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Stability and Oscillation of θ-methods for Differential Equation with Piecewise Constant Arguments

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Abstract

This paper studies the numerical properties of θ-methods for the alternately advanced and retarded differential equation $u'(t) = au(t)+bu(2[(t+1)/2])$. Using two classes of θ-methods, namely the linear θ-method and the one-leg θ-method, the stability regions of numerical methods are determined, and the conditions of oscillation for the θ-methods are derived. Moreover, we give the conditions under which the numerical stability regions contain the analytical stability regions. It is shown that the θ-methods preserve the oscillation of the analytic solution. In addition, the relationships between stability and oscillation are presented. Several numerical examples are given.

Keywords

θ-methods, Stability, Oscillation, Piecewise Constant Arguments.

1. Introduction

In the last few decades there has been an increasing interest in the study of differential equation with piecewise constant arguments (EPCA). EPCA describe hybrid dynamical systems, combine properties of both differential and difference equations. For instance, the equation [12] $u'(t) = u(t)(r - \sum_{j=0}^{m} d_j u([t-j]))$ can be regarded as a semi-discretization [7, 5] of logistic equation with multi-delay $u'(t) = u(t)(r - \sum_{j=0}^{m} d_j u(t-\tau_j)$.

EPCA have been introduced by Shah and Wiener [14], Cooke and Wiener [3]. They have many applications in biomedical models [1] and the stabilization of hybrid control systems with feedback discrete controller [8]. So more and more research activities on EPCA are carried out. There exists an extensive literature dealing with EPCA, such as the stability [2], the oscillation [21], the periodicity [17] and contractivity [13]. The general theory and basic results for EPCA have been thoroughly developed in the book of Wiener [22].

In addition to the research on the qualitative property of EPCA, much studies have been concentrated on the numerical analysis of EPCA recently. The convergence and the stability of numerical methods for

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a variety of EPCA have been addressed in [16, 23, 19, 9, 24]. In [10, 11], oscillation of numerical solutions for the same equation $x'(t) + ax(t) + a_1 x([t-1]) = 0$ are investigated, respectively. However, until now the author are not aware of any published results on the stability and oscillation of numerical solutions of EPCA simultaneously except for [18, 15]. In [18, 15], EPCA with $[t + 1/2]$ and $2[(t + 1)/2]$ were considered, respectively. Different from them, in our paper, we consider both stability and oscillation of the θ-methods for EPCA with $2[(t + 1)/2]$, and discuss their relationships quantitatively.

Consider the EPCA

$$u'(t) = au(t) + bu\left(2\left[\frac{t+1}{2}\right]\right), \quad u(0) = u_0, \tag{1}$$

where a, b, u_0 are real constants and u_0 is a given initial value. Here as usual $[\cdot]$ denotes the greatest integer function. Equations like (1) have stimulated great interest and have been considered by Cooker and Wiener [4], Jayasree and Deo [6], Wiener and Aftabizadeh [20]. In (1) the argument deviation $t - 2[(t + 1)/2]$ is a piecewise linear periodic function. Moreover, $T(t)$ is negative for $t \in [2n - 1, 2n)$ and positive for $t \in [2n, 2n + 1)$. Therefore, (1) is advanced type on $[2n - 1, 2n)$ and retarded type on $[2n, 2n + 1)$. So (1) is EPCA of alternately advanced and retarded type. The aim of the current paper is to study the stability and oscillation of the numerical solutions in the θ-methods for (1). The problems of numerical method preserves stability and oscillation and the relationships between stability and oscillation are also investigated.

2. Stability and Oscillation of Analytic Solution

For convenience, we give some known results that are required later.

