{aligned}
u_1(x,t)=&\mp\frac{\frac{(k+l)\tau^2}{\sqrt{6k^4v}}}{\frac{kx^\alpha\Gamma(\beta+1)}{\alpha}+\frac{lt^\alpha\Gamma(\beta+1)}{\alpha}-\tau-z_0}\pm\frac{\sqrt{\frac{6k^2}{v}}}{\frac{kx^\alpha\Gamma(\beta+1)}{\alpha}+\frac{lt^\alpha\Gamma(\beta+1)}{\alpha}-z_0}\\
&\mp\frac{(k+l)\tau}{\sqrt{6k^4v}},\quad z_0\in\mathbb{C},
\end{aligned}
$$

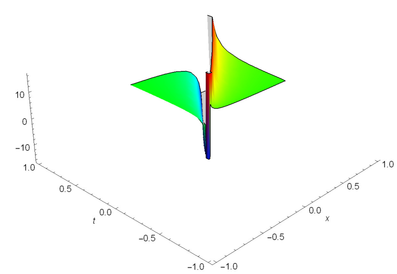
and

$$
u_2(x,t)=\frac{\pm\sqrt{\frac{6k^2}{v}}}{\frac{kx^\alpha\Gamma(\beta+1)}{\alpha}+\frac{lt^\alpha\Gamma(\beta+1)}{\alpha}-z_0},\quad z_0\in\mathbb{C}.
$$

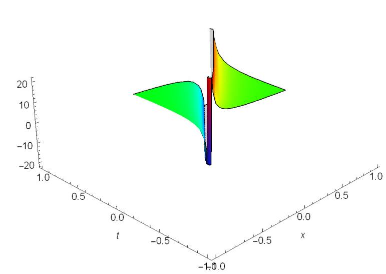
The images of rational solution $u_{2,1}(x,t)=\frac{\sqrt{\frac{6k^2}{v}}}{\frac{kx^\alpha\Gamma(\beta+1)}{\alpha}+\frac{lt^\alpha\Gamma(\beta+1)}{\alpha}-z_0}$ are as follows.



(a) $\alpha=0.2$



(b) $\alpha=0.5$



(c) $\alpha=0.8$

Figure 1: $u_{2,1}(x,t)$ with $k=1, l=-1, v=4, \beta=0.09, z_0=0$.

3.2. Simple periodic solutions

Let $W(z) = R(\eta), \eta = e^{\gamma z}$($\gamma$ is a constant), Eq.(2) turns to

$$(k+l)R - \frac{1}{3}kvR^3 + k^3\gamma^2(R''\eta^2 + R'\eta) + b = 0. \tag{5}$$

Its solutions with $z = 0$ as the pole are in the following form

$$R(\eta) = \frac{c_{11}}{\eta - 1} + \frac{c_{21}}{\eta - \delta} + c_{10}, \tag{6}$$

where c_{11}, c_{21}, c_{10} are undetermined coefficients, and $\delta = e^{\gamma\xi}$, ξ is any complex number. Bring Eq.(6) into Eq.(5) and we get the expression

