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On the Use of Green's Functions in Solving Boundary Value Problems

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Abstract

There is no doubt that Green's functions have a long history in their use in many fields of applied mathematics and physics and especially in solving boundary value problems, hence we thought that it is worthwhile to write this article to summarize some important results in this concern emphasizing the beauty behind using them and the elegant mathematical techniques used as tools in conjunction with them. Famous problems related to wave propagation and potential theory will be tackled in some details, giving the solutions of the partial differential equation which are connected with the problem. There remains also to mention that Green's functions are used in many other applications as will be pointed out in the conclusions.

Keywords:

Green's Functions, Boundary Value Problems, Discrete, Domain.

1. Introduction

One of the early boundary value problem was the vibrating string studied by Euler, Bernoulli, and D'Alembert around the year 1750 A.D. using separation of variables. In the nineteen century around the year 1828 A.D. ,Green used certain functions to solve the wave equation where he obtained important formulae in his solution named after him [1].Green's function proved to be a powerful tool to get solutions of partial differential equations describing problems in applied fields especially in physics and when dealing with somewhat complicated problems [2] .

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Accordingly, we believe that a review article is in order where we revise some of old problems and discuss recent ones. In the next section, some important concepts related to Green's functions are given; the properties of these functions with green's theorem are given in section 3.In the section to follow Green's function in different dimensions are exposed to .In section 5 discrete Green's function is presented; in section 6 the solutions of partial differential equations encountered in physics for some elementary problems are given, moreover and in the same section more complicated problems were tackled and which showed that Green's function was so efficient to help in solving Schrodinger, Poisson differential equations and inverse problems [3].Discrete Green's function and their use to solve difference equations are discussed in section 7 [4].Finally we give concluding remarksregarding the mean developments in the subject.

2.Basic Concepts

Definition 1

Assume that we have the boundary value problem

$$L(y) \equiv p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \tag{1}$$

with the boundary conditions

$$V_k(y) = \alpha_k y(a) + \alpha_k^{(1)} y^{(1)}(a) + \cdots + \alpha_k^{(n-1)} y^{(n-1)}(a) + \cdots \beta_k y(b) + \beta_k^{(1)} y^{(1)}(b) + \cdots + \beta_k^{(n-1)} y^{(n-1)}(b) \ ,k = 1,2,..n \tag{2}$$

Where the $p's\ are\ functions\ of\ x, \alpha's\ and\ \beta'sare\ scalars$. Moreover the$V'$sare linearly independent in$y(a)\ and\ y(b)$ and the various derivatives of y at $a\ and\ b$,then Green's function for the fore-mentioned problem given by Equation(1) and Equation(2) is a function $G(x,\rho)$ such that $a < \rho < b$ and satisfying the following four conditions:

(i)$G(x,\rho)$ is continuous at the point x and has continuous derivatives up to order $n-2$ for all x such that $a \le x \le b$.

(ii)Its derivative of order $n-1$ is discontinuous at the point $x = \rho$ such that

$$\frac{\partial^{(n-1)}G(x,\rho)}{\partial x^{n-1}}\Big|_{x=\rho+0} - \frac{\partial^{(n-1)}G(x,\rho)}{\partial x^{n-1}}\Big|_{x=\rho-0} = \frac{1}{p_0(x)} \tag{3}$$

(iii)In the interval $[a,\rho) \cup (\rho,b]$, $G(x,\rho)$ is a solution of Equation(1),i.e. $L(G) = 0$.

(iv) $G(x,\rho)$ satisfies the boundary conditions $V_k(G) = 0\ ,k = 1,2,3,\dots n$.

where V_k are given by Equation(2).

Note that$G(x,\rho)$ has the following properties

a-If $G(x,\rho)$ satisfies the boundary value problem(BVP) given by Equation(1) and Equation(2) then the only solution to the problem and only one is Green's function $G(x,\rho)$.

b-If $G(x,\rho)$ is the solution to the above mentioned BVP ,then $Y(x)$ can be written as

$$Y(x) = \int_a^b G(x,\rho)f(\rho)d\rho \tag{4}$$

c-If we consider a string fixed along the x-axis with the end points fixed at $x = 0, x = a$ and is under a certain tension Tcaused by a weight (or a density $f(x)$)causing a deviation $g(x)$ which satisfies the equation

$$-T\frac{d^2g}{dx^2} = f(x) \qquad ,0 < x < a \tag{5}$$

with the boundary conditions $g(0) = 0$,$g(a) = 0$;then the solution of this BVP is Green's function which vanishes at the end points [1] [2] .

d-The above BVP for the stretched string can also be written as

$$-T\frac{d^2g}{dx^2} = \delta(x - x_0) \ ; g(x,0) = 0 = g(x,a) \ and \ x_0 \in (0,a) \tag{6}$$

where $\delta(x - x_0)$ is Dirac delta function. Moreover, $g(x,x_0)$ is Green's function and is the solution of Equation(6).

