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A New Family of Optimal Eighth-Order Iterative Scheme for Solving Nonlinear Equations

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Abstract

The objective of this manuscript is to introduce a new family of optimal eight-order iterative methods for computing the numerical zeros of a nonlinear univariate equation that is not dependent on the second derivative. The family was designed to enhance the order of convergence by merging Bawazir's method and Newton's method as a third step. To demonstrate the performance of the offered scheme, assorted numerical comparisons have been investigated. In addition, the efficiency index of the new family is 1.6818.

Keywords: Convergence order; Efficiency index; Iterative methods; Nonlinear equations; Optimal eighth-order.

1. Introduction

Numerical analysis has several applications in the pure and practical engineering field that may be addressed within nonlinear equations $f(x)=0$,where $f:D\subset R\rightarrow R$ for an open interval [1–17]. (1)

The classical Newton's technique is a well-known one-step approach for determining the root of nonlinear equations

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

This approach generates a series of approximations that quadratically converge to a simple zero of the function f(x). Traub [1] began categorizing iterative methods, and the first third-order iterative approach was suggested by him. Due to their importance, various eight-order schemes for solving nonlinear equations have been analyzed and investigated by experienced researchers, as seen in [2–6].

The efficiency index [7] (EI) which may be calculated by $k^{1/w}$, where k is the iterative scheme's convergence order and w is the number of functions that must be obtained at each iteration, is one of the most frequent techniques to compare the efficiency of iterative procedures. The iterative strategy with the number of functional evaluations equal to w is optimal, according to the conjecture of Kung et al [8], if the order of convergence equals 2^{w-1}. Numerous experts have developed the optimum iterative algorithms for various convergence orders. Numerous experts have developed the optimum iterative algorithms for various convergence orders. The composition technique is implemented in conjunction with various interpolations and approximations to reduce the required functional evaluations per iteration.

Theorem 1[9]: Let $\varphi_1(x),\varphi_2(x),\ldots,\varphi_s(x)$ be iterative functions with the order $k_1,k_2,\ldots,k_s$, respectively. Then the composition of iterative functions $\varphi_1\left(\varphi_2(\ldots\varphi_s(x)\ldots)\right)$, defines the iterative method of the order $k_1,k_2,\ldots,k_s$.

The remainder of this paper's contents is provided in what follows. Section 2 presents the design of our new family of the optimum eight-order iterative scheme. Section 3 shows Convergence Analysis. Section 4 includes several numerical examples that demonstrate the performance of the new optimum class of iterative algorithms described in this work. The last section concludes with a brief of the findings.

2. Construction of New Method

The objective of the new method is to have a new family of optimal eight order and free second derivatives. Bawazir[10] obtained two steps iterative method by depending on double Newton's method:

$$\begin{cases} y_n=x_n-\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1}=y_n-\dfrac{f(y_n)}{f'(y_n)}\left(1+\dfrac{f(y_n)[f'(x_n)-f'(y_n)]}{2f(x_n)f'(y_n)}\right). \end{cases} \tag{3}$$

This scheme is not optimal since it requires the evaluation of four functions. We reduce the number of functions that must be evaluated in order to achieve optimality by using an approximation which's obtained by King [11]:

$$f'(y_n)\approx\frac{f'(x_n)[f(x_n)+(\beta-2)f(y_n)]}{f(x_n)+\beta f(y_n)}. \tag{4}$$

By substituting (4) in (3), a family of fourth-order methods will be defined as:

$$\begin{cases} y_n=x_n-\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1}=y_n-\dfrac{\left(f(x_n)^2+(\beta-1)f(x_n)f(y_n)+f(y_n)^2\right)\left(\beta f(y_n)+f(x_n)\right)}{f(x_n)\left(f(x_n)+(\beta-2)f(y_n)\right)^2}\dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{5}$$

The composition technique will be applied to extend (5) to the eight-order of convenience by adding a third step as Newton's method:

$$\begin{cases} y_n=x_n-\dfrac{f(x_n)}{f'(x_n)}, \\ z_n=y_n-\dfrac{\left(f(x_n)^2+(\beta-1)f(x_n)f(y_n)+f(y_n)^2\right)\left(\beta f(y_n)+f(x_n)\right)}{f(x_n)\left(f(x_n)+(\beta-2)f(y_n)\right)^2}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1}=z_n-\dfrac{f(z_n)}{f'(z_n)}. \end{cases} \tag{6}$$