$$\sum_{i=0}^{6} \frac{A_i\eta^i}{3(\eta-1)^3(\eta-\delta)^3} = 0,$$

where

$$\begin{aligned}
A_0 =& 3b\delta^3 + 3c_{10}\delta^3k - 3c_{11}\delta^3k - 3c_{21}\delta^2k - kc_{10}^3\delta^3v + c_{11}^3\delta^3kv - 3c_{10}c_{11}^2\delta^3kv + 3c_{10}^2c_{11}\delta^3kv + 3c_{10}^2c_{21}\delta^2kv \\
&+ 3c_{11}^2c_{21}\delta^2kv - 6c_{10}c_{11}c_{21}\delta^2kv - 3c_{10}c_{21}^2\delta kv + 3c_{11}c_{21}^2\delta kv + c_{21}^3kv + 3c_{10}\delta^3l - 3c_{11}\delta^3l - 3c_{21}\delta^2l, \\
A_1 =& -9b\delta^3 - 9b\delta^2 - 3\gamma^2c_{11}\delta^3k^3 - 3\gamma^2c_{21}\delta k^3 - 9c_{10}\delta^3k + 6c_{11}\delta^3k - 9c_{10}\delta^2k + 9c_{11}\delta^2k + 9c_{21}\delta^2k + 6c_{21}\delta k \\
&+ 3c_{10}^3\delta^3kv + 3c_{10}c_{11}^2\delta^3kv - 6c_{10}^2c_{11}\delta^3kv + 3c_{10}^3\delta^2kv - 3c_{11}^3\delta^2kv + 9c_{10}c_{11}^2\delta^2kv - 9c_{10}^2c_{11}\delta^2kv \\
&- 9c_{10}^2c_{21}\delta^2kv - 3c_{11}^2c_{21}\delta^2kv + 12c_{10}c_{11}c_{21}\delta^2kv + 9c_{10}c_{21}^2\delta kv - 6c_{11}c_{21}^2\delta kv - 6c_{10}^2c_{21}\delta kv - 6c_{11}^2c_{21}\delta kv \\
&+ 12c_{10}c_{11}c_{21}\delta kv - 3c_{21}^3kv + 3c_{10}c_{21}^2kv - 3c_{11}c_{21}^2kv - 9c_{10}\delta^3l + 6c_{11}\delta^3l - 9c_{10}\delta^2l + 9c_{11}\delta^2l + 9c_{21}\delta^2l \\
&+ 6c_{21}\delta l, \\
A_2 =& 9b\delta^3 + 27b\delta^2 + 9b\delta - 3\gamma^2c_{11}\delta^3k^3 + 9\gamma^2c_{11}\delta^2k^3 + 9\gamma^2c_{21}\delta k^3 - 3\gamma^2c_{21}k^3 + 9c_{10}\delta^3k - 3c_{11}\delta^3k + 27c_{10}\delta^2k \\
&- 18c_{11}\delta^2k - 9c_{21}\delta^2k + 9c_{10}\delta k - 9c_{11}\delta k - 18c_{21}\delta k - 3c_{10}^3\delta^3kv + 3c_{10}^2c_{11}\delta^3kv - 9c_{10}^3\delta^2kv - 9c_{10}c_{11}^2\delta^2kv \\
&+ 18c_{10}^2c_{11}\delta^2kv + 9c_{10}^2c_{21}\delta^2kv - 6c_{10}c_{11}c_{21}\delta^2kv - 3c_{10}^3\delta kv + 3c_{11}^3\delta kv - 9c_{10}c_{11}^2\delta kv - 9c_{10}c_{21}^2\delta kv \\
&+ 3c_{11}c_{21}^2\delta kv + 9c_{10}^2c_{11}\delta kv + 18c_{10}^2c_{21}\delta kv + 6c_{11}^2c_{21}\delta kv - 24c_{10}c_{11}c_{21}\delta kv + 3c_{21}^3kv - 9c_{10}c_{21}^2kv \\
&+ 6c_{11}c_{21}^2kv + 3c_{10}^2c_{21}kv + 3c_{11}^2c_{21}kv - 6c_{10}c_{11}c_{21}kv - 3c_{21}k + 9c_{10}\delta^3l - 3c_{11}\delta^3l + 27c_{10}\delta^2l - 18c_{11}\delta^2l \\
&- 9c_{21}\delta^2l + 9c_{10}\delta l - 9c_{11}\delta l - 18c_{21}\delta l - 3c_{21}l, \\
A_3 =& -3b\delta^3 - 27b\delta^2 - 27b\delta - 3b + 9\gamma^2c_{11}\delta^2k^3 - 9\gamma^2c_{11}\delta k^3 - 9\gamma^2c_{21}\delta k^3 + 9\gamma^2c_{21}k^3 - 3c_{10}\delta^3k - 27c_{10}\delta^2k \\
&+ 9c_{11}\delta^2k + 3c_{21}\delta^2k - 27c_{10}\delta k + 18c_{11}\delta k + 18c_{21}\delta k + c_{10}^3\delta^3kv + 9c_{10}^3\delta^2kv - 9c_{10}^2c_{11}\delta^2kv - 3c_{10}^2c_{21}\delta^2kv \\
&+ 9c_{10}^3\delta kv + 9c_{10}c_{11}^2\delta kv + 3c_{10}c_{21}^2\delta kv - 18c_{10}^2c_{11}\delta kv - 18c_{10}^2c_{21}\delta kv + 12c_{10}c_{11}c_{21}\delta kv + c_{10}^3kv - c_{11}^3kv \\
&- c_{21}^3kv + 3c_{10}c_{11}^2kv + 9c_{10}c_{21}^2kv - 3c_{11}c_{21}^2kv - 3c_{10}^2c_{11}kv - 9c_{10}^2c_{21}kv - 3c_{11}^2c_{21}kv + 12c_{10}c_{11}c_{21}kv \\
&- 3c_{10}k + 3c_{11}k + 9c_{21}k - 3c_{10}\delta^3l - 27c_{10}\delta^2l + 9c_{11}\delta^2l + 3c_{21}\delta^2l - 27c_{10}\delta l + 18c_{11}\delta l + 18c_{21}\delta l \\
&- 3c_{10}l + 3c_{11}l + 9c_{21}l, \\
A_4 =& 9b\delta^2 + 27b\delta + 9b - 9\gamma^2c_{11}\delta k^3 + 3\gamma^2c_{21}\delta k^3 + 3\gamma^2c_{11}k^3 - 9\gamma^2c_{21}k^3 + 9c_{10}\delta^2k + 27c_{10}\delta k - 9c_{11}\delta k \\
&- 6c_{21}\delta k - 3c_{10}^3\delta^2kv - 9c_{10}^3\delta kv + 9c_{10}^2c_{11}\delta kv + 6c_{10}^2c_{21}\delta kv - 3c_{10}^3kv - 3c_{10}c_{11}^2kv - 3c_{10}c_{21}^2kv \\
&+ 6c_{10}^2c_{11}kv + 9c_{10}^2c_{21}kv - 6c_{10}c_{11}c_{21}kv + 9c_{10}k - 6c_{11}k - 9c_{21}k + 9c_{10}\delta^2l + 27c_{10}\delta l - 9c_{11}\delta l - 6c_{21}\delta l \\
&+ 9c_{10}l - 6c_{11}l - 9c_{21}l, \\
A_5 =& -9b\delta - 9b + 3\gamma^2c_{11}k^3 + 3\gamma^2c_{21}k^3 - 9c_{10}\delta k + 3c_{10}^3\delta kv + 3c_{10}^3kv - 3c_{10}^2c_{11}kv - 3c_{10}^2c_{21}kv - 9c_{10}k \\
&+ 3c_{11}k + 3c_{21}k - 9c_{10}\delta l - 9c_{10}l + 3c_{11}l + 3c_{21}l,
\end{aligned}$$