Definition 1. *[22] A solution of (1) on $[0, \infty)$ is a function $u(t)$ which satisfies the following conditions :*

(i) *$u(t)$ is continuous on $[0, \infty)$,*

(ii) *The derivative $u'(t)$ exists at each point t in $[0, \infty)$, with the possible exception of the points $t = 2n - 1$ for $n \in \mathbb{N}$, where one-sided derivatives exist,*

(iii) *(1) is satisfied on each interval $[2n - 1, 2n + 1)$ for $n \in \mathbb{N}$.*

Theorem 1. *[22] Equation (1) has on $[0, \infty)$ a unique solution $u(t)$ given by*

$$\begin{aligned} u(t) &= \Omega_1(T(t))\left(\tfrac{\Omega_1(1)}{\Omega_1(-1)}\right)^{[\frac{t+1}{2}]} u_0, \quad a \neq 0, \\ u(t) &= \Omega_2(T(t))\left(\tfrac{\Omega_2(1)}{\Omega_2(-1)}\right)^{[\frac{t+1}{2}]} u_0, \quad a = 0, \end{aligned} \tag{2}$$

where

$$\Omega_1(t) = e^{at} + (e^{at} - 1)a^{-1}b, \Omega_2(t) = 1 + bt, T(t) = t - 2\left[\frac{t+1}{2}\right].$$

Theorem 2. *[22] The solution $u(t) = 0$ of (1) is asymptotically stable if and only if any one of the following conditions is satisfied:*

$$\begin{aligned} &-\tfrac{a(e^{2a}+1)}{(e^a-1)^2} < b < -a, \quad \textit{for} \quad a > 0, \\ &b > -\tfrac{a(e^{2a}+1)}{(e^a-1)^2}, \textit{or} \quad b < -a, \quad \textit{for} \quad a < 0, \\ &b < 0, \quad \textit{for} \quad a = 0. \end{aligned} \tag{3}$$

Definition 2. *A nontrivial solution of (1) is said to be oscillatory if there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ as $k \to \infty$ and $u(t_k)u(t_{k-1}) \leq 0$. Otherwise, it is called non-oscillatory. We say (1) is oscillatory if all nontrivial solutions of (1) are oscillatory. We say (1) is non-oscillatory if all nontrivial solutions of (1) are non-oscillatory.*

Theorem 3. *[22] A necessary and sufficient condition for all solutions of (1) to be oscillatory is*

$$\begin{aligned} &b < -\tfrac{ae^a}{e^a-1} \quad \textit{or} \quad b > \tfrac{a}{e^a-1}, \quad a \neq 0, \\ &b < -1 \quad \textit{or} \quad b > 1, \quad a = 0. \end{aligned}$$

3. Stability and Oscillation of the θ-Methods

3.1. The Difference Scheme and Convergence

Let $h = 1/m$ be a given stepsize with integer $m \geq 1$ and the gridpoints t_n be defined by $t_n = nh$ $(n = 1, 2, 3, \cdots)$. We consider the application of the linear θ-method to (1),

$$u_{n+1} = u_n + h\left\{\theta\left(au_{n+1} + bu^h(2[\frac{(n+1)h+1}{2}])\right) + (1-\theta)\left(au_n + bu^h(2[\frac{nh+1}{2}])\right)\right\} \tag{4}$$

and the one-leg θ-method to (1),

$$u_{n+1} = u_n + h\left\{a\left(\theta u_{n+1} + (1-\theta)u_n\right) + bu^h\left(2\left[\frac{(n+\theta)h+1}{2}\right]\right)\right\}, \tag{5}$$

where $0 \leq \theta \leq 1$ is a parameter, u_n is approximation to the exact solution $u(t)$ at the gridpoints t_n $(n = 1, 2, 3, \cdots)$, $u^h(2[(nh+1)/2])$ and $u^h(2[((n+1)h+1)/2])$ are approximations to $u(2[(t+1)/2])$ at t_n and t_{n+1}, respectively. Let $n = 2km + l, l = -m, -m+1, \cdots, m-2, m-1$ for $k \geq 1$ and $l = 0, 1, \cdots, m-1$ for $k = 0$. Then $u^h(2[(nh+1)/2])$, $u^h(2[((n+1)h+1)/2])$ and $u^h(2[((n+\theta)h+1)/2])$ can be defined as u_{2km} according to Definition 1. As a result, relations (4) and (5) are reduced to the same recurrence relation

$$\begin{aligned} &u_{2km+l+1} = R(z)u_{2km+l} + \tfrac{b}{a}(R(z)-1)u_{2km}, \quad a \neq 0, \\ &u_{2km+l+1} = u_{2km+l} + hbu_{2km}, \quad a = 0, \end{aligned} \tag{6}$$

where $z = ha$ and $R(z) = 1 + z/(1-\theta z)$ is the stability function of the θ-methods.