Example 1

In this example we study the deviation of the string from equilibrium $f(x)$ and with homogenous boundary conditions at these ends.The Differential equation is then given by $\emptyset'' = -\frac{f(x)}{T}$;hence Green's function(GF) obeys the equation $G''(x,x_0) = -\delta(x - x_0) = -\frac{T\delta(x-x_0)}{T}$,where GF represents the deviation at x_0 due to the tension at x.From equilibrium we get $T = Tsin\alpha + Tsin\beta$,where α and β are the angles the string makes with the x-axis at the ends points $x = 0 \ and \ x = a$ respectively. Using simple geometric processes one will reach a formula for the related Green's function as $G(x;x_0) = \begin{cases} \frac{(a-x)x_0}{a} & ,0 \le x_0 < x \\ \frac{x(a-x_0)}{a} & ,x < x_0 \le a \end{cases}$ [1].

3.Properties

Definition 2

For any real number ρ and any differential operator L of the order n , the solution of the differential equation

$$Lt(x) = \delta(x - \rho) \tag{7}$$

Is called a principal solution for the operator L with a pole ρ [3].

The principal solutions are actually weak solutions and the solution for the above equation consists of general solution(GS) of the homogenous equation plus the actual particular solution(PS). The GS can obtained by solving the homogenous equation

$$Lt(x) = 0\,; x = \rho \;\; is\, the\, point\, of\, discontinuity \qquad (8)$$

and is given by

$$t = \{\begin{matrix} u(x)\,, x < \rho \\ v(x)\,, x > \rho \end{matrix} \qquad (9)$$

with $u(x)$ and $v(x)$ with their corresponding derivatives equal to each other at the point $x = \rho$ [1].

Example 2

The solution of the equation $-t'' = \delta(x - \rho)$ is $t = -\frac{|x-\rho|}{2}$ and from the results of Example 1 one gets $G(x,\rho) = \{\begin{matrix} (1-\rho)x \;\,, x < \rho \\ (1-x)\rho\,, x > \rho \end{matrix}$ .Note that we put $T = a = 1\; and\; x_0 = \rho$and the Green's function one obtains is also a principal solution of the given differential equation. Moreover ,we can see that the full solution to the problem is $G(x,\rho) - t(x,\rho) = {}^{x}\!/_{2} + {}^{\rho}\!/_{2} - \rho x$ [1] .

3.1 The Modified Green's Function

Since the system

$$-\frac{d^2 g}{dx^2} = \delta(x - \rho)\,, g'(0) = 0 = g'(a) \qquad (10)$$

cannot be solved due to the fact that $\int_0^a \delta(x - \rho)dx \neq 0$ when $\rho \in (0, a)$. hence in order to get a solution one adds the constant $-\frac{1}{a}$ to get $\int_0^a \left[\delta(x-\rho) - \frac{1}{a}\right] dx = 0$ when $\rho \in (0, a)$ .Accordingly,one gets the modified Green's function(MGF) which is the solution of the system

$$-\frac{d^2 g_m(x,\rho)}{dx^2} = \delta(x - \rho) - \frac{1}{a} \;\;, \frac{dg_m}{dx}(0) = 0 = \frac{dg_m}{dx}(a) \qquad (11)$$

It also satisfies the equation

$$-\frac{d^2 g_m}{dx^2}(x,\rho) = -\frac{1}{a} \;\;, x \neq \rho \qquad (12)$$

Moreover the MGF can be used to solve the system$-\frac{d^2 y}{dx^2} = f(x)\,, y'(0) = 0 = y'(a)$ to get $y(x)$ as

$$y(x) = A + \int_0^a g_m(x,\rho) f(\rho) d\rho \qquad (13)$$

A is a constant.

3.2 Adjoint Green's Function

If Green's function $g(x,\rho)$ with the differential operator L of the second order with homogenous boundary conditions is a solution of the system

$$Lg = \delta(x-\rho)\,; a < x\,, \rho < b \tag{14}$$

with boundary conditions given by

$$B_1(g) = 0: \ \alpha_{11}g(a) + \alpha_{12}g'(a) + \beta_{11}g(b) + \beta_{12}g'(b) = 0$$

$$B_2(g) = 0: \ \alpha_{21}g(a) + \alpha_{22}g'(a) + \beta_{21}g(b) + \beta_{22}g'(b) = 0 \tag{15}$$

and where α_{ij}, β_{ij} , $i = 1,2; j = 1,2$are real numbers and the two row vectors$(\alpha_{11}, \alpha_{12}, \beta_{11}, \beta_{12})$and $(\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$are independent of each other, then the adjoint Green' function $h(x,\rho)$ is defined as the solution of the system[3]

$$L^{*}h = \delta(x-\rho)\,; a < x\,, \rho < b \tag{16}$$

with boundary conditions given by

$$B_1^{*}(h) = 0: \ \alpha_{11}^{*}h(a) + \alpha_{12}^{*}h'(a) + \beta_{11}^{*}h(b) + \beta_{12}^{*}h'(b) = 0$$

$$B_2^{*}(h) = 0: \ \alpha_{21}^{*}h(a) + \alpha_{22}^{*}h'(a) + \beta_{21}^{*}h(b) + \beta_{22}^{*}h'(b) = 0 \tag{17}$$

Now putting $\rho = \tau$ in Equation(16) one gets

$$\int_a^b [hLg - gL^{*}h]dx = \int_a^b h(x,\tau)\delta(x-\rho)dx - \int_a^b g(x,\rho)\delta(x-\tau)dx = h(\rho,\tau) - g(\tau,\rho)\,; a < \rho\,, \tau < b \tag{18}$$

Since the adjoint Green's function $h(x,\rho)$ satisfies the adjoint boundary conditions one obtains

$$h(x,\rho) = g(\rho,x) \ ; a < \rho \ , x < b \tag{19}$$

So that $g(x,\rho)$ is the solution of the system(14) ,this shows that

$$g(x,\rho) = g(\rho,x) \ ; a < \rho \ , x < b \tag{20}$$

which means that Green's function is a symmetric function in the two variables x and ρ .This result is an important one.