$$A_6 = 3b - kc_{10}^3 v + 3c_{10}k + 3c_{10}l.$$

Calculate the equations of $A_i = 0$ and get $c_{11} = \pm\frac{\sqrt{6}k\gamma}{\sqrt{v}}, c_{21} = 0, c_{10} = \pm\sqrt{\frac{3}{2v}}k\gamma, l = \frac{k(-2+k^2\gamma^2)}{2}, b = 0$ and $c_{11} = \pm\frac{\sqrt{6}k\gamma(\delta-1)}{\sqrt{v(\delta-1)^2}}, c_{21} = \mp\frac{\sqrt{6}k\gamma\delta(\delta-1)}{\sqrt{v(\delta-1)^2}}, c_{10} = \mp\frac{\sqrt{3}k\gamma(\delta+1)}{\sqrt{2v(\delta-1)^2}}, l = -k + \frac{k^3\gamma^2(1+\delta(10+\delta))}{2(\delta-1)^2}, b = \pm\frac{2\sqrt{6}k^4 v\gamma^3\delta(1+\delta)}{(v(-1+\delta)^2)^{\frac{3}{2}}}$.

So, all the simple periodic solutions of Eq.(2) are

$$W_3(z) = \pm\frac{\frac{\sqrt{6}k\gamma}{\sqrt{v}}}{e^{\gamma(z-z_0)} - 1} \pm \sqrt{\frac{3}{2v}}k\gamma, \ \ z_0 \in \mathbb{C},$$

and

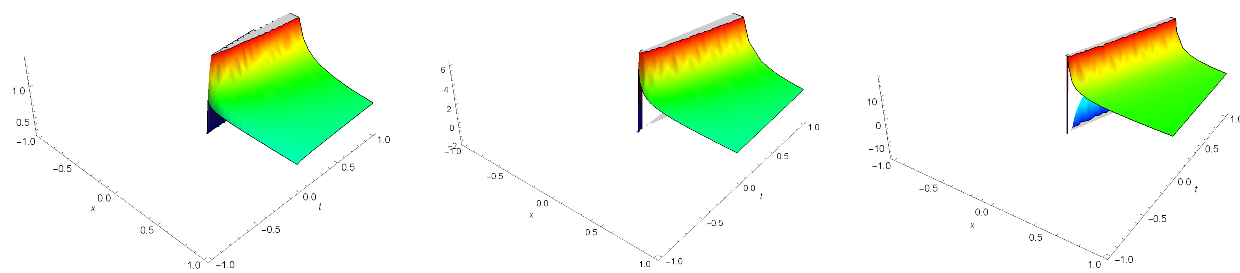
$$W_4(z) = \pm\frac{\frac{\sqrt{6}k\gamma(\delta-1)}{\sqrt{v(\delta-1)^2}}}{e^{\gamma(z-z_0)} - 1} \mp \frac{\frac{\sqrt{6}k\delta\gamma(\delta-1)}{\sqrt{v(\delta-1)^2}}}{e^{\gamma(z-z_0)} - e^{\gamma\xi}} \mp \frac{\sqrt{3}k\gamma(\delta+1)}{\sqrt{2v(\delta-1)^2}}, \ \ z_0 \in \mathbb{C}.$$