In fact, in each interval $[2n-1, 2n+1)$, (1) can be seen as ordinary differential equation, so the convergence of the θ-methods is obtained.

Theorem 4. *[16] The θ-methods applied to (1) are of order 1 when $\theta \neq 1/2$ and of order 2 when $\theta = 1/2$.*

3.2. Numerical Stability

In the rest of the paper, we always suppose $h < 1/|a|$, then it follows from (6) that

$$\begin{aligned} &u_{2km+l} = (1 + hlb)u_{2km}, \\ &u_{2(k+1)m} = \tfrac{1+b}{1-b}u_{2km}, \end{aligned}$$

for $a = 0$ and

$$u_{2km+l} = W(l)u_{2km}, \tag{7}$$

$$u_{2(k+1)m} = \lambda u_{2km}, \tag{8}$$

for $a \neq 0$, where

$$W(l) = R(z)^l + \frac{b}{a}(R(z)^l - 1), \lambda = \frac{W(m)}{W(-m)}.$$

Definition 3. *The θ-methods are called asymptotically stable at (a, b) if there exists a constant M such that u_n defined by relation (4) or (5) tends to zero as $n \to \infty$ for all $h = 1/m$ and any given u_0.*

Definition 4. *The set of all points (a, b) at which the θ-methods are asymptotically stable is called the asymptotic stability region denoted by S.*

Hence we have the following theorem for stability.

Theorem 5. *The θ-methods are asymptotically stable if any one of the following conditions is satisfied:*

$$\begin{array}{l} -\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2} < b < -a, \quad \textit{for} \quad a > 0, \\ b < -a \quad \textit{or} \quad b > -\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2}, \quad \textit{for} \quad a < 0, \\ b < 0, \quad \textit{for} \quad a = 0. \end{array} \tag{9}$$

Proof. It is easy to see from (7) and (8) that $u_n \to 0$ as $n \to \infty$ if and only if $u_{2km} \to 0$ as $k \to \infty$. Therefore, the θ-methods are asymptotically stable if and only if

$$\begin{cases} \left| \frac{R(z)^m + \frac{b}{a}(R(z)^m - 1)}{R(z)^{-m} + \frac{b}{a}(R(z)^{-m}-1)} \right| < 1, \quad \text{for} \quad a \neq 0, \\ \left| \frac{1+b}{1-b} \right| < 1, \quad \text{for} \quad a = 0, \end{cases}$$

thus we have

$$-1 < \frac{(a+b)R(z)^{2m} - bR(z)^m}{a+b-bR(z)^m} < 1, \quad \text{for} \quad a \neq 0, \tag{10a}$$

$$b < 0, \quad \text{for} \quad a = 0. \tag{10b}$$

Case 1. If $a + b - bR(z)^m > 0$, then (10a) becomes

$$b > -\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2}, (a+b)R(z)^{2m} < a+b. \tag{11}$$

If $a > 0$ then $R(z) > 1$, in view of (11) we obtain $b < -a$, hence

$$-\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2} < b < -a, \quad \text{for} \quad a > 0. \tag{12}$$

If $a < 0$ then $0 < R(z) < 1$, by (11) we have $b > -a$. It is obvious that

$$-\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2} > -a > 0,$$

then by (11) we have

$$b > -\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2}, \quad \text{for} \quad a < 0. \tag{13}$$

