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Generalized John Numbers

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Abstract

In this paper, we define and investigate the generalized John sequences and we deal with, in detail, two special cases, namely, John and John-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between John and John-Lucas numbers and Pell, Pell-Lucas numbers.

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Keywords

John numbers, John-Lucas numbers, Tribonacci numbers, Pell numbers, Pell-Lucas numbers.

1. Introduction

Pell sequence $\{P_n\}_{n\geq 0}$ (OEIS: A000129, [11]) and Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ (OEIS:

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A002203, [11]) are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} + P_{n-2}, \qquad P_0 = 0, P_1 = 1 \tag{1.1}$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \qquad Q_0 = 2, Q_1 = 2. \tag{1.2}$$

The sequences $\{P_n\}_{n\geq 0}$ **and** $\{Q_n\}_{n\geq 0}$ **can be extended to negative subscripts by defining**

$$P_{-n} = -2P_{-(n-1)} + P_{-(n-2)}$$

and

$$Q_{-n} = -2Q_{-(n-1)} + Q_{-(n-2)}$$

for $n = 1, 2, 3, ...$ **respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer** n.

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,2,3,4,5,6,9,10]. For higher order Pell sequences, see [7,8,12,13,14,15].

Now, we define two sequences related to Pell and Pell-Lucas numbers. John and John-Lucas numbers are defined as

$$J_n = 2J_{n-1} + J_{n-2} + 1, \quad \textbf{with} \quad J_0 = 0, J_1 = 1, \quad n \geq 2,$$

and

$$H_n = 2H_{n-1} + H_{n-2} - 2, \quad \textbf{with} \quad H_0 = 3, H_1 = 3, \qquad n \geq 2,$$

respectively. The first few values of John and John-Lucas numbers are

$$0, 1, 3, 8, 20, 49, 119, 288, 696, 1681, 4059, \ldots$$

and

$$3, 3, 7, 15, 35, 83, 199, 479, 1155, 2787, 6727, \ldots$$

respectively. The sequences $\{J_n\}$ **and** $\{H_n\}$ **satisfy the following third order linear recurences:**

$$\begin{aligned} J_n &= 3J_{n-1} - J_{n-2} - J_{n-3}, \qquad J_0 = 0, J_1 = 1, J_2 = 3, \\ H_n &= 3H_{n-1} - H_{n-2} - H_{n-3}, \qquad H_0 = 3, H_1 = 3, H_2 = 7. \end{aligned}$$

There are close relations between John and John-Lucas and Pell, Pell-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} J_n &= \frac{1}{2}(P_{n+2} - P_{n+1} - 1), \\ H_n &= Q_n + 1, \end{aligned}$$

and

$$J_n = \frac{1}{4}(Q_{n+1} - 2),$$
$$H_n = 2P_{n+1} - 2P_n + 1.$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., John, John-Lucas numbers). First, we recall some properties of generalized Tribonacci numbers.

The generalized (r, s, t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.3}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [17]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.3) holds for all integer n. As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.4}$$

whose roots are

$$\alpha = \frac{r}{3} + A + B,$$
$$\beta = \frac{r}{3} + \omega A + \omega^2 B,$$
$$\gamma = \frac{r}{3} + \omega^2 A + \omega B,$$

where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3},$$
$$\Delta = \Delta(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1. *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$) Binet's formula of generalized Tribonacci numbers is*

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= A_1\alpha^n + A_2\beta^n + A_3\gamma^n, \end{aligned} \tag{1.5}$$

where

$$p_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \; p_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \; p_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0,$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)}, \\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

2 Generalized John Sequence

In this paper, we consider the case $r = 3, s = -1, t = -1$. A generalized John sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-3} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + 3W_{-(n-2)} - W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

(1.5) can be used to obtain Binet formula of generalized John numbers. Binet formula of generalized John numbers can be given as