Now, if you bring in the transformation about z, you will get

$$u_3(x,t) = \pm\frac{\frac{\sqrt{6}k\gamma}{\sqrt{v}}}{e^{\gamma(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha} - z_0)} - 1} \pm \sqrt{\frac{3}{2v}}k\gamma, \ \ z_0 \in \mathbb{C},$$

and

$$\begin{aligned} u_4(x,t) = &\pm \frac{\frac{\sqrt{6}k\gamma(\delta-1)}{\sqrt{v(\delta-1)^2}}}{e^{\gamma(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha} - z_0)} - 1} \mp \frac{\frac{\sqrt{6}k\delta\gamma(\delta-1)}{\sqrt{v(\delta-1)^2}}}{e^{\gamma(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha} - z_0)} - e^{\gamma\xi}} \\ &\mp \frac{\sqrt{3}k\gamma(\delta+1)}{\sqrt{2v(\delta-1)^2}}, \ \ z_0 \in \mathbb{C}. \end{aligned}$$



(a) $\alpha = 0.2$ (b) $\alpha = 0.5$ (c) $\alpha = 0.8$

Figure 2: $u_{3,1}(x,t)$ with $k = 1, l = -\frac{1}{2}, v = 4, \gamma = 1, \beta = 0.01, z_0 = 0$.

The dynamic properties of the simple periodic solution $u_{3,1}(x,t) = \frac{\frac{\sqrt{6}k\gamma}{\sqrt{v}}}{e^{\gamma(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha} - z_0)} - 1} + \sqrt{\frac{3}{2v}}k\gamma$ is shown in Fig.2.

Both Fig.1 and Fig.2 accurately represent the properties of meromorphic functions in complex space. It can be clearly seen that there is a line in the figure, and the values on both sides tend to ∞. This is the line composed of the poles of meromorphic functions on a two-dimensional complex plane.

3.3. Elliptic function solutions

The form of elliptic function solution is $W(z)=\frac{c_{-11}}{2}\frac{\wp'(z)+B_i}{\wp(z)-A_i}+c_0$, where $B_i^2=4A_i^3-g_2A_i-g_3$, A_i and g_2 are arbitrary constants. We deduce that $c_0=0, A_i=B_i=0$, and then combine it with rantional solutions, we can get $c_{-11}=\pm\sqrt{\frac{6k^2}{v}}$. Naturally, all elliptic function solutions are expressed as

$$W_5(z)=\pm\frac{\sqrt{6k^2}}{2\sqrt{v}}\frac{\wp'(z-z_0,g_2,0)}{\wp(z-z_0,g_2,0)},\ \ z_0\in\mathbb{C},$$

and

$$u_5(x,t)=\pm\frac{\sqrt{6k^2}}{2\sqrt{v}}\frac{\wp'(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha}+\frac{lt^\alpha\Gamma(\beta+1)}{\alpha}-z_0,g_2,0)}{\wp(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha}+\frac{lt^\alpha\Gamma(\beta+1)}{\alpha}-z_0,g_2,0)},\ \ z_0\in\mathbb{C}.$$

Remark 2. *$\wp$ has the following definition.*

Let ω_1,ω_2 be two given complex numbers, such that $Im\frac{\omega_1}{\omega_2}>0, L=L[2\omega_1,2\omega_2]$ be discrete subset $L[2\omega_1,2\omega_2]=\{\omega\mid\omega=2n\omega_1+2m\omega_2,n,m\in\mathbb{Z}\}$, which is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. The discriminant $\Delta=\Delta(c_1,c_2):=c_1^3-27c_2^2$ and

$$s_n=s_n(L):=\sum_{\omega\in L\backslash\{0\}}\frac{1}{\omega^n}.$$

The Weierstrass elliptic function $\wp(z):=\wp(z,g_2,g_3)$ is a meromorphic function with double periods $2\omega_1,2\omega_2$ and satisfying:

$$[\wp'(z)]^2=4\wp(z)^3-g_2\wp(z)-g_3,$$

where $g_2=60s_4, g_3=140s_6$, and $\Delta(g_2,g_3)\neq 0$.

4. Generalized trigonometric solutions with extended direct algebraic method

In this section, we simplify the integral constant b to zero. The solutions of nonlinear ordinary differential equations obtained by extended direct algebraic method [20] can be expressed as $W(z)=\sum_{j=0}^N b_jQ^j(z), b_N\neq 0$, where $Q'(z)=ln(A)(\lambda+\rho Q(z)+\tau Q^2(z)), A\neq 0,1$.

For Eq.(2), if the coefficients of the highest order term is balanced, $j=1$ can be known. So we suppose that $W(z)=b_0+b_1Q(z)$.