Case 2. If $a + b - bR(z)^m < 0$, then (10a) yields

$$b < -\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2}, (a+b)R(z)^{2m} > a+b.$$

In a way analogues to the discussion in Case 1, we obtain

$$b < -a, \quad \text{for} \quad a < 0. \tag{14}$$

By means of (10), (12), (13) and (14), the proof is finished. □

3.3. Numerical Oscillation and Non-oscillation

Theorem 6. *The following statements are equivalent:*

(i) $\{u_n\}$ *is oscillatory,*

(ii) $\{u_{2km}\}$ *is oscillatory,*

(iii) $b < -\frac{aR(z)^m}{R(z)^m-1}$ *or* $b > \frac{a}{R(z)^m-1}$ *for* $a \neq 0$ *and* $b < -1$ *or* $b > 1$ *for* $a = 0$.

Proof. Firstly, when $a \neq 0$, we only need to prove that the following two statements: (a) $\{u_n\}$ is not oscillatory and (b) $\{u_{2km}\}$ is not oscillatory are equivalent. It is easy to know that (a) implies (b). Conversely, if (b) holds, we have $\lambda = (R(z)^m + b(R(z)^m - 1)/a)/(R(z)^{-m} + b(R(z)^{-m} - 1)/a) > 0$, from which we obtain

$$R(z)^m + \frac{b}{a}(R(z)^m - 1) > 0, R(z)^{-m} + \frac{b}{a}(R(z)^{-m} - 1) > 0,$$

or

$$R(z)^m + \frac{b}{a}(R(z)^m - 1) < 0, R(z)^{-m} + \frac{b}{a}(R(z)^{-m} - 1) < 0,$$

by simple computation, we have

$$-\frac{aR(z)^m}{R(z)^m - 1} < b < \frac{a}{R(z)^m - 1}.$$

Thus for any $l = 1, 2, \cdots, m-1$, we get

$$-\frac{aR(z)^l}{R(z)^l - 1} < -\frac{aR(z)^m}{R(z)^m - 1} < b < \frac{a}{R(z)^m - 1} < \frac{a}{R(z)^l - 1},$$

which is equivalent to

$$R(z)^l + \frac{b}{a}(R(z)^l - 1) > 0, R(z)^{-l} + \frac{b}{a}(R(z)^{-l} - 1) > 0.$$

From (7) we know that $\{u_n\}$ is not oscillatory. Thus (a) and (b) are equivalent, that is to say, (i) and (ii) are equivalent. In what follows, we prove (ii) and (iii) are equivalent. It is known to us that $\{u_{2km}\}$ is oscillatory if and only if $\lambda = (R(z)^m + b(R(z)^m - 1)/a)/(R(z)^{-m} + b(R(z)^{-m} - 1)/a) < 0$, which is equivalent to

$$b < -\frac{aR(z)^m}{R(z)^m - 1} \quad \text{or} \quad b > \frac{a}{R(z)^m - 1},$$

so (ii) and (iii) are equivalent. The case of $a = 0$ can be got in the same way. Therefore the theorem is proved. □

4. Preservation of Stability and Oscillation

Let us consider the following four problems

$$u'(t) = 2u(t) - 2.7u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{15}$$

$$u'(t) = -3u(t) + 3.4u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{16}$$

$$u'(t) = 4u(t) - 4.2u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{17}$$

$$u'(t) = -5u(t) + 5.1u(2[\frac{t+1}{2}]), \quad u(0) = 1. \tag{18}$$

We apply the θ-methods to (15)-(18), respectively. In view of Theorem 4 we know that the θ-methods are convergent. From Figure 1 (the figure of the analytic solution of (18) should be seen by zooming in), we can see that the analytic solutions of (15) and (16) are both asymptotically stable, but their numerical solutions are not both asymptotically stable. The analytic solutions of (17) and (18) are both oscillatory, but their numerical solutions are not both oscillatory. That is to say, for one problem, the analytic solutions and the numerical solutions may have the same or the different stability and oscillatory properties. It is known to