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^n + z_2\beta^n}{4} - \frac{z_3}{2} \\ &= \frac{z_1\alpha^n + z_2\beta^n - 2z_3}{4} \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} z_1 &= W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 = W_2 - (\beta+1)W_1 + \beta W_0, \\ z_2 &= W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 = W_2 - (\alpha+1)W_1 + \alpha W_0, \\ z_3 &= W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 = W_2 - 2W_1 - W_0, \end{aligned}$$

i.e.,

$$W_n = \frac{(W_2 - (\beta+1)W_1 + \beta W_0)\alpha^n + (W_2 - (\alpha+1)W_1 + \alpha W_0)\beta^n - 2(W_2 - 2W_1 - W_0)}{4}.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 3x^2 + x + 1 = (x^2 - 2x - 1)(x - 1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= 1+\sqrt{2}, \\ \beta &= 1-\sqrt{2}, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma &= 3, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= 1, \\ \alpha\beta\gamma &= -1, \end{aligned}$$

i.e

$$\alpha+\beta = 2,\ \alpha\beta = -1,\ \alpha-\beta = 2\sqrt{2}$$

and

$$\begin{aligned} (\alpha-\beta)(\alpha-\gamma) &= 4, \\ (\beta-\alpha)(\beta-\gamma) &= 4, \\ (\gamma-\alpha)(\gamma-\beta) &= -2. \end{aligned}$$

The first few generalized John numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized John numbers

n	W_n	W_{-n}
0	W_0	$W_0 = W_0$
1	W_1	$W_{-1} = 3W_1 - W_0 - W_2$
2	W_2	$W_{-2} = 4W_0 - 4W_1 + W_2$
3	$W_3 = 3W_2 - W_1 - W_0$	$W_{-3} = 13W_1 - 8W_0 - 4W_2$
4	$W_4 = 8W_2 - 4W_1 - 3W_0$	$W_{-4} = 21W_0 - 28W_1 + 8W_2$
5	$W_5 = 20W_2 - 11W_1 - 8W_0$	$W_{-5} = 71W_1 - 49W_0 - 21W_2$
6	$W_6 = 49W_2 - 28W_1 - 20W_0$	$W_{-6} = 120W_0 - 168W_1 + 49W_2$
7	$W_7 = 119W_2 - 69W_1 - 49W_0$	$W_{-7} = 409W_1 - 288W_0 - 120W_2$
8	$W_8 = 288W_2 - 168W_1 - 119W_0$	$W_{-8} = 697W_0 - 984W_1 + 288W_2$
9	$W_9 = 696W_2 - 407W_1 - 288W_0$	$W_{-9} = 2379W_1 - 1681W_0 - 697W_2$
10	$W_{10} = 1681W_2 - 984W_1 - 696W_0$	$W_{-10} = 4060W_0 - 5740W_1 + 1681W_2$
11	$W_{11} = 4059W_2 - 2377W_1 - 1681W_0$	$W_{-11} = 13861W_1 - 9800W_0 - 4060W_2$
12	$W_{12} = 9800W_2 - 5740W_1 - 4059W_0$	$W_{-12} = 23661W_0 - 33460W_1 + 9800W_2$
13	$W_{13} = 23660W_2 - 13859W_1 - 9800W_0$	$W_{-13} = 80783W_1 - 57121W_0 - 23661W_2$

Now we define two special cases of the sequence $\{W_n\}$. John sequence $\{J_n\}_{n\geq 0}$ and John-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$J_n = 3J_{n-1} - J_{n-2} - J_{n-3}, \qquad J_0 = 0, J_1 = 1, J_2 = 3, \tag{2.3}$$

$$H_n = 3H_{n-1} - H_{n-2} - H_{n-3}, \qquad H_0 = 3, H_1 = 3, H_2 = 7. \tag{2.4}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n} = -J_{-(n-1)} + 3J_{-(n-2)} - J_{-(n-3)}$$

$$H_{-n} = -H_{-(n-1)} + 3H_{-(n-2)} - H_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (2.3)-(2.4) hold for all integer n.

