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Some New Discrete Inequalities of Opial and Lasota's Type

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Abstract

In this paper, we establish some new discrete inequalities of Opial and Lasota's type which reduce to some inequalities in [4].

Keywords: Hölder's inequality; Opial inequality; Lasota inequality; Forward difference operator; Backward difference operator.

1. Introduction

In this paper, we denote $\{x_i\}_{i=0}^{N}$ by a sequence of real numbers, the operators Δ and ∇ by $\Delta x_i = x_{i+1} - x_i$ and $\nabla x_i = x_i - x_{i-1}$, and $[\cdot]$ by the greatest integer function. The empty sums is taken to be 0.

In 1960, Opial [7] established the following important integral inequality:

Theorem A. *Let $f(x) \in C^1[0,h]$ be such that $f(0) = f(h) = 0$, and $f(x) > 0$ in $(0,h)$. Then*

$$\int_0^h |f(x)f'(x)|dx \le \frac{h}{4}\int_0^h \left(f'(x)\right)^2 dx, \tag{1.1}$$

where $\frac{h}{4}$ is the best possible.

The inequality (1.1) is known in the literature as Opial inequality. For some results which generalize, improve and extend this famous integral inequality (see [1]-[11]).

In [4], Lasota provided discrete versions of Opial inequality (1.1) about the forward difference operator as following:

Theorem B. *Let $\{x_i\}_{i=0}^{N}$ be a sequence of numbers, and $x_0 = x_N = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \le \frac{1}{2}\left[\frac{N+1}{2}\right]\sum_{i=0}^{N-1} (\Delta x_i)^2. \tag{1.2}$$

If N is even, then the inequality (1.2) is sharp.

Also, we have the following three Theorems C-E (see [1]):

Theorem C. *Let* $\{x_i\}_{i=0}^{\tau}$ *be a sequence of numbers, and* $x_0 = 0$. *Then, the following inequality holds*

$$\sum_{i=1}^{\tau-1} |x_i \Delta x_i| \leq \frac{\tau-1}{2} \sum_{i=0}^{\tau-1} (\Delta x_i)^2. \tag{1.3}$$

Theorem D. *Let* $\{x_i\}_{i=\tau}^{N}$ *be a sequence of numbers, and* $x_N = 0$. *Then, the following inequality holds*

$$\sum_{i=\tau}^{N-1} |x_i \Delta x_i| \leq \frac{N-\tau+1}{2} \sum_{i=\tau}^{N-1} (\Delta x_i)^2. \tag{1.4}$$

Theorem E. *Let* $\{x_i\}_{i=0}^{\tau}$ *be a sequence of numbers, and* $x_0 = 0$. *Then, the following inequality holds*

$$\sum_{i=1}^{\tau} |x_i \nabla x_i| \leq \frac{\tau+1}{2} \sum_{i=1}^{\tau} (\nabla x_i)^2. \tag{1.5}$$

We shall establish some new results which are the generalizations of Theorems B-E.

2. Main Results

Throughout this section, let $m, n > 0$ and $c(m,n) = \frac{1}{m+n} \max\{m,n\}$.

We state and prove the following theorems:

Theorem 1. *Let* $\{x_i\}_{i=0}^{l}$ *be a sequence of real numbers with* $x_0 = 0$. *Then we have the following inequality*

$$\sum_{i=1}^{l} |x_i| |\Delta x_i|^m \leq c(m,1) l \sum_{i=0}^{l} |\Delta x_i|^{m+1}. \tag{2.1}$$

Proof. Since $x_0 = 0$, we have the following identity

$$x_i = \sum_{j=0}^{i-1} \Delta x_j, \quad i = 1,2,\dots,l. \tag{2.2}$$

Hence the following inequality holds

$$\sum_{i=1}^{l} |x_i| |\Delta x_i|^m \leq \sum_{i=1}^{l} \left[\sum_{j=0}^{i-1} |\Delta x_j| |\Delta x_i|^m \right]. \tag{2.3}$$