3.3 Green's Theorem

Let

$$\vec{A} = U\nabla W \tag{21}$$

Where $U \ and \ W$ are arbitrary functions in three dimensions,then

$$\nabla.\vec{A} = U\nabla^2 W + \nabla U.\nabla W \tag{22}$$

From Gauss Theorem we see that

$$\iiint_V \nabla.\vec{V}dV = \iint_S \vec{A}.\overrightarrow{dS} \qquad (23)$$

Hence

$$\iiint_V (U\nabla^2 W + \nabla U.\nabla W)dV = \iint_S U\nabla W dS \qquad (24)$$

Equation(24) leads to the so-called first form of Green's theorem; using this result one can obtain the second form of Green's theorem(or simply Green' Theorem) as

$$\iiint_V (U\nabla^2 W - W\nabla^2 U)dV = \iint_S (U\nabla W - W\nabla U).\overrightarrow{dS} \qquad (25)$$

V represents the volume surrounded by the surface S.

The two forms are very useful in electrodynamics and Fluid Mechanics [3] .

3.4 The Complex Form of Green's Function

If z is a complex variable ,then it is clear that one **can** get the relations between the various differential operators, when applied to a certain function $\emptyset$, as

$$D_x\emptyset = D_z\emptyset + D_{\bar{z}}\emptyset \; ; \; D_y\emptyset = j(D_z\emptyset - D_{\bar{z}}\emptyset) \qquad (26)$$

From Equation(26) one gets

$$(2D_{\bar{z}}\emptyset)(2D_z\emptyset) = \nabla^2\emptyset \qquad (27)$$

Hence, we get from Green's theorem in two dimensions

$$2j \oint_C (Pdx - Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \qquad (28)$$

and putting $Q = \emptyset - P = j\emptyset$,one obtains

$$2j \iint_S D_{\bar{z}}\emptyset dxdy = \oint_C \emptyset dz \qquad (29)$$

Equation(29) is known as " The complex form of Green's function".

4. Green's Function in Different Dimensions

Consider the following BVP, which is encountered in studying oscillations resulting from a stretched string with fixed end points [3] :

$$\frac{d^2\psi}{dx^2} + k^2 = -f(x)\,; 0 \le x \le a\,; \psi(0) = 0 = \psi(a) \qquad (30)$$

Assume that the solution is of the form

$$\psi(x) = A(x)sinkx + B(x)coskx \qquad (31)$$

Then on differentiating Equation(31) with respect to x once and twice respectively, and assuming that

$$A'(x)sinkx + B'(x)coskx = 0 \tag{32}$$

This will lead to

$$kA'(x)sinkx - kB'(x)coskx = -f(x) \tag{33}$$

With few manipulations we get the solution as

$$\psi(x) = \frac{-sinkx}{k}\int_{c_1}^{x} f(y)coskydy + \frac{coskx}{k}\int_{c_2}^{x} f(y)sinkydy \tag{34}$$

(c_1 and c_2 are constants to be determined).

Taking into account the boundary conditions,the solution in its final form is obtained as

$$\psi(x) = \int_0^a f(y)G(x,y)dy \tag{35}$$

where the Green's function $G(x,y)$ is given by

$$G(x,y) = \begin{cases} \frac{sinkysink\ (a-x)}{ksinka} & ;0 \le y \le x \\ \frac{sinkxsink\ (a-y)}{ksinka} & ;\ x \le y \le a \end{cases} \tag{36}$$

Note that the Green's function we got here is of dimension one and is independent of $f(x)$;namely, it depends on the type of the differential equation and the value of k and the boundary conditions. Moreover, the solutions for such problems with different $f(x)$ are possible if the integral $\int_0^a f(y)G(x,y)dy$ exists [3] .

Now ,assume that the problem under consideration , of two dimensions, is

$$\nabla^2\psi + \lambda\psi = -f(x,y) \tag{37}$$

with the boundary conditions $\psi(0) = 0$ or $\frac{d\psi}{dn} = 0$ or $\frac{d\psi}{dn} + \alpha\psi = 0$ on the curve C bounding the region of interest R .[$\frac{d\psi}{dn}$ is the normal derivative on the curve C]

Let us consider the following equation

$$\iint_R f(x,y)G(\rho,\tau,x,y)dxdy = \iint_R (\nabla^2\psi + \lambda\psi)Gdxdy \tag{38}$$

Where $G(\rho,\tau,x,y)$ is Green's function in two dimensions and is discontinuous at the points $x = \rho\ ;\ y = \tau$ which will be avoided from the region R through the determination of a small circle $(x-\rho)^2 + (y-\tau)^2 = r^2$ having a radius r and center at the point (ρ,τ).Hence we can write

$$\iint_R (\nabla^2\psi + \lambda\psi)Gdxdy = \lim_{r\to 0}\iint_{R'} (\nabla^2\psi + \lambda\psi)Gdxdy \tag{39}$$