Next, we present the first few values of the John and John-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
J_n	0	1	3	8	20	49	119	288	696	1681	4059	9800	23660	57121
J_{-n}	0	0	−1	1	−4	8	−21	49	−120	288	−697	1681	−4060	9800
H_n	3	3	7	15	35	83	199	479	1155	2787	6727	16239	39203	94643
H_{-n}	3	−1	7	−13	35	−81	199	−477	1155	−2785	6727	−16237	39203	−94641

For all integers n, John and John-Lucas numbers can be expressed using Binet's formulas as

$$
\begin{aligned}
J_n &= \frac{\alpha^{n+1}+\beta^{n+1}-2}{4}, \\
H_n &= \alpha^n+\beta^n+1,
\end{aligned}
$$

respectively. Note that Binet's formulas of Pell and Pell-Lucas numbers, respectively, are

$$
\begin{aligned}
P_n &= \frac{\alpha^n-\beta^n}{\alpha-\beta}, \\
Q_n &= \alpha^n+\beta^n,
\end{aligned}
$$

so

$$
J_n = \frac{1}{2}(P_{n+2}-P_{n+1}-1), \tag{2.5}
$$

$$
H_n = Q_n+1. \tag{2.6}
$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 2. *Suppose that $f_{W_n}(x)=\sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized John sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$
\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-3W_0)x+(W_2-3W_1+W_0)x^2}{1-3x+x^2+x^3}.
$$

Proof. Take $r=3, s=-1, t=-1$ in Soykan [17, Lemma 1.1]. □

The previous lemma gives the following results as particular examples.

Corollary 3. *Generated functions of John and John-Lucas numbers are*

$$
\begin{aligned}
\sum_{n=0}^{\infty} J_n x^n &= \frac{x}{1-3x+x^2+x^3}, \\
\sum_{n=0}^{\infty} H_n x^n &= \frac{3-6x+x^2}{1-3x+x^2+x^3},
\end{aligned}
$$

respectively.

3 Simson Formulas

There is a well-known Simson Identity (formula) for Pell sequence $\{F_n\}$, namely,

$$
F_{n+1}F_{n-1}-F_n^2=(-1)^n
$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized John sequence $\{W_n\}_{n\geq 0}$.

Theorem 4 (Simson Formula of Generalized John Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = (-1)^n(W_2 - 2W_1 - W_0)(-W_2^2 - 2W_1^2 + W_0^2 + 4W_1W_2 - 2W_0W_2).$$

Proof. Take $r = 3, s = -1, t = -1$ in Soykan [16, Theorem 2.2]. □

The previous theorem gives the following results as particular examples.

Corollary 5. *For all integers n, Simson formula of John and John-Lucas numbers are given as*

$$\begin{vmatrix} J_{n+2} & J_{n+1} & J_n \\ J_{n+1} & J_n & J_{n-1} \\ J_n & J_{n-1} & J_{n-2} \end{vmatrix} = (-1)^n,$$

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = 32 \times (-1)^n,$$

respectively.

4 Some Identities

In this section, we obtain some identities of John and John-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{J_n\}$.

Lemma 6. *The following equalities are true:*

(a) $W_n = (13W_1 - 8W_0 - 4W_2)J_{n+4} + (28W_0 - 43W_1 + 13W_2)J_{n+3} + (28W_1 - 21W_0 - 8W_2)J_{n+2}.$

(b) $W_n = (4W_0 - 4W_1 + W_2)J_{n+3} + (15W_1 - 13W_0 - 4W_2)J_{n+2} + (8W_0 - 13W_1 + 4W_2)J_{n+1}.$

(c) $W_n = (3W_1 - W_0 - W_2)J_{n+2} + (4W_0 - 9W_1 + 3W_2)J_{n+1} + (4W_1 - 4W_0 - W_2)J_n.$