Bring $W(z)=b_0+b_1Q(z)$ into Eq.(2), and we obtain $\sum_{i=0}^3 A_iQ(z)^i=0$, where

$$A_0=b_1\lambda k^3\rho\ln^2(A)-\frac{1}{3}b_0^3kv+b_0k+b_0l,$$

$$A_1=2b_1\lambda k^3\tau\ln^2(A)+b_1k^3\rho^2\ln^2(A)-b_0^2b_1kv+b_1k+b_1l,$$

$$A_2=3b_1k^3\rho\tau\ln^2(A)-b_0b_1^2kv,$$

$$A_3=2b_1k^3\tau^2\ln^2(A)-\frac{1}{3}b_1^3kv.$$

Let $A_i=0$ to obtain $b_0=\pm\frac{\sqrt{\frac{3}{2}}k\rho\ln(A)}{\sqrt{v}}, b_1=\pm\frac{\sqrt{6}k\tau\ln(A)}{\sqrt{v}}$, and $l=\frac{1}{2}k\left(k^2\ln^2(A)\left(\rho^2-4\lambda\tau\right)-2\right)$. Combining with the definition of $Q'(z)$, some representative solutions of Eq.(2) and Eq.(1) are obtained:

(1) $\rho^2-4\lambda\tau<0,\tau\neq 0$

$$W_6(z)=b_0+b_1(-\frac{\rho}{2\tau}+\frac{\sqrt{4\lambda\tau-\rho^2}}{2\tau}tan_A(\frac{\sqrt{4\lambda\tau-\rho^2}}{2}z)),$$

$$u_6(x,t) = b_1(-\frac{\rho}{2\tau} + \frac{\sqrt{4\lambda\tau-\rho^2}}{2\tau} tan_A(\frac{\sqrt{4\lambda\tau-\rho^2}}{2}(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha}))) + b_0,$$

(2) $\rho^2 - 4\lambda\tau > 0, \tau \neq 0$

$$W_7(z) = b_0 + b_1(-\frac{\rho}{2\tau} - \frac{\sqrt{\rho^2-4\lambda\tau}}{2\tau} tanh_A(\frac{\sqrt{\rho^2-4\lambda\tau}}{2} z)),$$

$$u_7(x,t) = b_1(-\frac{\rho}{2\tau} - \frac{\sqrt{\rho^2-4\lambda\tau}}{2\tau} tanh_A(\frac{\sqrt{\rho^2-4\lambda\tau}}{2}(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha}))) + b_0,$$

(3) $\lambda\tau > 0, \rho = 0$

$$W_8(z) = b_1(\sqrt{\frac{\lambda}{\tau}} tan_A(\sqrt{\lambda\tau})z),$$

$$u_8(x,t) = b_1(\sqrt{\frac{\lambda}{\tau}} tan_A(\sqrt{\lambda\tau})(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha})),$$

(4) $\lambda\tau < 0, \rho = 0$

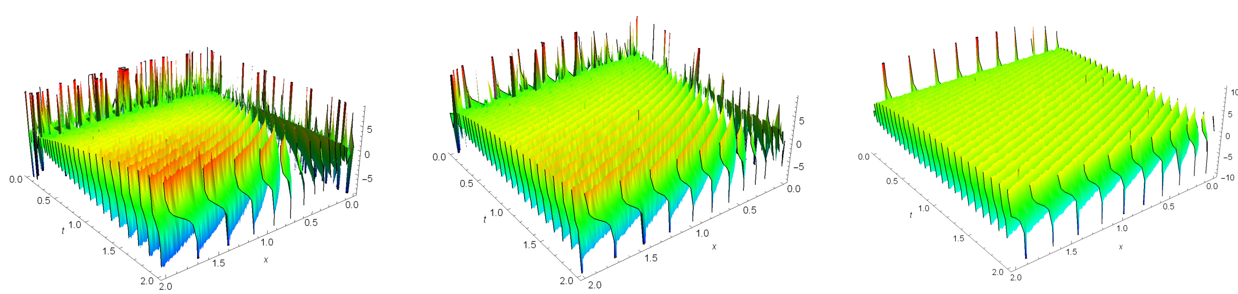
$$W_9(z) = b_1(-\sqrt{\frac{-\lambda}{\tau}} tanh_A(\sqrt{-\lambda\tau})z),$$

$$u_9(x,t) = b_1(-\sqrt{\frac{-\lambda}{\tau}} tanh_A(\sqrt{-\lambda\tau})(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha})).$$

Remark 3. *tan_A and $tanh_A$ represent generalized triangular and hyperbolic functions [21]:*

$$tan_A(z) = -i\frac{pA^{iz} - qA^{-iz}}{pA^{iz} + qA^{-iz}}, \quad tanh_A(z) = \frac{pA^{mz} - qA^{-mz}}{pA^{mz} + qA^{-mz}}.$$

If $p = q, m = 1, A = e$, $tan_A(z)$ and $tanh_A(z)$ degenerate into the general function $tan(z)$ and $tanh(z)$.



(a) $\alpha = 0.2$ (b) $\alpha = 0.5$ (c) $\alpha = 0.8$

Figure 3: $u_{6,1}(x,t)$ with different α.