Where R' is the region in between the two curves C and C' (which is the circle). Now [1] [3] ,

$$\iint_{R'}(\nabla^2\psi+\lambda\psi)Gdxdy=\iint_{R'}(\nabla^2G+\lambda G)\psi dxdy+\oint_C\left(G\frac{d\psi}{dn}-\psi\frac{dG}{dn}\right)ds-\oint_{C'}\left(G\frac{d\psi}{dr}-\psi\frac{dG}{dr}\right)ds \tag{40}$$

Note that $r=\sqrt{(x-\rho)^2+(y-\tau)^2}$; $ds=rd\theta$ and $\theta=tan^{-1}\left(\frac{y-\tau}{x-\rho}\right)$ for $0\leq\theta\leq 2\pi$.

Assuming that $\nabla^2G+\lambda G=0$ in the region R except at $x=\rho$ and $y=\tau$ and that Gsatisfies the boundary conditions on C ,then

$$\iint_R(\nabla^2G+\lambda G)\psi dxdy=0 \tag{41}$$

and

$$\oint_C\left(G\frac{d\psi}{dn}-\psi\frac{dG}{dn}\right)ds=0 \tag{42}$$

The integral in Equation(38) becomes

$$\iint_R f(x,y)G(\rho,\tau,x,y)dxdy=\lim_{r\to 0}\oint_0^{2\pi}\left(G\frac{d\psi}{dn}-\psi\frac{dG}{dn}\right)d\theta \tag{43}$$

Such that $\lim_{r\to 0}\oint_0^{2\pi}\left(G\frac{d\psi}{dn}-\psi\frac{dG}{dn}\right)d\theta\neq 0$ and $\lim_{r\to 0}rG=0$. This means that G behaves like $kLogr$ when $r\to 0$; accordingly, and since $Y_0(\sqrt{\lambda}r)$ satisfies Helmholtz equation and behaves as $\frac{2}{\pi}Logr$ as $r\to 0$, the solution is given by

$$G(\rho,\tau,x,y)=-\frac{1}{4}Y_0\left(\sqrt{\lambda}r\right)+H(\rho,\tau,x,y) \tag{44}$$

Where the function H with its first and second derivatives are continuous at (x,y) in R.

In three dimensions, once the point (ρ,τ,μ) is kept away by eliminating what is in a small ball with center at that point and radius R ,then

$$\psi(\rho,\tau,\mu)=\iiint_V f(x,y,z)G(\rho,\tau,\mu,x,y,z)dxdydz=-\iiint_V(\nabla^2\psi+\lambda\psi)Gdxdydz=$$

$$-\lim_{r\to 0}\iiint_{V'}(\nabla^2\psi+\lambda\psi)Gdxdydz \tag{45}$$

WhereV' is the resulting region inside the ball which is kept away. Hence

$$\psi(\rho,\tau,\mu)=$$

$$-\lim_{r\to 0}\iiint_{V'}(\nabla^2\psi+\lambda\psi)Gdxdydz+\iint_{S'}\left(\psi\frac{dG}{dn}-G\frac{d\psi}{dn}\right)dS+\lim_{r\to 0}R^2\int_0^{2\pi}\int_0^{2\pi}\left(G\frac{\partial\psi}{\partial r}-\right.$$

$$\psi\partial G\partial r)r=Rsin\theta d\theta d\varphi \tag{46}$$

Now we assume that $\lim_{r\to 0} r^2 G = 0$ and$\lim_{r\to 0} r^2 (\frac{\partial G}{\partial r})_{r=R} = 0$, therefore one can see that Green's function G behaves as kr^{-1} when $r \to 0$ and since$r^{-1} cos\sqrt{\lambda r}$ satisfies Helmholtz equation and it behaves as $\frac{1}{r}$ $as\ r \to 0$ then

$$G(\rho,\tau,\mu,x,y,z) = \frac{cos\sqrt{\lambda r}}{4\pi r} + H(\rho,\tau,\mu,x,y,z) \tag{47}$$

The function H with its first and second derivatives are continuous in the region V .This function is unique in this case as long as λ is not an eigenvalue for the homogenous problem.

At the end of this section we summarize the important properties of Green's function as follows:-

a-Green's function satisfies the homogeneous differential equation

$$\nabla^2 G + k^2 G = 0 \tag{48}$$

In the interval $[0, y) \cup (y, a]$.

b-G is continuous at $x = y$,i.e.

$$\lim_{x\to y-} G(x,y) = \frac{sinkxsink\ (a-x)}{ksinka} = \lim_{x\to y+} G(x,y) \tag{49}$$

c-The first derivative of GF is discontinuous at$x = y$ and

$$G'(x,x^-) = \frac{coskx\ \ sink\ (a-x)}{sinka} \tag{50}$$

$$G'(x,x^+) = \frac{-sinkx\ \ cosk\ (a-x)}{sinka} \tag{51}$$

Moreover

$$G'(x,x+) - G'(x,x-) = -1 \tag{52}$$

d-Green's function in two variables is symmetric in the two variables, i.e.

$$G(x,y) = G(y,x) \tag{53}$$

e-Green's function satisfies the boundary conditions $G(0,y) = 0 = G(a,y)$.

f-Green's function is used to solve inhomogeneous boundary value problems of the kind shown in Equation(30).