Finally, we aim to rewrite $f'(z_n)$ by using the approximation which's proved by Solaiman and Hashim [12]:

$$f'(z_n)\approx q'(z_n)=f[z_n,x_n]\left(2+\frac{x_n-z_n}{y_n-z_n}\right)-f[x_n,y_n]\frac{(x_n-z_n)^2}{(x_n-y_n)(y_n-z_n)}+f'(x_n)\left(\frac{y_n-z_n}{x_n-y_n}\right). \tag{7}$$

By superseding the approximation into (6), we obtain new family of optimal eight-order scheme:

$$\begin{cases} y_n=x_n-\dfrac{f(x_n)}{f'(x_n)}, \\ z_n=y_n-\dfrac{\left(f(x_n)^2+(\beta-1)f(x_n)f(y_n)+f(y_n)^2\right)\left(\beta f(y_n)+f(x_n)\right)}{f(x_n)\left(f(x_n)+(\beta-2)f(y_n)\right)^2}\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1}=z_n-\dfrac{f(z_n)}{q'(z_n)}. \end{cases} \tag{8}$$

Depending on the conjecture of Kung et al [8], the scheme (8) attains optimality and has EI $(8)^{\frac{1}{4}}=1.6818$.

3. Convergence Analysis

This province uses Maple scripts to verify the order of convergence of the offered method (ZSM), which is given by (8).

Theorem 2: consider α is a simple zero of (1). In an open interval I, let $e_n=x_n-\alpha$ be a genuine adequately differentiable function. Let$x_n$be sufficiently close to α. Therefore, the optimal method was introduced in (8) has at least order of convergence eight and it meets the error equation

$$e_{n+1}=\alpha+\left(4\beta^2c_2^7-4\beta c_2^5c_3+2\beta c_2^4c_4+c_2^3c_3^2-c_2^2c_3c_4\right)e_n^8+Oe_n^9, \tag{9}$$

where, $e_n=x_n-\alpha, c_j\dfrac{f^j(\alpha)}{i!\,f'(\alpha)}$.

Proof: Utilizing Taylor's series expansion of x_n and the zero α, we have the ensuing equations:

$$f(x_n)=f'(\alpha)\left(Oe_n^{9}+c_8e_n^{8}+c_7e_n^{7}+c_6e_n^{6}+c_5e_n^{5}+c_4e_n^{4}+c_3e_n^{3}+c_2e_n^{2}+e_n\right), \tag{10}$$

and

$$f'(x_n)= f'(\alpha)\left(Oe_n^{9}+9c_9e_n^{8}+8c_8e_n^{7}+7c_7e_n^{6}+6c_6e_n^{5}+5c_5e_n^{4}+4c_4e_n^{3}+3c_3e_n^{2}+2c_2e_n+1\right). \tag{11}$$

Now from (10), and (11), we get

$$\frac{f(x_n)}{f'(x_n)}= e_n-c_2e_n^{2}+\left(2c_2^2-2c_3\right)e_n^{3}+\ldots+(-64c_2^7+304c_2^5c_3-176c_2^4c_4+\ldots+27c_3c_6+31c_4c_5-7c_8)e_n^{8}+Oe_n^{9}. \tag{12}$$

From (12), we have

$$y_n=\alpha+c_2e_n^{2}+\left(-2c_2^2+2c_3\right)e_n^{3}+\ldots+(64c_2^7-304c_2^5c_3+176c_2^4c_4+\ldots+27c_3c_6-31c_4c_5+7c_8)e_n^{8}+Oe_n^{9}. \tag{13}$$

From (13), and Using the Taylor series, we have:

$$f(y_n)= f'(\alpha)[c_2e_n^{2}+\left(-2c_2^2+2c_3\right)e_n^{3}+\ldots+\left(144c_2^7-552c_2^5c_3+297c_2^4c_4+\ldots+27c_3c_6-31c_4c_5+7c_8\right)e_n^{8}+Oe_n^{9}]. \tag{14}$$

Substituting equations (10), (11), (13), and (14) in the second step of the method, we have

$$z_n= \alpha+\left(2\beta c_2^3-c_2c_3\right)e_n^{4}+\ldots+\left(2\beta^5c_2^7+17\beta^4c_2^7-24\beta^4c_2^5c_3+\ldots+5c_2c_7-13c_3c_6-17c_4c_5\right)e_n^{8}+Oe_n^{9}. \tag{15}$$