Using the inequality

$$\alpha\beta^m \leq \frac{1}{m+1} \alpha^{m+1} + \frac{m}{m+1} \beta^{m+1} (\alpha, \beta > 0) \tag{2.4}$$

and the definition of $c(m,1)$, we have the following inequality

$$\begin{aligned} &\sum_{i=1}^{l}\left[\sum_{j=0}^{i-1}|\Delta x_j||\Delta x_i|^m\right] \qquad (2.5)\\ &\le \sum_{i=1}^{l}\left[\sum_{j=0}^{i-1}\left(\frac{|\Delta x_j|^{m+1}}{m+1}+\frac{m}{m+1}|\Delta x_i|^{m+1}\right)\right]\\ &\le c(m,1)\sum_{i=1}^{l}\left(\sum_{j=0}^{i-1}|\Delta x_j|^{m+1}+i|\Delta x_i|^{m+1}\right)\\ &= c(m,1)\sum_{i=0}^{l}[(l-i)|\Delta x_i|^{m+1}+i|\Delta x_i|^{m+1}]\\ &= c(m,1)l\sum_{i=0}^{l}|\Delta x_i|^{m+1}. \end{aligned}$$

Using the inequalities (2.3) and (2.5), we derive the inequality (2.1). This completes the proof.

Remark 1. *In the inequality* (2.1)*, let* $m = 1$ *and* $l = \tau - 1$*. Then the inequality* (2.1) *reduces to the inequality* (1.3)*.*

Theorem 2. *Let* $\{x_i\}_{i=0}^{l}$ *be a sequence of real numbers with* $x_0 = 0$*. Then we have the following inequality*

$$\sum_{i=0}^{l-1}|x_{i+1}||\Delta x_i|^m \le c(m,1)(l+1)\sum_{i=0}^{l-1}|\Delta x_i|^{m+1}. \qquad (2.6)$$

Proof. Since $x_0 = 0$, we have the following identity

$$x_{i+1} = \sum_{j=0}^{i}\Delta x_j, \quad i = 0,1,\dots,l-1. \qquad (2.7)$$

Using the inequality (2.4), the identity (2.7) and the definition of $c(m,n)$, we have the following inequality

$$\begin{aligned} \sum_{i=0}^{l-1}|x_{i+1}||\Delta x_i|^m &\le \sum_{i=0}^{l-1}\left[\sum_{j=0}^{i}|\Delta x_j||\Delta x_i|^m\right]\\ &\le \sum_{i=0}^{l-1}\left[\sum_{j=0}^{i}\left(\frac{|\Delta x_j|^{m+1}}{m+1}+\frac{m}{m+1}|\Delta x_i|^{m+1}\right)\right]\\ &\le c(m,1)\sum_{i=0}^{l-1}\left(\sum_{j=0}^{i}|\Delta x_j|^{m+1}+(i+1)|\Delta x_i|^{m+1}\right) \end{aligned}$$

$$= c(m,1)\sum_{i=0}^{l-1}[(l-i)|\Delta x_i|^{m+1} + (i+1)|\Delta x_i|^{m+1}]$$

$$= c(m,1)(l+1)\sum_{i=0}^{l-1}|\Delta x_i|^{m+1}$$

which is the inequality (2.6). This completes the proof.

Theorem 3. *Let* $\{x_i\}_{i=l+1}^{N}$ *be a sequence of real numbers with* $x_N = 0$. *Then we have the following inequality*

$$\sum_{i=l+1}^{N-1}|x_i||\Delta x_i|^m \le c(m,1)(N-l)\sum_{i=l+1}^{N-1}|\Delta x_i|^{m+1}. \tag{2.8}$$

Proof. Let $x_i = y_{N-i}$ where $i = l+1, l+2, \dots, N$. Then

$$\Delta x_i = -\Delta y_{N-i-1} \text{ and } y_0 = 0$$

where $i = l+1, l+2, \dots, N-1$. Using the inequality (2.6), we have the following inequality

$$\sum_{i=l+1}^{N-1}|x_i||\Delta x_i|^m = \sum_{i=0}^{N-l-2}|y_{i+1}||\Delta y_i|^m$$

$$\le c(m,1)(N-l)\sum_{i=0}^{N-l-2}|\Delta y_i|^{m+1}$$

$$= c(m,1)(N-l)\sum_{i=l+1}^{N-1}|\Delta x_i|^{m+1}$$

which is the inequality (2.8). This completes the proof.

Remark 2. *In the inequality* (2.8), *let* $m = 1$ *and* $l = \tau - 1$. *Then the inequality* (2.8) *reduces to the inequality* (1.4).