(d) $W_n = W_0J_{n+1} + (W_1 - 3W_0)J_n + (W_0 - 3W_1 + W_2)J_{n-1}.$

(e) $W_n = W_1 J_n + (W_2 - 3W_1)J_{n-1} - W_0 J_{n-2}.$

(f) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)J_n = -(W_1^2+W_2^2+W_0W_1-3W_1W_2)W_{n+4} + (4W_1^2+3W_2^2+3W_0W_1-W_0W_2-9W_1W_2)W_{n+3} - (W_0^2+W_1^2+W_2^2+2W_0W_1-3W_0W_2-2W_1W_2)W_{n+2}.$

(g) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)J_n = (W_1^2-W_0W_2)W_{n+3} - (W_0^2+W_0W_1-3W_0W_2+W_1W_2)W_{n+2} + (W_1^2+W_2^2+W_0W_1-3W_1W_2)W_{n+1}.$

(h) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)J_n = -(W_0^2-3W_1^2+W_0W_1+W_1W_2)W_{n+2} + (W_2^2+W_0W_1+W_0W_2-3W_1W_2)W_{n+1} + (-W_1^2+W_0W_2)W_n.$

(i) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)J_n = -(3W_0^2-9W_1^2-W_2^2+2W_0W_1-W_0W_2+6W_1W_2)W_{n+1} + (W_0^2-4W_1^2+W_0W_1+W_0W_2+W_1W_2)W_n + (W_0^2-3W_1^2+W_0W_1+W_1W_2)W_{n-1}.$

(j) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)J_n = (-8W_0^2+23W_1^2+3W_2^2-5W_0W_1+4W_0W_2-17W_1W_2)W_n + (4W_0^2-12W_1^2-W_2^2+3W_0W_1-W_0W_2+7W_1W_2)W_{n-1} + (3W_0^2-9W_1^2-W_2^2+2W_0W_1-W_0W_2+6W_1W_2)W_{n-2}.$

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times J_{n+4} + b \times J_{n+3} + c \times J_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times J_4 + b \times J_3 + c \times J_2 \\ W_1 &= a \times J_5 + b \times J_4 + c \times J_3 \\ W_2 &= a \times J_6 + b \times J_5 + c \times J_4 \end{aligned}$$

we find that $a = 13W_1 - 8W_0 - 4W_2, b = 28W_0 - 43W_1 + 13W_2, c = 28W_1 - 21W_0 - 8W_2$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{H_n\}$.

Lemma 7. *The following equalities are true:*

(a) $8W_n = (5W_0 - 14W_1 + 5W_2)H_{n+4} - 2(10W_0 - 21W_1 + 7W_2)H_{n+3} + (19W_0 - 20W_1 + 5W_2)H_{n+2}.$

(b) $8W_n = -(5W_0 - W_2)H_{n+3} + 2(7W_0 - 3W_1)H_{n+2} - (5W_0 - 14W_1 + 5W_2)H_{n+1}.$

(c) $8W_n = -(W_0 + 6W_1 - 3W_2)H_{n+2} + 2(7W_1 - 3W_2)H_{n+1} + (5W_0 - W_2)H_n.$

(d) $8W_n = -(3W_0 + 4W_1 - 3W_2)H_{n+1} + 2(3W_0 + 3W_1 - 2W_2)H_n + (W_0 + 6W_1 - 3W_2)H_{n-1}.$

(e) $8W_n=-(3W_0+6W_1-5W_2)H_n+2(2W_0+5W_1-3W_2)H_{n-1}+(3W_0+4W_1-3W_2)H_{n-2}.$