5. Discrete Green's Function

Finite difference methods are very important and effective in solving many boundary value problems; moreover, it helps in the improvement of error estimation .Green's function can also play a vital role in this concern especially in studying convergence criteria [4]

In addition to continuous problems we have seen in the last section Green's function methods in the discrete problems became very important and effective in the determination of the rate of convergence of finite difference solution to the exact one of the differential equation as the discretization parameter approaches zero [4].

For the Laplace difference operator, and from the existing results obtained via discrete Green's function method, the maximum error estimation was of the order $O(h^4), h\ is\ the\ step\ lenght$ [4].while a maximum error of order $O(h^6)$ for the case Poisson' equation on the square grid was obtained [4].

Due to the importance of this topic of discrete Green's function, a more exposition to the subject will be presented later in this paper.

6. Sample Applications

In this section some applications on Green's function in boundary value problems are presented. To start with we give the simple problem of a stretched string [1].

6.1 Stretched String

In this case the boundary value problem is expressed as

$$\frac{d^2y}{dx^2} + k^2 = -f(x) \quad ; y(0) = 0 = y(a) \tag{54}$$

For simplicity we exchange the function $f(x)$ with Dirac delta function $\delta(x - x')$ to get

$$\frac{d^2y}{dx^2} + k^2 = -\delta(x - x') \tag{55}$$

Using Laplace transform technique (and inverse Laplace transform) to solve Equation(55), one gets

$$y(x, x') = \begin{cases} \frac{sink\,x' sink\ (a-x)}{ksinka} & ; 0 \leq x' \leq x \\ \frac{sinkxsink\ \ (a-x')}{ksinka} & ;\ x \leq x' \leq a \end{cases} \tag{56}$$

Hence the solution of the given BVP is [5]

$$y(x) = \int_0^a f(x')y(x, x')dx' \tag{57}$$

6.2 Deviation of a Beam

Here the deviation of a fixed beam from equilibrium is studied and which is positive and given by $y(x)$ [1] [5].The equation for steady deviation is given by

$$\frac{dF(x)}{dx} = -\omega(x) \tag{58}$$

where $\omega(x)$ is the applied weight function in the direction of deviation, and for maximum deviation the equation for the deviation is

$$A\frac{d^4y}{dx^4} = \omega(x) \tag{59}$$

where $A$is a constant related to the elasticity coefficient [1].Again using Laplace transform ,one gets

$$y(x,x') = \begin{cases} \frac{-x^3+(x-x')^3}{6A} + \frac{x'x^2}{2A} \,; 0 \le x' \le x \\ -\frac{x^3}{6A} + \frac{x'x^2}{2A} \,; x \le x' \le a \end{cases} \tag{60}$$

and the solution is then given by

$$y(x) = \int_0^a \omega(x')y(x,x')dx' \tag{61}$$

Example 3

Let us consider the problem of determining the static deviation of the beam with a weight function $\omega(x) = \omega_0 u(x)\,; \omega_0 = constant$, then from Equation(60) and Equation(61) the deviation is given by $y(x) = \frac{\omega_0}{24A}x^2(6a^2 - 4ax + x^2)$ [1].

6.3 Potential Due to a Homogeneous Cylinder

In cylindrical coordinates assume one has the following transformations [6]

$$z_1 = z + h\,, z_2 = z - h\,, r_1 = \sqrt{z_1^2 + r^2}\,, r_2 = \sqrt{z_2^2 + r^2}\,, r_a = \sqrt{r^2 + a^2}\,, \rho_1 = \sqrt{z_1^2 + r_a^2}\,, \rho_2 = \sqrt{z_2^2 + r_a^2}\,, Z = z' - z\,, R_0 = \sqrt{Z^2 + r^2}\,, R_a = \sqrt{Z^2 + r_a^2}\,, R = \sqrt{Z^2 + r^2 + r'^2}, \quad H = -2rr'\cos(\varphi' - \varphi) \tag{62}$$

Where (r, φ, z) and (r', φ', z') are two points in cylindrical coordinates and $a, 2h$ are the radius and the length of the cylinder .Using Green's function method one can compute the potential V due to the presence of the homogeneous cylinder at a point on its axis, assuming having cylindrical symmetry ,as [1] [6]

$$V = \pi\left\{a^2 \ln\left[\frac{z_1+\sqrt{z_1^2+a^2}}{z_2+\sqrt{z_2^2+a^2}}\right] + z_1\sqrt{z_1^2 + a^2} - z_2\sqrt{z_2^2 + a^2} + B\right\} \tag{63}$$

where

$$B = \begin{cases} -4zh & , z > h \\ -2(z^2 + h^2) & , -h < z < h \\ 4zh & , z < -h \end{cases} \tag{64}$$

Note that the quantity $\frac{1}{|\vec{r}-\vec{r}'|}$ can be written as

$$\frac{1}{|\vec{r}-\vec{r}'|} = [(Z' - Z)^2 + r'^2 + r^2 - 2rr' \cos(\varphi' - \varphi)]^{-1/2} = (R^2 + H)^{-1/2} \quad (65)$$