Theorem 4. *Let* $\{x_i\}_{i=0}^{N}$ *be a sequence of real numbers with* $x_0 = x_N = 0$. *Then we have the following inequality*

$$\sum_{i=1}^{N-1}|x_i||\Delta x_i|^m \le c(m,1)\left[\frac{N+1}{2}\right]\sum_{i=0}^{N-1}|\Delta x_i|^{m+1}, \tag{2.9}$$

where $\left[\frac{N+1}{2}\right]$ *is the Gaussian integer of* $\frac{N+1}{2}$.

If N *is even, then the inequality* (2.9) *is sharp.*

Proof. (1) Let $l = \left[\frac{N+1}{2}\right]$. Then $N - l \le l$. Using the inequalities (2.1) and (2.8), we have the following inequality

$$\sum_{i=1}^{N-1}|x_i||\Delta x_i|^m = \sum_{i=1}^{l}|x_i||\Delta x_i|^m + \sum_{i=l+1}^{N-1}|x_i||\Delta x_i|^m$$
$$\leq c(m,1)\left\{l\sum_{i=0}^{l}|\Delta x_i|^{m+1} + (N-l)\sum_{i=l+1}^{N-1}|\Delta x_i|^{m+1}\right\}$$
$$\leq c(m,1)l\sum_{i=0}^{N-1}|\Delta x_i|^{m+1}$$

which is the inequality (2.9).

(2) Suppose $m = 1$ and N is even. Then $\frac{1}{2}\left[\frac{N+1}{2}\right] = \frac{N}{4}$. Let

$$x_i = \frac{1}{2}N - \left|i - \frac{1}{2}N\right| \,(0 \leq i \leq N-1).$$

Then, we have

$$x_0 = x_N = 0, |\Delta x_i| = 1 \;\; (0 \leq i \leq N-1),$$

$$\sum_{i=1}^{N-1}|x_i\Delta x_i| = \frac{1}{4}N^2 \text{ and } \sum_{i=0}^{N-1}|\Delta x_i|^2 = N.$$

Hence, the equality holds in the inequality (2.9) and from which the inequality (2.9) is sharp. This completes the proof.

Remark 3. *In the inequality* (2.9), *let* $m = 1$. *Then the inequality* (2.9) *reduces to the inequality* (1.2).

Theorem 5. *Let* $\{x_i\}_{i=0}^{l}$ *be a sequence of real numbers with* $x_0 = 0, n > 1$. *Then we have the following inequality*

$$\sum_{i=1}^{l}|x_i|^n|\Delta x_i|^m \leq c(m,1)l^n\sum_{i=0}^{l}|\Delta x_i|^{m+n}. \tag{2.10}$$

Proof. Using the identity (2.2), the Hölder's inequality with indices $n/(n-1)$,n, and the inequality

$$\alpha^n\beta^m \leq \frac{n}{m+n}\alpha^{m+n} + \frac{m}{m+n}\beta^{m+n} \leq c(m,n)(\alpha^{m+n} + \beta^{m+n}) \tag{2.11}$$

where $\alpha, \beta > 0$, we have the following inequality

$$\sum_{i=1}^{l}|x_i|^n|\Delta x_i|^m \leq \sum_{i=1}^{l}\left[\left(\sum_{j=0}^{i-1}|\Delta x_j|\right)^n|\Delta x_i|^m\right]$$

$$\leq \sum_{i=1}^{l}\left[\left(\sum_{j=0}^{i-1} 1\right)^{\frac{n-1}{n}}\left(\sum_{j=0}^{i-1}|\Delta x_j|^n\right)^{\frac{1}{n}}\right]^n |\Delta x_i|^m$$

$$= \sum_{i=1}^{l} i^{n-1}\sum_{j=0}^{i-1}|\Delta x_j|^n|\Delta x_i|^m$$

$$\leq l^{n-1}\sum_{i=1}^{l}\sum_{j=0}^{i-1}|\Delta x_j|^n|\Delta x_i|^m$$

$$\leq c(m,n)l^{n-1}\sum_{i=1}^{l}\left[\sum_{j=0}^{i-1}\left(|\Delta x_j|^{m+n}+|\Delta x_i|^{m+n}\right)\right]$$

$$= c(m,n)l^{n-1}\sum_{i=0}^{l}[(l-i)|\Delta x_i|^{m+n}+i|\Delta x_i|^{m+n}]$$

$$= c(m,n)l^{n}\sum_{i=0}^{l}|\Delta x_i|^{m+n}$$

which is the inequality (2.10). This completes the proof.