From (15), we get

$$f(z_n)= f'(\alpha)\left[\left(13c_3c_6-17c_4c_5\right)e_n^{4}+\ldots+\left(13c_3c_6-17c_4c_5-24\beta^4c_2^5c_3+\ldots+5c_2c_7-13c_3c_6-17c_4c_5\right)e_n^{8}+Oe_n^{9}\right]. \tag{16}$$

Hence, from (10), (13), and (14)

$$f[x_n,y_n]= f'(\alpha)[1+c_2e_n+\left(c_3+c_2^2\right)e_n^{2}+\ldots+(116c_2^2c_3c_5-313c_2^3c_3c_4+56c_2c_3^2c_4+\ldots+4c_5^2-36c_2c_6c_3-40c_2c_5c_4)e_n^8+Oe_n^{9}. \tag{17}$$

Hence, from (10), (15), and (16)

$$f[z_n,x_n]= f'(\alpha)[1+c_2e_n+c_3e_n^{2}+\ldots+(21c_2^2c_3c_4-14c_2c_3c_5+22\beta^3c_2^5c_3+\ldots+168\beta c_2c_3^2c_4-18c_2c_6c_3-22c_2c_5c_4)e_n^8+Oe_n^9. \tag{18}$$

Combining (11), (13), (15), (17), and (18), then substitute in the approximation

$$q'(z_n)= f'(\alpha)+\left(4\beta f'(\alpha)c_2^4-2f'(\alpha)c_2^2c_3+f'(\alpha)c_2c_4\right)e_n^4+\ldots+Oe_n^7. \tag{19}$$

Finally, the error equation is given by using (15),(16),and(19) in the last step of (8), we have:

$$e_{n+1}=\alpha+\left(4\beta^2c_2^7-4\beta c_2^5c_3+2\beta c_2^4c_4+c_2^3c_3^2-c_2^2c_3c_4\right)e_n^8+Oe_n^9. \tag{20}$$

The method is proved that is the eight-order.

4. Numerical Examples

We give various numerical tests in this part to demonstrate the efficacy of the proposed method. The performance of two situations of the novel optimum eighth-order scheme was compared

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{\left(f(x_n)^2 + (\beta-1)f(x_n)f(y_n) + f(y_n)^2\right)\left(\beta f(y_n) + f(x_n)\right)}{f(x_n)\left(f(x_n) + (\beta-2)f(y_n)\right)^2} \dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{q'(z_n)}. \end{cases} \tag{21}$$

Where β=1 (ZSM1), and where β=-1 (ZSM2), with Kung-Traub's method [8] (KTM),

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{f(y_n) - f(x_n)}{sf(y_n) - f(x_n)} \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \left\{\{1 + t_1^2 + 2t_1^3 + \alpha t_1^4\} + \{-1 + \beta t_2\} + \{1 + 2t_3 + \gamma t_3^2\}\right\} \dfrac{f(z_n)}{f[y_n,]}, \end{cases} \tag{22}$$

where α, β and $\gamma = 0$, with Kung-Traub's method [8] (KTM):

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(y_n)f(x_n)}{\left(f(x_n) - f(y_n)\right)^2} \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(x_n)}{f'(x_n)} \dfrac{f(x_n)f(y_n)f(z_n)}{\left(f(x_n) - f(y_n)\right)^2} \dfrac{f(x_n)^2 + f(y_n)\left(f(y_n) - f(z_n)\right)}{\left(f(x_n) - f(z_n)\right)^2 \left(f(y_n) - f(z_n)\right)}. \end{cases} \tag{23}$$

Method given by Liu et al [13] (LWM):

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n)}{f(x_n) - 2f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(x_n)} \left(\left(\dfrac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \right)^2 + \dfrac{f(z_n)}{f(y_n) - f(z_n)} + \dfrac{4f(z_n)}{f(x_n) + f(z_n)} \right). \end{cases} \tag{24}$$

Method proposed by Sharma et al [14] (SAM):

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left(3 - 2\dfrac{f[y_n,x_n]}{f'(x_n)}\right)\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(x_n)}\left(\dfrac{f(x_n) - f[y_n,x_n] + f[z_n,y_n]}{2f[z_n,y_n] - f[z_n,x_n]}\right). \end{cases} \tag{25}$$

Method proposed by Parimala et al [16] (PMJ):

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)(z_n - y_n)}{f(z_n) - f(y_n)}(1 + 2\eta)\left(1 + \tau^2 + 2\tau^3 + \left(\dfrac{7}{24}\right)\tau^4\right), \end{cases} \tag{26}$$

where $\eta = \dfrac{f(z_n)}{f(x_n)}$, *and* $\tau = \dfrac{f(y_n)}{f(x_n)}$.