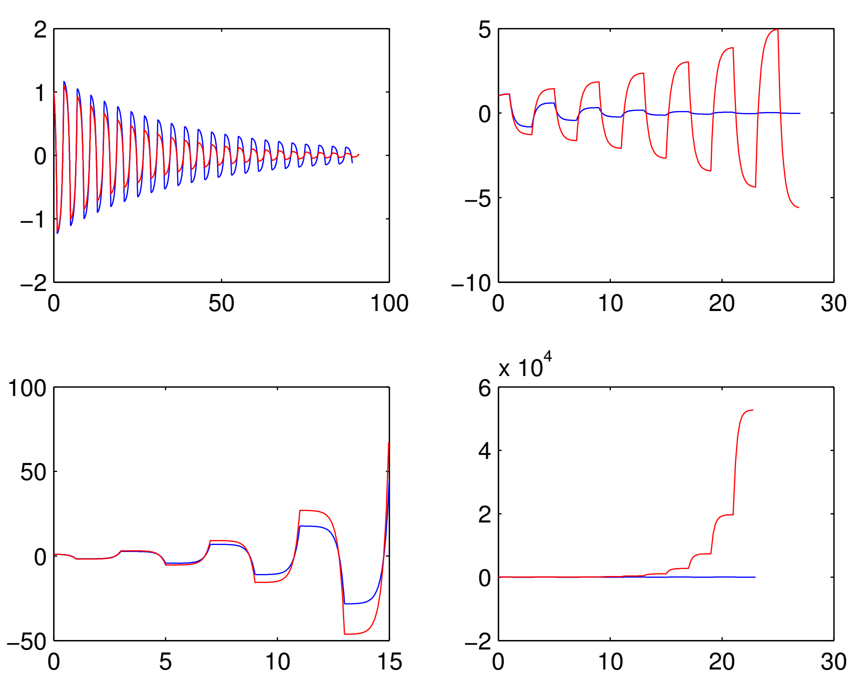

Figure 1: The analytic solution (blue line) and the numerical solution (red line). Left of first row: (15) with $\theta = 0.3$ and $m = 50$, right of first row: (16) with $\theta = 0.8$ and $m = 10$, left of second row: (17) with $\theta = 0.6$ and $m = 40$, right of second row: (18) with $\theta = 1$ and $m = 5$.

us that the numerical method which can preserve the corresponding properties of original problem is useful and practical. Therefore, it is necessary to study the conditions under which the numerical solution and the analytic solution have the same stability and oscillatory properties.

In this section, we investigate the conditions under which the analytical stability region is contained in the numerical stability region and the conditions under which the numerical solution and the analytic solution are oscillatory simultaneously.

4.1. Preservation of Stability

We introduce the set consisting of all points $(a, b) \in \mathbb{R}^2$ at which (1) is asymptotically stable as H. In the following we will investigate which conditions lead to $H \subseteq S$. For convenience, we divide H and S into three parts, respectively:

$$H_0 = \{(a,b) \in H : a = 0\}, H_1 = \{(a,b) \in H \setminus H_0 : a < 0\},$$

$$H_2 = \{(a,b) \in H \setminus H_0 : a > 0\}, S_0 = \{(a,b) \in S : a = 0\},$$

$$S_1 = \{(a,b) \in S \setminus S_0 : a < 0\}, S_2 = \{(a,b) \in S \setminus S_0 : a > 0\}.$$

Obviously, $H = H_0 \cup H_1 \cup H_2, S = S_0 \cup S_1 \cup S_2$ and

$$H_i \cap H_j = \emptyset, S_i \cap S_j = \emptyset, H_i \cap S_j = \emptyset, \quad i \neq j, i, j = 0, 1, 2.$$

Thus $H \subseteq S$ is equivalent to $H_i \subseteq S_i$ $(i = 0, 1, 2)$.