Remark 4. *In the inequality* (2.10), *let* $n \to 1^+$. *Then the inequality* (2.10) *reduces to the inequality* (2.1).

Theorem 6. *Let* $\{x_i\}_{i=0}^{l}$ *be a sequence of real numbers with* $x_0 = 0, n > 1$. *Then we have the following inequality*

$$\sum_{i=0}^{l-1}|x_{i+1}|^n|\Delta x_i|^m \leq c(m,n)l^{n-1}(l+1)\sum_{i=0}^{l-1}|\Delta x_i|^{m+n}. \tag{2.12}$$

Proof. Using the identity (2.7), the Hölder's inequality with indices $n/(n-1), n$, and the inequality (2.11), we have the following inequality

$$\sum_{i=1}^{l-1}|x_{i+1}|^n|\Delta x_i|^m \leq \sum_{i=0}^{l-1}\left[\left(\sum_{j=0}^{i}|\Delta x_j|\right)^n |\Delta x_i|^m\right]$$

$$\leq \sum_{i=0}^{l-1}\left[\left(\sum_{j=0}^{i} 1\right)^{\frac{n-1}{n}}\left(\sum_{j=0}^{i}|\Delta x_j|^n\right)^{\frac{1}{n}}\right]^n |\Delta x_i|^m$$

$$= \sum_{i=0}^{l-1}(i+1)^{n-1}\sum_{j=0}^{i}|\Delta x_j|^n|\Delta x_i|^m$$

$$\leq l^{n-1}\sum_{i=0}^{l-1}\sum_{j=0}^{i}|\Delta x_j|^n|\Delta x_i|^m$$

$$\leq c(m,n)l^{n-1}\sum_{i=0}^{l-1}\left[\sum_{j=0}^{i}\left(|\Delta x_j|^{m+n}+|\Delta x_i|^{m+n}\right)\right]$$

$$= c(m,n)l^{n-1}\sum_{i=0}^{l-1}[(l-i)|\Delta x_i|^{m+n} + (i+1)|\Delta x_i|^{m+n}]$$

$$= c(m,n)l^{n-1}(l+1)\sum_{i=0}^{l-1}|\Delta x_i|^{m+n}$$

which is the inequality (2.12). This completes the proof.

Remark 5. *In the inequality* (2.12), *let* $n \to 1^+$. *Then the inequality* (2.12) *reduces to the inequality* (2.6).

Theorem 7. *Let* $\{x_i\}_{i=l+1}^{N}$ *be a sequence of real numbers with* $x_N = 0, n > 1$. *Then we have the following inequality*

$$\sum_{i=l+1}^{N-1}|x_i|^n|\Delta x_i|^m \le c(m,n)(N-l-1)^{n-1}(N-l)\sum_{i=l+1}^{N-1}|\Delta x_i|^{m+n}. \tag{2.13}$$

Proof. Let $x_i = y_{N-i}$ where $i = l+1, l+2, \dots, N$. Then

$$\Delta x_i = -\Delta y_{N-i-1} \text{ and } y_0 = 0$$

where $i = l+1, l+2, \dots, N-1$. Using the inequality (2.12), we have the following inequality

$$\sum_{i=l+1}^{N-1}|x_i|^n|\Delta x_i|^m = \sum_{i=0}^{N-l-2}|y_{i+1}|^n|\Delta y_i|^m$$

$$\le c(m,n)(N-l-1)^{n-1}(N-l)\sum_{i=0}^{N-l-2}|\Delta y_i|^{m+n}$$

$$= c(m,n)(N-l-1)^{n-1}(N-l)\sum_{i=l+1}^{N-1}|\Delta x_i|^{m+n}$$

which is the inequality (2.13). This completes the proof.

Remark 6. *In the inequality* (2.13), *let* $n \to 1^+$. *Then the inequality* (2.13) *reduces to the inequality* (2.8).