(f) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)H_n=(W_0^2+10W_1^2+7W_2^2+8W_0W_1-6W_0W_2-20W_1W_2)W_{n+4}-2(3W_0^2+14W_1^2+11W_2^2+14W_0W_1-12W_0W_2-30W_1W_2)W_{n+3}+(7W_0^2+14W_1^2+13W_2^2+20W_0W_1-22W_0W_2-32W_1W_2)W_{n+2}.$

(g) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)H_n=-(3W_0^2-2W_1^2+W_2^2+4W_0W_1-6W_0W_2)W_{n+3}+2(3W_0^2+2W_1^2+3W_2^2+6W_0W_1-8W_0W_2-6W_1W_2)W_{n+2}-(W_0^2+10W_1^2+7W_2^2+8W_0W_1-6W_0W_2-20W_1W_2)W_{n+1}.$

(h) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)H_n=-(3W_0^2-10W_1^2-3W_2^2-2W_0W_2+12W_1W_2)W_{n+2}+2(W_0^2-6W_1^2-3W_2^2-2W_0W_1+10W_1W_2)W_{n+1}+(3W_0^2-2W_1^2+W_2^2+4W_0W_1-6W_0W_2)W_n.$

(i) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)H_n=(-7W_0^2+18W_1^2+3W_2^2-4W_0W_1+6W_0W_2-16W_1W_2)W_{n+1}+2(3W_0^2-6W_1^2-W_2^2+2W_0W_1-4W_0W_2+6W_1W_2)W_n+(3W_0^2-10W_1^2-3W_2^2-2W_0W_2+12W_1W_2)W_{n-1}.$

(j) $(W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)H_n=(-15W_0^2+42W_1^2+7W_2^2-8W_0W_1+10W_0W_2-36W_1W_2)W_n+2(5W_0^2-14W_1^2-3W_2^2+2W_0W_1-4W_0W_2+14W_1W_2)W_{n-1}+(7W_0^2-18W_1^2-3W_2^2+4W_0W_1-6W_0W_2+16W_1W_2)W_{n-2}.$

Now, we give a few basic relations between $\{J_n\}$ and $\{H_n\}$.

Lemma 8. *The following equalities are true*

$$\begin{aligned}
8J_n &= H_{n+4}-5H_{n+2},\\
8J_n &= 3H_{n+3}-6H_{n+2}-H_{n+1},\\
8J_n &= 3H_{n+2}-4H_{n+1}-3H_n,\\
8J_n &= 5H_{n+1}-6H_n-3H_{n-1},\\
8J_n &= 9H_n-8H_{n-1}-5H_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
H_n &= -13J_{n+4}+46J_{n+3}-35J_{n+2},\\
H_n &= 7J_{n+3}-22J_{n+2}+13J_{n+1},\\
H_n &= -J_{n+2}+6J_{n+1}-7J_n,\\
H_n &= 3J_{n+1}-6J_n+J_{n-1},\\
H_n &= 3J_n-2J_{n-1}-3J_{n-2}.
\end{aligned}$$

5 Relations Betwen Special Numbers

In this section, we present identities on John, John-Lucas numbers and Pell, Pell-Lucas numbers. We know that

$$\begin{aligned} J_n &= \frac{1}{2}(P_{n+2}-P_{n+1}-1), \\ H_n &= Q_n+1. \end{aligned}$$

We also note that from Lemma 8, we have

$$\begin{aligned} 8J_n &= 3H_{n+2}-4H_{n+1}-3H_n, \\ H_n &= -J_{n+2}+6J_{n+1}-7J_n. \end{aligned}$$

Using the above identities we see that

$$\begin{aligned} J_n &= \frac{1}{4}(Q_{n+1}-2), \\ H_n &= 2P_{n+1}-2P_n+1. \end{aligned} \tag{5.1}$$