The properties of generalized functions, such as

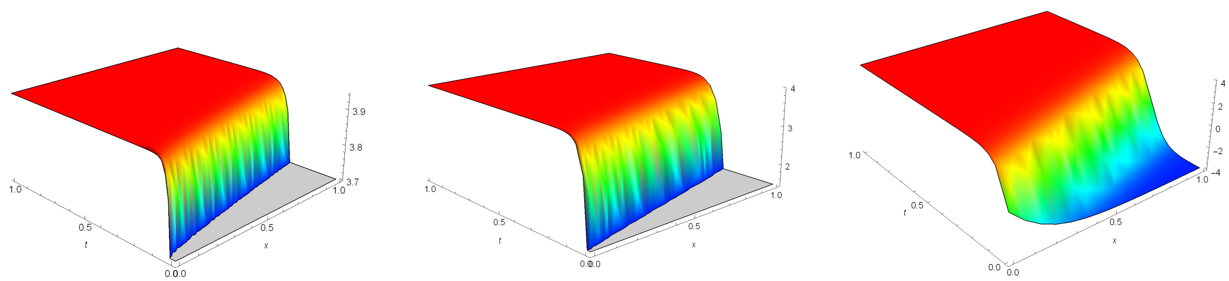
$$u_{6,1}(x,t) = \frac{\sqrt{6}k\tau\ln(A)}{\sqrt{v}}(-\frac{\rho}{2\tau} + \frac{\sqrt{4\lambda\tau-\rho^2}}{2\tau} tan_A(\frac{\sqrt{4\lambda\tau-\rho^2}}{2}(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha} + \frac{lt^\alpha\Gamma(\beta+1)}{\alpha}))) + \frac{\sqrt{\frac{3}{2}}k\rho\ln(A)}{\sqrt{v}} = -\frac{3\ln(2)\tan\left(\frac{6\sqrt{3}\ln(2)(5t^\alpha - 2x^\alpha)}{\alpha}\right)}{\sqrt{2}},$$

where $p=q, A=2, k=1, l=-\frac{5}{2}, v=1, \rho=1, \tau=1, \lambda=1, \beta=4$, and

$$\begin{aligned} u_{9,1}(x,t) &= \frac{\sqrt{6}k\tau\ln(A)}{\sqrt{v}}(-\sqrt{\frac{-\lambda}{\tau}}tanh_A(\sqrt{-\lambda\tau})(\frac{kx^\alpha\Gamma(\beta+1)}{\alpha}+\frac{lt^\alpha\Gamma(\beta+1)}{\alpha})) \\ &= \sqrt{6}\ln(5)\tanh\left(\frac{9.65106t^\alpha-3.21702x^\alpha}{\alpha}\right), \end{aligned}$$

where $p=q, A=5, m=2, k=1, l=-3, v=1, \tau=1, \lambda=-1, \beta=0.001$, are shown in Fig.3 and Fig.4.

By observing Fig.3 and Fig.4, we can see that when $p=q$, no matter how A and m change, the generalized triangular and hyperbolic function can still be sorted into a general form, and the images are similar to the three-dimensional graph of ordinary trigonometric functions.



(a) $\alpha=0.6$ (b) $\alpha=0.75$ (c) $\alpha=0.9$

Figure 4: $u_{9,1}(x,t)$ with different α.

5. Numerical solutions with ODM

In this section, we consider searching the approximate numerical solutions of Eq.(2). In addition, the accuracy of the numerical solutions are determined by comparing them with the exact solutions. Using ODM [22], the linear approximation function $F(L[W],W)$ of Eq.(2) can be written as

$$\begin{aligned} F(L[W],W) &= \frac{d^2}{dz^2}W(z)+\frac{(k+l)}{k^3}W(z)-\frac{v}{3k^2}W(z)^3 \\ &\approx \frac{d^2}{dz^2}W(z)+[\frac{(k+l)}{k^3}-\frac{v}{k^2}W^2(0)]W(z). \end{aligned}$$

Here, $L=\frac{d^2}{dz^2}$ is a linear differential operator, and $L[W(z)]=N[W(z)]+\varphi(z)$. $N[W(z)]$ stands for the nonlinear terms in Eq.(2) and $\varphi(z)$ is a given function. From the above formula we can also derive an important constant $C_0=\frac{(k+l)}{k^3}-\frac{v}{k^2}W^2(0)$.

Suppose the solution of Eq.(2) is an infinite series $W(z)=\sum_{k=0}^{\infty}y_k(z)$. Next, our main task is to calculate $y_k(z)$. By ODM, we can do the following calculation:

$$\begin{cases} y_0(z)=f_0(z) \\ y_1(z)=f_1(z)+L^{-1}(Q_0(z)) \\ y_2(z)=L^{-1}(Q_1(z)+C_0y_1(z)) \\ y_{k+1}(z)=L^{-1}(Q_k(z)+C_0(y_k(z)-y_{k-1}(z))), k\geq 2, \end{cases}$$

L^{-1} is the integral operator, $L^{-1}=\int_0^z\int_0^z(\cdot)dzdz$. Use inverse operator L^{-1} for the formula

$$L[W(z)]=N[W(z)]+\varphi(z),$$

we have

$$W(z) = L^{-1}([N[W(z)]) + f(z),$$

where $f(z) = L^{-1}(\varphi(z)) + W(0) + W'(0)z$. Denote $f_0(z) = W(0), f_1(z) = W^{'}(0)z$, and

$$Q_k(z) = \frac{1}{k!}\frac{d^k}{d\theta^k}[N(\sum_{i=0}^{k}\theta^i y_i(z))]\big|_{\theta=0}.$$