Theorem 8. *Let* $\{x_i\}_{i=0}^{N}$ *be a sequence of real numbers with* $x_0 = x_N = 0, n > 1$. *Then we have the following inequality*

$$\sum_{i=1}^{N-1}|x_i|^n|\Delta x_i|^m \le c(m,n)\left[\frac{N+1}{2}\right]^n\sum_{i=0}^{N-1}|\Delta x_i|^{m+n}. \tag{2.14}$$

Proof. Let $l = \left[\frac{N+1}{2}\right]$. Then $N-l-1 \le N-l \le l$. Using the inequalities (2.10) and (2.13), we have the following inequality

$$\sum_{i=1}^{N-1}|x_i|^n|\Delta x_i|^m = \sum_{i=1}^{l}|x_i|^n|\Delta x_i|^m + \sum_{i=l+1}^{N-1}|x_i|^n|\Delta x_i|^m$$

$$\leq c(m,n)\left[l^n \sum_{i=0}^{l} |\Delta x_i|^{m+n} + (N-l-1)^{n-1}(N-l) \sum_{i=l+1}^{N-1} |\Delta x_i|^{m+n}\right]$$

$$\leq c(m,n) l^n \sum_{i=0}^{N-1} |\Delta x_i|^{m+n}$$

which is the inequality (2.14). This completes the proof.

Under the conditions of Theorems 2 and 6, we have the following corollaries and remarks.

Corollary 1. *Let* $\{x_i\}_{i=0}^{l}$ *be a sequence of real numbers with* $x_0 = 0$. *Then we have the following inequality*

$$\sum_{i=1}^{l} |x_i||\nabla x_i|^m \leq c(m,1)(l+1) \sum_{i=1}^{l} |\nabla x_i|^{m+1}. \tag{2.15}$$

Proof. Since

$$\sum_{i=1}^{l} |x_i||\nabla x_i|^m = \sum_{i=0}^{l-1} |x_{i+1}||\Delta x_i|^m,$$

it follows from the inequality (2.6) that

$$\sum_{i=1}^{l} |x_i||\nabla x_i|^m \leq c(m,1)(l+1) \sum_{i=0}^{l-1} |\Delta x_i|^{m+1} = c(m,1)(l+1) \sum_{i=1}^{l} |\nabla x_i|^{m+1}$$

which is the inequality (2.15). This completes the proof.

Remark 7. *In the inequality* (2.15), *let* $m = 1$ *and* $l = \tau$. *Then the inequality* (2.15) *reduces to the inequality* (1.5).

Corollary 2. *Let* $\{x_i\}_{i=0}^{l}$ *be a sequence of real numbers with* $x_0 = 0, n > 1$. *Then we have the following inequality*

$$\sum_{i=1}^{l} |x_i|^n |\nabla x_i|^m \leq c(m,1)(l+1)^n \sum_{i=1}^{l} |\nabla x_i|^{m+n}. \tag{2.16}$$

Proof. Since

$$\sum_{i=1}^{l} |x_i|^n |\nabla x_i|^m = \sum_{i=0}^{l-1} |x_{i+1}|^n |\Delta x_i|^m,$$

it follows from the inequality (2.12) that

$$\sum_{i=1}^{l}|x_i|^n|\nabla x_i|^m \leq c(m,n)l^{n-1}(l+1)\sum_{i=0}^{l-1}|\Delta x_i|^{m+n}$$
$$\leq c(m,n)(l+1)^n\sum_{i=0}^{l-1}|\Delta x_i|^{m+n}$$
$$= c(m,n)(l+1)^n\sum_{i=1}^{l}|\nabla x_i|^{m+n}$$

which is the inequality (2.16). This completes the proof.

Remark 8. *For* $x_i = i, 0 \leq i \leq l$ *in the inequality* (2.16), *we note that*

$$\sum_{i=1}^{l} i^n \leq c(m,n)(l+1)^n l$$
$$< max\{m,n\}\frac{(l+1)^{n+1}-1}{n+1}$$
$$< max\{m,n\}\int_1^{l+1} t^n dt,$$

which shows that the inequality (2.16) gives a better estimate than that obtained by simply comparing areas. Moreover, for $n \to 1^+$, *this gives the well-known identity* $\sum_{i=1}^{l} i = l(l+1)/2$.

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