Using the above identities (and Lemma 6), we obtain the following Binet's formula of generalized John numbers in the following forms:

$$\begin{aligned} W_n &= (3W_1-W_0-W_2)J_{n+2}+(4W_0-9W_1+3W_2)J_{n+1}+(4W_1-4W_0-W_2)J_n \\ &= \frac{1}{4}((W_1-W_0)Q_{n+1}+(W_2-3W_1+2W_0)Q_n-2W_2+4W_1+2W_0) \\ &= \frac{1}{2}((W_2-2W_1+W_0)P_{n+1}-(W_2-4W_1+3W_0)P_n-W_2+2W_1+W_0). \end{aligned}$$

6 On the Recurrence Properties of Generalized John Sequence

Taking $r=3, s=-1, t=-1$ in Soykan [18, Theorem 2], we obtain the following Proposition.

Proposition 9. *For $n\in\mathbb{Z}$, generalized John numbers (the case $r=3, s=-1, t=-1$) have the following identity:*

$$W_{-n}=(-1)^{-n}(W_{2n}-H_nW_n+\frac{1}{2}(H_n^2-H_{2n})W_0).$$

From the above Proposition and Corollary 6 in [18], we have the following corollary which gives the connection between the special cases of generalized John sequence at the positive index and the negative index: for John and John-Lucas numbers: take $W_n=J_n$ with $J_0=0, J_1=1, J_2=3$ and take $W_n=H_n$ with $H_0=3, H_1=3, H_2=7$, respectively.

Corollary 10. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a) *John sequence:*

$$J_{-n} = (-1)^{-n}(J_{2n} - H_n J_n).$$

(b) *John-Lucas sequence:*

$$H_{-n} = \frac{1}{2(-1)^n}(H_n^2 - H_{2n}).$$

By using the identity $H_n = -J_{n+2} + 6J_{n+1} - 7J_n$ (and Proposition 9 or Corollary 10), we get

$$J_{-n} = (-1)^{-n}(J_{2n} + J_n J_{n+2} - 6J_n J_{n+1} + 7J_n^2).$$

Note also that since $J_n = \frac{1}{2}(P_{n+2} - P_{n+1} - 1)$ and $P_{-n} = (-1)^{n+1}P_n$, we get

$$J_{-n} = \frac{1}{2}((-1)^{n-1}(P_{n-1} + P_{n-2}) - 1)$$

and since $H_n = Q_n + 1$ and $Q_{-n} = (-1)^n Q_n$ we obtain

$$H_{-n} = (-1)^n Q_n + 1.$$

7 Sums

The following Corollary gives sum formulas of Pell and Pell-Lucas numbers.

Corollary 11. *For $n \geq 0$, Pell and Pell-Lucas numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} P_k = \frac{1}{2}(3P_n + P_{n-1} - 1)$.

(b) $\sum_{k=0}^{n} P_{2k} = \frac{1}{4}(5P_{2n} - P_{2n-2} - 2)$.

(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{4}(5P_{2n+1} - P_{2n-1})$.

2.

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{2}(3Q_n + Q_{n-1})$.

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{4}(5Q_{2n} - Q_{2n-2} + 4)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{4}(5Q_{2n+1} - Q_{2n-1} - 4)$.

Proof. It is given in Soykan [19, Corollary 4.9]. □

The following Corollary presents sum formulas of John and John-Lucas numbers.

Corollary 12. *For $n \geq 0$, John and John-Lucas numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} J_k = \frac{1}{2}(P_{n+2} - n - 2)$.

(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{4}(3P_{2n+1} + P_{2n} - 2n - 3)$.

(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{4}(7P_{2n+1} + 3P_{2n} - 2n - 3)$

2.

(a) $\sum_{k=0}^{n} H_k = \frac{1}{2}(Q_{n+1} + Q_n + 2(n+1))$.

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{2}(Q_{2n+1} + 2\,(n+2))$.

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{2}(Q_{2n+2} + 2n)$.