Table 1 contains a list of the test functions considered, their roots (α) with only15 decimal digits, and the beginning estimates (x_0) in the region of the zeros.

To carry out all computations with 1000 significant digits, MATLAB (R2018a) was utilized. Table 2 includes the number of iterations is denoted by (IT), the computational order of convergence (COC), the results of $|f(x_n)|$, and $|x_n-\alpha|$are computed.For computer programs, the following criteria for terminating them was applied:

$$|x_n - \alpha| < 10^{-300} \text{and } |f(x_n)| < 10^{-300}.$$

Moreover, the COC is determined using the formula [17]:

$$COC \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

Table 1:The test functions, their initial guesses, and their precise root

Test function	Initial Guesses (x_o)	Root (α)
$f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}$	-1.1	0.784895987661213
$f_2(x) = x^3 + \ln(1+x)$	0.25	0.0
$f_3(x) = \sin^{-1}(x^2-1) - (x/2) + 1$	-0.49	0.296550195139443
$f_4(x) = x^5 + x^4 + 4x^2 - 15$	1.5	1.34742809896830
$f_5(x) = \sqrt{x^2+2x+5} - 2\sin(x) - x^3 + 3$	1.9	1.62671060867628
$f_6(x) = (x+2)e^x - 1$	-0.5	0.442854401002389

Table 2: The convergence behavior of several iterative methods

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n-\alpha\rvert$	COC
		$f_1(x)$		
KTM	3	1.51587e-328	5.3809e-329	8
LWM	3	2.48718e-305	8.82879e-306	8
SAM	3	2.77011e-307	9.83313e-308	8
ASM	3	5.24386e-334	1.86142e-334	8
PMJ	3	1.30999e-334	4.65008e-335	8
ZSM1	3	2.89798e-376	1.0287e-376	8
ZSM2	3	4.93749e-428	1.75267e-428	8
		$f_2(x)$		
KTM	3	1.29102e-432	1.29102e-432	8
LWM	3	1.28561e-415	1.28561e-415	8
SAM	3	1.34383e-358	1.34383e-358	8
ASM	3	4.9606e-423	4.9606e-423	8
PMJ	3	1.70259e-422	1.70259e-422	8
ZSM1	3	6.92598e-449	6.92598e-449	8
ZSM2	3	4.76728e-436	4.76728e-436	8
		$f_3(x)$		
KTM	3	1.21424e-774	6.23851e-775	8
LWM	3	1.26156e-705	6.4816e-706	8
SAM	3	5.46328e-716	2.80691e-716	8
ASM	-	-	-	-
PMJ	-	-	-	-
ZSM1	3	1.01647e-719	5.22238e-720	8
ZSM2	3	8.93918e-711	4.59274e-711	8

$f_4(x)$				
KTM	3	5.70932e-386	1.54114e-387	8
LWM	3	2.36031e-398	6.37129e-400	8
SAM	3	1.11326e-349	3.00507e-351	8
ASM	3	3.21464e-305	8.67741e-307	8
PMJ	3	2.4082e-405	6.50054e-407	8
ZSM1	3	2.46262e-399	6.64746e-401	8
ZSM2	3	9.08559e-360	2.45251e-361	8
$f_5(x)$				
Method	**IT**	$\lvert f(x_n) \rvert$	$\lvert x_n-\alpha \rvert$	**COC**
KTM	3	8.2423e-402	1.17225e-402	8
LWM	3	1.42733e-409	2.03001e-410	8
SAM	3	1.36672e-367	1.94381e-368	8
ASM	-	-	-	-
PMJ	3	1.02172e-419	1.45313e-420	8
ZSM1	3	2.81374e-419	4.0018e-420	8
ZSM2	3	1.56415e-382	2.2246e-383	8
$f_6(x)$				
KTM	3	1.23851e-677	7.54178e-678	8
LWM	-	-	-	-
SAM	-	-	-	-
ASM	-	-	-	-
PMJ	-	-	-	-
ZSM1	3	1.6017e-685	9.75339e-686	8
ZSM2	3	1.13906e-656	6.93617e-657	8

5. Conclusion

In this research work, a new family of optimal scheme with eighth-order of convergence has been devolved. The proposed method's optimality was achieved using the composition technique with

Solaiman and Hashim's approximation and King's approximation. Overall, the strength of the scheme was illustrated by using some numerical examples and compared with the examined iterative methods of equal order of convergence.

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