Theorem 7. *$H_1 \subseteq S_1$ if and only if $0 \leq \theta \leq 1/2$ and $H_2 \subseteq S_2$ if and only if $0 \leq \theta \leq \psi(1)$, where $\psi(x) = 1/x - 1/(e^x - 1)$.*

Proof. By virtue of Theorems 2 and 5, we obtain that $H_1 \subseteq S_1$ if and only if

$$-\frac{a(R(z)^{2m} + 1)}{(R(z)^m - 1)^2} \leq -\frac{a(e^{2a} + 1)}{(e^a - 1)^2},$$

that is

$$\frac{R(z)^{2m} + 1}{(R(z)^m - 1)^2} \leq \frac{e^{2a} + 1}{(e^a - 1)^2}, \tag{19}$$

it is a simple matter to verify that the function $f(x) = (x^2 + 1)/(x - 1)^2$ is increasing in $[0, 1)$ and decreasing in $(1, \infty)$, so (19) reduces to $R(z) \leq e^z$, as a consequence of Lemma 3 in [16], we have $0 \leq \theta \leq 1/2$. Similarly, the case of $H_2 \subseteq S_2$ can be proved. □

Theorem 8. *For all θ with $0 \leq \theta \leq 1$, we have $H_0 = S_0$.*

4.2. Preservation of Oscillation

Definition 5. *We call that the θ-methods preserve oscillation of (1) if (1) oscillates, which implies that there is an h_0 such that (7) oscillates for $h < h_0$.*

Theorem 9. *If $a \neq 0$, then the θ-methods preserve the oscillation of (1) if either of the following conditions is satisfied:*

(i) $1/2 \leq \theta \leq 1$ *for* $a > 0$ *and* $h < h_1$,

(ii) $0 \leq \theta \leq 1/2$ *for* $a < 0$ *and* $h < h_2$,

where $h_1 = 1/a$, $h_2 = -1/a$.

Proof. According to Theorems 3 and 6, the θ-methods preserve the oscillation of (1) if and only if

$$-\frac{ae^a}{e^a-1} \leq -\frac{aR(z)^m}{R(z)^m-1} \quad \text{or} \quad \frac{a}{e^a-1} \geq \frac{a}{R(z)^m-1}. \tag{20}$$

When $a>0$, we get

$$\frac{e^a}{e^a-1} \geq \frac{R(z)^m}{R(z)^m-1} \quad \text{or} \quad e^a \leq R(z)^m. \tag{21}$$

Since the function $x/(x-1)$ is decreasing, so it follows from (21) that $R(z)^m \geq e^a$, then by Lemma 3 in [16], we get $1/2 \leq \theta \leq 1$. The other case can be proved in a similar way. □

As a direct application of Theorem 9, we obtain the following theorem naturally.

Theorem 10. *If $a \neq 0$, then the θ-methods preserve the non-oscillation of (1) if either of the following conditions is satisfied:*

(i) $0 \leq \theta \leq \psi(1)$ *for* $a>0$ *and* $h<h_1$,

(ii) $\psi(-1) \leq \theta \leq 1$ *for* $a<0$ *and* $h<h_2$,

where $\psi(x)=1/x-1/(e^x-1)$, $h_1=1/a$ *and* $h_2=-1/a$.

Theorem 11. *If $a=0$, then the θ-methods preserve the oscillation and non-oscillation of (1) for any θ.*

5. Relationships Between Stability and Oscillation

Stability and oscillation are very important aspects in the research of differential equations and numerical analysis, thus it is meaningful to consider the relationships between them. Let

$$A_1=\frac{a}{e^a-1}, A_2=-\frac{ae^a}{e^a-1}, A_3=-\frac{a(e^{2a}+1)}{(e^a-1)^2},$$

and

$$A_1(m)=\frac{a}{R(z)^m-1}, A_2(m)=-\frac{aR(z)^m}{R(z)^m-1}, A_3(m)=-\frac{a(R(z)^{2m}+1)}{(R(z)^m-1)^2}.$$

Combing Theorems 2, 3, 5 and 6, we obtain the following theorems.