Proof. The proof follows from Corollary 11 and the identities (2.5) and (2.6), i.e.,

$$\begin{aligned} J_n &= \frac{1}{2}(P_{n+2} - P_{n+1} - 1), \\ H_n &= Q_n + 1. \quad \square \end{aligned}$$

8 Matrices Related With Generalized John numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{8.1}$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -1$. **From (2.1) we have**

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{8.2}$$

and from (8.1) (or using (8.2) and induction) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W = J$ in (8.2) we have

$$\begin{pmatrix} J_{n+2} \\ J_{n+1} \\ J_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} J_{n+1} \\ J_n \\ J_{n-1} \end{pmatrix}. \tag{8.3}$$

We also define

$$B_n = \begin{pmatrix} J_{n+1} & -J_n - J_{n-1} & -J_n \\ J_n & -J_{n-1} - J_{n-2} & -J_{n-1} \\ J_{n-1} & -J_{n-2} - J_{n-3} & -J_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -W_n - W_{n-1} & -W_n \\ W_n & -W_{n-1} - W_{n-2} & -W_{n-1} \\ W_{n-1} & -W_{n-2} - W_{n-3} & -W_{n-2} \end{pmatrix}.$$

Theorem 13. *For all integers m, n, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 3, s = -1, t = -1$ in Soykan [17, Theorem 5.1.]. □

Some properties of matrix A^n can be given as

$$A^n = 3A^{n-1} - A^{n-2} - A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-1)^n$$

for all integer m and n.

Corollary 14. *For all integers n, we have the following formulas for John and John-Lucas numbers.*

(a) *John Numbers.*

$$A^n = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} J_{n+1} & -J_n - J_{n-1} & -J_n \\ J_n & -J_{n-1} - J_{n-2} & -J_{n-1} \\ J_{n-1} & -J_{n-2} - J_{n-3} & -J_{n-2} \end{pmatrix}.$$

(b) *John-Lucas Numbers.*

$$A^{n}=\frac{1}{8}\begin{pmatrix} 3H_{n+3}-4H_{n+2}-3H_{n+1} & -6H_{n+2}+10H_{n+1}+4H_{n} & -3H_{n+2}+4H_{n+1}+3H_{n} \\ 3H_{n+2}-4H_{n+1}-3H_{n} & -6H_{n+1}+10H_{n}+4H_{n-1} & -3H_{n+1}+4H_{n}+3H_{n-1} \\ 3H_{n+1}-4H_{n}-3H_{n-1} & -6H_{n}+10H_{n-1}+4H_{n-2} & -3H_{n}+4H_{n-1}+3H_{n-2} \end{pmatrix}.$$

Proof.

(a) It is given in Theorem 13 (a).

(b) Note that, from Lemma 8, we have

$$8J_{n}=3H_{n+2}-4H_{n+1}-3H_{n}.$$

Using the last equation and (a), we get required result. □

Using the above last Corollary and the identities (2.5) and (2.6), i.e.,

$$\begin{aligned} J_{n} &= \frac{1}{2}(P_{n+2}-P_{n+1}-1), \\ H_{n} &= Q_{n}+1, \end{aligned}$$

we obtain the following identities for Pell and Pell-Lucas numbers.

Corollary 15. *For all integers n, we have the following formulas for Pell and Pell-Lucas numbers.*

(a) *Pell Numbers.*

$$A^{n}=\begin{pmatrix} \frac{1}{2}(P_{n+3}-P_{n+2}-1) & \frac{1}{2}(P_{n}-P_{n+2}+2) & \frac{1}{2}(P_{n+1}-P_{n+2}+1) \\ \frac{1}{2}(P_{n+2}-P_{n+1}-1) & \frac{1}{2}(P_{n-1}-P_{n+1}+2) & \frac{1}{2}(P_{n}-P_{n+1}+1) \\ \frac{1}{2}(P_{n+1}-P_{n}-1) & \frac{1}{2}(P_{n-2}-P_{n}+2) & \frac{1}{2}(P_{n-1}-P_{n}+1) \end{pmatrix}.$$