Where $\vec{r}$ and $\vec{r}'$ are the position vectors of the points of interest. Using the last equation and making use of Bessel's functions one obtains

$$\frac{\mathbf{1}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^n \Gamma\left(n+\frac{1}{2}\right)}{n!} \cdot \frac{r^n r'^n}{R^{2n+1}} \cos^n(\varphi - \varphi') \quad (66)$$

Hence the potential of the cylinder is obtained as [6]

$$V = 2\sqrt{\pi} \sum_{m=0}^{\infty} \frac{r^{2m} \Gamma\left(2m+\frac{1}{2}\right)}{(m!)^2} \iint \frac{r'^{2m+1}}{R^{4m+1}} dr' dz' \quad (67)$$

Evaluating the integral in the last equation and taking into account the various transformations in Equation(62), one obtains the potential at the center of the cylinder as [6]

$$V(r=0) = \pi[a^2 \ln\left[\frac{(h^2+a^2)^{\frac{1}{2}}+h}{(h^2+a^2)^{\frac{1}{2}}-h}\right] + 2h(h^2+a^2)^{\frac{1}{2}} - 2h] \quad (68)$$

6.4 Green's Function for a Hollow Earthed Cube

Consider the partial differential equation

$$\nabla^2 \Psi(\vec{r}) + (f(\vec{r}) + \lambda)\Psi(\vec{r}) = 0 \quad (69)$$

Where $\Psi(\vec{r})$ is the solution which satisfies homogeneous Dirichlet boundary condition on the surface S of the cube with volume V [7]. $f(\vec{r})$ Is the source function. The solution can be obtained using the eigenfunction expansion technique, in this case Ψ_n and λ_n are the eigenfunctions and eigenvalues respectively. On the other hand ,the corresponding Green's function $G(\vec{r},\vec{r}')$ satisfies the equation

$$\nabla_{\vec{r}}^2 G(\vec{r},\vec{r}') + (f(\vec{r}) + \lambda)G(\vec{r},\vec{r}') = -4\pi\delta(\vec{r}-\vec{r}') \quad (70)$$

with the same boundary conditions.

Writing Green's function as

$$G(\vec{r},\vec{r}') = \sum_n a_n(\vec{r})\Psi_n(\vec{r}) \quad (71)$$

Using Equation (70) and Equation(71) and with a few manipulations one gets

$$G(\vec{r},\vec{r}') = 4\pi \sum_n \frac{\Psi_n(\vec{r})\Psi_n(\vec{r}')}{\lambda_n - \lambda} \quad (72)$$

Now to complete the solution of the problem, we note that we are dealing with Poisson's equation of the form

$$(\nabla^2 + k^2)\Psi(\vec{r}) = 0 \tag{73}$$

whose solution can be obtained through the use of the method of separation of variables and the eigenfunctions are given by

$$\Psi_{lmn} = \sqrt{\frac{8}{a^3}} \sin\left(\frac{l\pi}{a}\right) x \sin\left(\frac{m\pi}{a}\right) y \sin\left(\frac{n\pi}{a}\right) z \tag{74}$$

a is the cube side. Accordingly, the final form of Green's function in this case is [7]

$$G(\vec{r}, \vec{r}') = \frac{32}{\pi a^3} \sum_{l,m,n=1}^{\infty} \frac{\sin\left(\frac{l\pi}{a}\right)x \sin\left(\frac{m\pi}{a}\right)y \sin\left(\frac{n\pi}{a}\right)z \sin\left(\frac{l\pi}{a}\right)x' \sin\left(\frac{m\pi}{a}\right)y' \sin\left(\frac{n\pi}{a}\right)z'}{(\frac{l}{a})^2 + (\frac{m}{a})^2 + (\frac{n}{a})^2} \tag{75}$$

Note that this result lead to the calculation of Madelung constant for Sodium Chloride (Na Cl) which is very important in the study of ionic crystals [8].

6.5 Green's Function and the Solution of Schrodinger Equation

Schrodinger equation describes the wave motion related to the particle motion (de Broglie waves), the radial part of this equation is given by

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2}\right) \Psi_l(r) = \int_0^{\infty} K(r, r')\Psi_l(r')dr' \tag{76}$$

Assume now that $\Psi_l(K, r) = N\emptyset_l(K, r)$,N *is a constant* and $\emptyset_l(K, r)$ satisfies the equation

$$\frac{d^2\emptyset_l(K,r)}{dr^2} + \left(k^2 - \frac{l(l+1)}{r^2}\right) \emptyset_l(K, r) = g_l(r);\ \emptyset_l(K, 0) = 0 \tag{77}$$

The solution of the above equation is

$$\emptyset_l(K, r) = \int_0^{\infty} G_l(r, r')g_l(r')dr' \tag{78}$$

where $G_l(r, r')$ is Green's function and which satisfies the equation

$$\frac{d^2G_l(r,r')}{dr^2} + \left(K^2 - \frac{l(l+1)}{r^2}\right) G_l(r, r') = 0 \tag{79}$$

With some simple transformations $G_l(r, r')$ can be found to be

$$G_l(r, r') = \begin{cases} A_l r J_l(Kr) & ; r < r' \\ B_l r \eta_l(Kr) & ; r > r' \end{cases} \tag{80}$$

$J_l(Kr)$ and $\eta_l(Kr)$ are Bessel's functions.