So for Eq.(2), we can figure out

$$\begin{cases} Q_0(z) = -\dfrac{(k+l)}{k^3}y_0(z) + \dfrac{v}{3k^2}y_0^3(z) \\ Q_1(z) = -\dfrac{(k+l)}{k^3}y_1(z) + \dfrac{v}{k^2}y_0^2(z)y_1(z) \\ Q_2(z) = -\dfrac{(k+l)}{k^3}y_2(z) + \dfrac{v}{k^2}(y_0(z)y_1(z)^2 + y_0(z)^2 y_2(z)) \\ \cdots\cdots \end{cases}$$

5.1. Obtaining initial values from a meromorphic solution

For example, for the solution in $W_3(z)$, we choose $W_{3,1}(z) = \frac{\frac{\sqrt{6}k\gamma}{\sqrt{v}}}{e^{\gamma(z-z_0)}-1} + \sqrt{\frac{3}{2v}}k\gamma$, where $l = \frac{k(-2+k^2\gamma^2)}{2}, b = 0$.

Suppose $W(z) = \sum_{k=0}^{\infty} y_k(z)$. Let's assign several values, $z_0 = 1, k = 2, \gamma = 1, v = 4, l = 2$.

$$\begin{cases} y_0(z) = f_0(z) = W_{3,1}(0) = \dfrac{\sqrt{\frac{3}{2}}(1+e)}{1-e} \\ y_1(z) = f_1(z) + \displaystyle\int_0^z\int_0^z Q_0(z)dzdz = W'_{3,1}(0)z + \int_0^z\int_0^z Q_0(z)dzdz \\ \qquad = -\dfrac{\sqrt{\frac{3}{2}}ez(z+e(z+2)-2)}{(e-1)^3} \\ y_2(z) = \displaystyle\int_0^z\int_0^z (Q_1(z) + C_0 y_1(z))dzdz = 0 \\ y_3(z) = \displaystyle\int_0^z\int_0^z [Q_2(z) + C_0(y_2(z) - y_1(z))]dzdz \\ \qquad = \dfrac{ez^3\left(-3e(1+e)^3z^3 - 18(e-1)e(1+e)^2z^2\right)}{20\sqrt{6}(e-1)^7} \\ \qquad - ez^3\dfrac{5(e-1)^2(1+e)(1+e(10+e))z - 20(e-1)^3(1+e(4+e))}{20\sqrt{6}(e-1)^7} \\ \cdots\cdots \end{cases}$$

Sort out the results from the above,

$$\begin{aligned} W_{10}(z) &= y_0(z) + y_1(z) + y_2(z) + y_3(z) + \cdots\cdots \\ &= \frac{\sqrt{\frac{3}{2}}(1+e)}{1-e} - \frac{\sqrt{6}ez}{(e-1)^2} - \frac{\sqrt{\frac{3}{2}}e(1+e)z^2}{(e-1)^3} - \frac{e(1+e(4+e))z^3}{\sqrt{6}(e-1)^4} - \frac{e(1+e)(1+e(10+e))z^4}{4\sqrt{6}(e-1)^5} + \cdots\cdots \end{aligned}$$

5.2. Obtaining initial values from a generalized trigonometric solution

In order to get the numerical solution, we choose

$$W_{6,1}(z) = \frac{\sqrt{\frac{3}{2}}k\rho}{\sqrt{v}} + \frac{\sqrt{6}k\tau}{\sqrt{v}}(-\frac{\rho}{2\tau} + \frac{\sqrt{4\lambda\tau - \rho^2}}{2\tau}tan(\frac{\sqrt{4\lambda\tau - \rho^2}}{2}z)),$$

where $p = q, A = e$ to provide the initial values.

Similarly, by ODM, let $W(z) = \sum_{k=0}^{\infty} y_k(z)$ and assign several values, $\rho = 1, \lambda = 1, \tau = 1, k = 2, l = -14, v = 1$.

$$\begin{cases} y_0(z) = f_0(z) = W_{6,1}(0) = 0 \\ y_1(z) = f_1(z) + \int_0^z \int_0^z Q_0(z)dzdz = W'_{6,1}(0)z = 3\sqrt{\frac{3}{2}}z \\ y_2(z) = \int_0^z \int_0^z (Q_1(z) + C_0 y_1(z))dzdz = 0 \\ y_3(z) = \int_0^z \int_0^z [Q_2(z) + C_0(y_2(z) - y_1(z))]dzdz = \frac{3}{4}\sqrt{\frac{3}{2}}z^3 \\ y_4(z) = \int_0^z \int_0^z [Q_3(z) + C_0(y_3(z) - y_2(z))]dzdz = \frac{27}{160}\sqrt{\frac{3}{2}}z^5 \\ y_5(z) = \int_0^z \int_0^z [Q_4(z) + C_0(y_4(z) - y_3(z))]dzdz = \frac{9}{160}\sqrt{\frac{3}{2}}z^5 \\ \cdots\cdots \end{cases}$$