(b) *Pell-Lucas Numbers.*

$$A^{n}=\frac{1}{4}\begin{pmatrix} 2Q_{n+1}+Q_{n}-2 & -Q_{n+1}-Q_{n}+4 & -Q_{n+1}+2 \\ Q_{n+1}-2 & -Q_{n+1}+Q_{n}+4 & -Q_{n}+2 \\ Q_{n}-2 & Q_{n+1}-3Q_{n}+4 & -Q_{n+1}+2Q_{n}+2 \end{pmatrix}.$$

Theorem 16. *For all integers m,n, we have*

$$W_{n+m}=W_{n}J_{m+1}+\left(-W_{n-1}-W_{n-2}\right)J_{m}-W_{n-1}J_{m-1} \tag{8.4}$$

Proof. Take $r=3,s=-1,t=-1$ in Soykan [17, Theorem 5.2.]. □

By Lemma 6, we know that

$$
\begin{aligned}
& (W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)J_m \\
= & -(W_0^2-3W_1^2+W_0W_1+W_1W_2)W_{m+2} \\
& +(W_2^2+W_0W_1+W_0W_2-3W_1W_2)W_{m+1}+(-W_1^2+W_0W_2)W_m
\end{aligned}
$$

so (8.4) can be written in the following form

$$
\begin{aligned}
& (W_0+2W_1-W_2)(W_0^2-2W_1^2-W_2^2-2W_0W_2+4W_1W_2)W_{n+m} \\
= & W_n(-(W_0^2-3W_1^2+W_0W_1+W_1W_2)W_{m+3} \\
& +(W_2^2+W_0W_1+W_0W_2-3W_1W_2)W_{m+2}+(-W_1^2+W_0W_2)W_{m+1}) \\
& +(-W_{n-1}-W_{n-2})\left(-(W_0^2-3W_1^2+W_0W_1\right. \\
& \left.+W_1W_2)W_{m+2}+(W_2^2+W_0W_1+W_0W_2-3W_1W_2)W_{m+1}+(-W_1^2+W_0W_2)W_m\right) \\
& -W_{n-1}(-(W_0^2-3W_1^2+W_0W_1+W_1W_2)W_{m+1} \\
& +(W_2^2+W_0W_1+W_0W_2-3W_1W_2)W_m+(-W_1^2+W_0W_2)W_{m-1})
\end{aligned}
$$

Corollary 17. *For all integers m, n, we have*

$$
\begin{aligned}
J_{n+m} &= J_nJ_{m+1}+(-J_{n-1}-J_{n-2})J_m-J_{n-1}J_{m-1}, \\
H_{n+m} &= H_nJ_{m+1}+(-H_{n-1}-H_{n-2})J_m-H_{n-1}J_{m-1},
\end{aligned}
$$

and

$$
\begin{aligned}
32H_{m+n} = & (12H_{m+3}-16H_{m+2}-12H_{m+1})H_n+(-12H_{m+1}+16H_m+12H_{m-1})H_{n-1} \\
& +(-12H_{m+2}+16H_{m+1}+12H_m)(H_{n-1}+H_{n-2}).
\end{aligned}
$$

Taking $m=n$ in the last corollary, we obtain the following identities:

$$
\begin{aligned}
J_{2n} &= J_nJ_{n+1}-(J_{n-1}+J_{n-2})J_n-J_{n-1}^2, \\
H_{2n} &= H_nJ_{n+1}+(-H_{n-1}-H_{n-2})J_n-H_{n-1}J_{n-1}, \\
32H_{2n} &= (12H_{n+3}-16H_{n+2}-12H_{n+1})H_n+(-12H_{n+1}+16H_n+12H_{n-1})H_{n-1} \\
& +(-12H_{n+2}+16H_{n+1}+12H_n)(H_{n-1}+H_{n-2}).
\end{aligned}
$$

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