Equation (80) gives the Green's function for Schrodinger equation and this depends on the potential involved in the problem.

Equation (8) is so important in the calculations related to the subject of positive energy bound states [9] [10] [11].

6.6 The Solution of Poisson's Equation Using Green's Function

Let Ψbe the corresponding potential to the charge distribution $\rho(r)$ and which satisfies the equation

$$\nabla^2 \Psi = -\frac{\rho(r)}{\varepsilon_0} \tag{81}$$

We search for Green's function G which is a solution of Poisson's equation with a point source at r_1 ,namely

$$\nabla^2 G = -\delta(r_1 - r_2) \tag{82}$$

G is the potential corresponding to a unit charge at the point r_1 ;using Green's and Gauss's theorems and with a few steps of simplifications one obtains

$$G(r_1, r_2) = \frac{1}{4\pi |r_1 - r_2|} \tag{83}$$

And the solution of the given Poisson's differential equation is

$$\Psi(r_1) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r_2)}{|r_1 - r_2|} d\tau_2 \tag{84}$$

Physically ,this result means that $\Psi(r_1)$ represents the potential at r_1 due to a point charge at r_2 [2].

6.7 The Solution of the Inverse Problem

The relation between the elastic scattering matrix $S(b)$ and the phase shift χ is given by

$$S(b) = \exp(i\chi(b)) \tag{85}$$

where b is the impact parameter.

The last equation is very important in the calculation of the potential using the so-called optical model potential via the use of inverse scattering [12];the phase shift is then given by

$$\chi(b) = -\frac{k}{E} \int_b^\infty \frac{rV(r)}{\sqrt{r^2 - b^2}} dr \tag{86}$$

$E\ an\ V$ are the energy and potential. Now beginning with Schrodinger equation

$$\nabla^2 \Psi(\vec{r}) + k^2 \Psi(\vec{r}) = \frac{2m}{\hbar^2} V(\vec{r}) \Psi(\vec{r}) \tag{87}$$

with k is given by $k^2 = \frac{2mE}{\hbar^2}$ .To solve this equation let us consider the equation

$$Q\emptyset(\vec{r}) = f(\vec{r}) \tag{88}$$

Q is a differential operator, $f(\vec{r})$ is a known function and $\emptyset(\vec{r})$ is to be found.

Since for any function $f(\vec{r})$ there exist a corresponding solution $\emptyset(\vec{r})$ such that

$$\emptyset(\vec{r}) = Lf(\vec{r}) \tag{89}$$

L is an operator which is a function of Q.

Consider now Green's function $G(\vec{r},\vec{r}')$ which is assumed to be a solution of the equation

$$QG(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}') \tag{90}$$

Hence one can see that

$$G(\vec{r},\vec{r}') = L\delta(\vec{r}-\vec{r}') \tag{91}$$

and

$$\emptyset(\vec{r}) = \int G(\vec{r},\vec{r}')f(\vec{r}')d^3\vec{r}' \tag{92}$$

Note that if one operates with Q on the last equation the result will be

$$Q\emptyset(\vec{r}) = \int[QG(\vec{r},\vec{r}')]f(\vec{r}')d^3\vec{r}' = \int \delta(\vec{r}-\vec{r}')f(\vec{r}')d^3\vec{r}' = f(\vec{r}) \tag{93}$$

as expected.

Considering the solution of the corresponding homogeneous equation of Equation(88) and with some manipulations one gets

$$G(\vec{r},\vec{r}') = \frac{1}{(2\pi)^3}\iiint(\nabla^2+k^2)^{-1}\exp[i\vec{q}.(\vec{r}-\vec{r}')]\,d^3\vec{q} \tag{94}$$

From which one obtains

$$G(\vec{r},\vec{r}') = -\frac{1}{4\pi}\frac{\exp(ik|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} \tag{95}$$

Finally one gets the solution of Equation(87) as

$$\Psi(\vec{r}) = \exp\left(i\vec{k}.\vec{r}\right) - \frac{2m}{4\pi\hbar^2}\iiint\frac{\exp(ik|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}V(\vec{r}')\Psi(\vec{r}')d^3\vec{r}' \tag{96}$$

Equation(96) is an integral equation in three dimensions; and as mentioned before this study enabled to compute the optical model potential for the scattering matrix S in the case of Heavy-ion interactions at intermediate energies [12].

6.8 Green's Function and Difference Equations

The solution of the boundary value equation for a second-order difference equation will involve its representation in terms of Green's function. Consider the boundary value problem for the differential equation

$$Lu = \frac{d}{dx}\left(k(x)\frac{du}{dx}\right) - q(x)u = -f(x), \qquad 0 < x < 1 ,$$

$$u(0) = 0 , \quad u(1) = 0 , \ k(x) \geq c_1 > 0, \quad q(x) \geq 0, \tag{97}$$

The solution of this problem can be written in an integral form as

$$u(x) = \int_0^1 G(x,\xi)f(\xi)d\xi , \tag{98}$$

where $G(x,\xi)$ is the source function or Green's function .