Sort out the results from the above,

$$\begin{aligned} W_{11}(z) &= y_0(z) + y_1(z) + y_2(z) + y_3(z) + \cdots\cdots \\ &= 3\sqrt{\frac{3}{2}}z + \frac{3}{4}\sqrt{\frac{3}{2}}z^3 + \frac{27}{160}\sqrt{\frac{3}{2}}z^5 + \frac{9}{160}\sqrt{\frac{3}{2}}z^5 + \cdots\cdots \end{aligned}$$

5.3. Accuracy verification of numerical solutions

We use the first seven terms of the series to make the images of the numerical solutions. As can be seen from Fig.5, there is little difference between the numerical solutions and the exact solutions in a certain interval. In order to understand the specific errors between them, Table 1 and Table 2 are listed. From these data, we can conclude that the accuracy of the numerical solutions obtained in Section 5 can be guaranteed. And with the increase of series terms, the error will be smaller and smaller.

6. Conclusions

With the help of the properties of truncated M-fractional derivative, a new transformation is proposed, which is used to change the mBBM equation into ODE. Some meromorphic solutions and generalized trigonometric solutions of this equation are calculated by reliable methods. We use ODM to give the numerical solutions, by comparing them with the exact solutions, we come to the conclusion that the errors of the numerical solutions are acceptable. This also shows the superiority of the new decomposition method proposed by Zaid odibat in 2019. This is an interesting work, which may attract more scholars' attention.

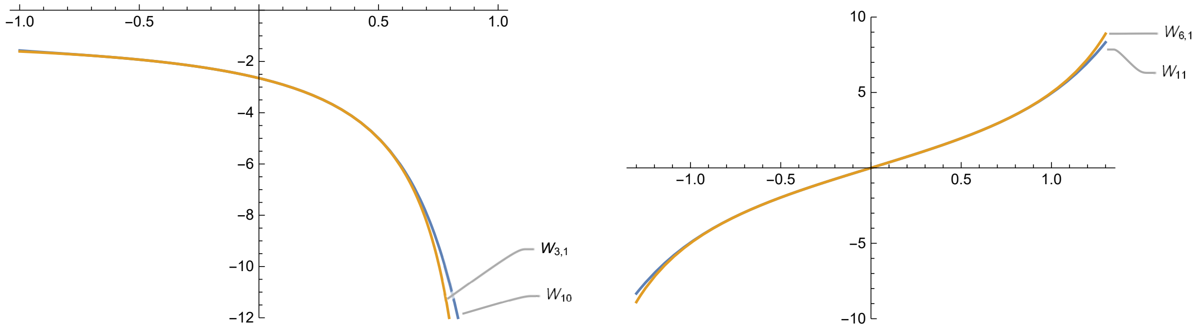
Acknowledgments

z	$W_{10}(z)$	$W_{3,1}(z)$	Error
-0.8	-1.70308	-1.70983	0.00674757
-0.6	-1.84376	-1.84439	0.000629789
-0.4	-2.02648	-2.02649	4.80765×10^{-6}
-0.2	-2.28051	-2.28051	-4.78092×10^{-7}
0.2	-3.22344	-3.22345	4.1188×10^{-6}
0.4	-4.20291	-4.20423	0.00131685
0.6	-6.14514	-6.20516	0.0600206
0.8	-10.7454	-12.2882	1.54286

Table 1: Values and errors (I)

z	$W_{11}(z)$	$W_{6,1}(z)$	Error
-0.9	-4.178	-4.19224	0.014244
-0.7	-2.94003	-2.94144	0.00140814
-0.5	-1.96118	-1.96126	0.0000729628
-0.3	-1.12776	-1.12776	1.0526×10^{6}
-0.1	-0.368345	-0.368345	2.71545×10^{-10}
0.1	0.368345	0.368345	-2.71545×10^{-10}
0.3	1.12776	1.12776	-1.0526×10^{6}
0.5	1.96118	1.96126	-0.0000729628
0.7	2.94003	2.94144	-0.00140814
0.9	4.178	4.19224	-0.014244

Table 2: Values and errors (II)



(a) $W_{3,1}(z)$ and $W_{10}(z)$

(b) $W_{6,1}(z)$ and $W_{11}(z)$

Figure 5: Comparison diagram of exact solutions and numerical solutions

We would like to thank the editor and reviewers for their helpful suggestions. This work was supported by National Natural Science Foundation of China [grant numbers 12071047, 12171050]; the Fundamental Research Funds for the Central Universities [grant number 500421126].

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