$u(x)$ in Equation(98) is a solution of Equation (97) subject to the boundary conditions $u(0) = 0$ and $u(1) = 0$; and Green's function
$G(x,\xi)$ as a function of x for fixed ξ satisfies the conditions

$$L_x G(x,\xi) = \frac{d}{dx}\left(k(x)\frac{dG(x,\xi)}{dx}\right) - q(x)G(x,\xi) = 0$$

$$x \neq \xi , \quad 0 < x < 1 , \quad G(0,\xi) = G(1,\xi) = 0 \tag{99}$$

$$[G] = G(\xi + 0,\xi) - G(\xi - 0,\xi) = 0 , \quad \left[k\frac{dG}{dx}\right] = -1 \ \text{ for } \ x = \xi .$$

$$G(x,\xi) \geq 0 \quad , \quad G(x,\xi) = G(\xi,x),$$

Then $G(x,\xi)$ can be written in the explicit form

$$G(x,\xi) = \begin{cases} \frac{\alpha(x)\beta(\xi)}{\alpha(1)} for x \leq \xi \\ \frac{\alpha(\xi)\beta(x)}{\alpha(1)} for x \geq \xi \end{cases} \tag{100}$$

where $\alpha(x)$ and $\beta(x)$ are solutions of the following problems [4] :

$$L\alpha = 0 , \quad 0 < x < 1 , \quad \alpha(0) = 0 , \quad k(0)\alpha'(0) = 1,$$

$$L\beta = 0 , \quad 0 < x < 1 , \quad \beta(1) = 0 , \quad k(1)\beta'(1) = -1. \tag{101}$$

Now we proceed to get the exact form of Green's function.

Using a closed rectangle $\bar{R} = \{(x,y): 0 \leqq x \leqq a , \ 0 \leqq y \leqq b\}$,

such that the ratio a/b is rational. The square grid on which the difference equation will be considered consists of the node points
(x_m, y_n) [4] :

$$\begin{aligned} x = x_m = mh , \qquad (m = 0,1, \dots ,M), \quad (Mh = a), \\ y = y_n = nh, \qquad (n = 0,1, \dots ,N) , \quad (Nh = b) \end{aligned} \tag{102}$$

Denoting a parameter point by (ξ,η) such that

$$\xi = \mu h, \quad \eta = vh, \ \left(0 \leqq \mu \leqq M \ , \ 0 \leqq v \leqq N\right). \tag{103}$$

and replacing Laplace's equation by its simple analogue, namely,

$$\Delta_h u(x,y) = \frac{1}{h^2}[u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)] = 0. \tag{104}$$

Green's function $G_h(x,y;\xi,\eta)$ is now defined on the grid by the difference equations [4]

$$\Delta_h G_h(x,y;\xi,\eta) = \begin{cases} 0\,, & when(x,y) \neq (\xi,\eta) \\ h^{-2}\,, & whenx = \xi \ and \ y = \eta\,, \end{cases} \tag{105}$$

From the condition that this must vanish on the boundary of the rectangle ,Green's function can be represented by the following expression

$$G_h(mh,nh;\mu h,vh) = \begin{cases} -\frac{2}{M}\sum_{k=1}^{M-1}\frac{\sin\mu\alpha_k \sin m\alpha_k \sin hv'\beta_k \sin hn\beta_k}{\sin h\beta_k \sin hN\beta_k} & (n \leqq v) \\ -\frac{2}{M}\sum_{k=1}^{M-1}\frac{\sin\mu\alpha_k \sin m\alpha_k \sin hv\beta_k \sin hn'\beta_k}{\sin h\beta_k \sin hN\beta_k} & (n \geqq v), \end{cases} \tag{106}$$

with

$$\alpha_k = \frac{k\pi}{M} = \frac{k\pi h}{a}, \qquad \cos h\,\beta_k = 2 - cos\alpha_k \quad , \tag{107}$$

From Equation(106) It is clear that the obtained discrete Green's function is symmetric with respect to its two kinds of variables (x, y) and (ξ, η) [4].

7. Concluding Remarks

Green's function is an important tool to solve boundary value problems faced with in applied fields such as physics and engineering ;the sample applications given in the last section are just few of thousands of published works on the subject and research on Green's function will keep producing interesting and fabulous publications.

Green's function played a very principal role in solving Helmholtz equation and in getting electronic spectral computations in periodic crystals [1].Moreover, the importance of the function in different branches of science was made clear in solving boundary value problems and in initial value ones such as Kirchhoff diffusion equation ,diffraction theory, and Helmholtz equation,… ect. [13].

On the other hand, using an integral representation for the first kind Hankel function one is lead to the so-called Basset formula that produces an application from which one can calculate a

Green' function associated with a second order partial differential equation involving a wave equation for a lossy two-dimensional medium [14].

Another area where Green's function play a good role is in the numerical computation direction, e.g. to construct an algorithm which enables to calculate the explicit form of Green's function for an n^{th} order linear ordinary differential equation with constant coefficients coupled with two-points linear boundary condition [15]. To add, Green's function can help in solving difference equations for particular boundary value problems [16].

Recently a beautiful article was published about the existence of positive solutions for a fourth-order three-point boundary value problem with sign-changing Green's function [17].

Hence we conclude with certainty that Green's function will continue to be a vital tool in solving boundary value problems and will always enrich research in various applied fields.

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