Theorem 12. *For $a>0$, the analytic solution of (1) is*

(i) *oscillatory and unstable if* $b \in (-\infty, A_3)$ *or* $b \in (A_1, +\infty)$,

(ii) *oscillatory and asymptotically stable if* $b \in (A_3, A_2)$,

(iii) *non-oscillatory and asymptotically stable if* $b \in (A_2, -a)$,

(iv) *non-oscillatory and unstable if* $b \in (-a, A_1)$,

for $a<0$, the analytic solution of (1) is

(i) *oscillatory and asymptotically stable if* $b \in (-\infty, A_2)$ *or* $b \in (A_3, +\infty)$,

(ii) *non-oscillatory and asymptotically stable if* $b \in (A_2, -a)$,

(iii) *non-oscillatory and unstable if* $b \in (-a, A_1)$,

(iv) *oscillatory and unstable if* $b \in (A_1, A_3)$.

Theorem 13. *For $a>0$, the numerical solution of (1) is*

(i) *oscillatory and unstable if* $b \in (-\infty, A_3(m))$ *or* $b \in (A_1(m), +\infty)$,

(ii) *oscillatory and asymptotically stable if* $b \in (A_3(m), A_2(m))$,

(iii) *non-oscillatory and asymptotically stable if* $b \in (A_2(m), -a)$,

(iv) *non-oscillatory and unstable if* $b \in (-a, A_1(m))$,

for $a < 0$, *the numerical solution of (1) is*

(i) *oscillatory and asymptotically stable if* $b \in (-\infty, A_2(m))$ *or* $b \in (A_3(m), +\infty)$,

(ii) *non-oscillatory and asymptotically stable if* $b \in (A_2(m), -a)$,

(iii) *non-oscillatory and unstable if* $b \in (-a, A_1(m))$,

(iv) *oscillatory and unstable if* $b \in (A_1(m), A_3(m))$.

Theorem 14. *For* $a = 0$, *the analytic solution and the numerical solution of (1) both are*

(i) *oscillatory and asymptotically stable if* $b \in (-\infty, -1)$,

(ii) *non-oscillatory and asymptotically stable if* $b \in (-1, 0)$,

(iii) *non-oscillatory and unstable if* $b \in (0, 1)$,

(iv) *oscillatory and unstable if* $b \in (1, +\infty)$.

6. Numerical Experiments

Firstly, we consider the following three problems

$$u'(t) = -1.5u(t) + 2.7u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{22}$$

$$u'(t) = 1.1u(t) - 2.6u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{23}$$

$$u'(t) = -4u(2[\frac{t+1}{2}]), \quad u(0) = 1. \tag{24}$$

From condition (3), we can see that $(-1.5, 2.7) \in H_1$, $(1.1, -2.6) \in H_2$ and $(0, -4) \in H_0$. We use the θ-methods with the stepsize $h = 1/m$ to get the numerical solution at $t = 12$, where the analytic solutions are $u(12) \approx 0.5612$, $u(12) \approx 0.5583$ and $u(12) \approx 0.0467$ for (22), (23) and (24), respectively. In Tables 1 and 2, we list the absolute errors (AE) and the relative errors (RE) between the numerical solution and the analytic solution at $t = 12$ and the ratio of the errors of the case $m = 50$ over that of $m = 100$. From these tables, we can see that the numerical solution converges to the analytic solution by the original order, that is to say, the θ-methods conserve their order of convergence, which is consistent with Theorem 4.

In Figures 2-4, we draw the numerical solutions with different parameters for these three problems, respectively. We can see that the stability of numerical solution is influenced by h and θ.

Next, consider the following six problems

$$u'(t) = -0.8u(t) + 4.3u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{25}$$

$$u'(t) = 2.7u(t) - 4u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{26}$$

$$u'(t) = -3.7u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{27}$$

	θ=0		θ=0.5		θ=1	
	AE	*RE*	*AE*	*RE*	*AE*	*RE*
$m = 3$	5.5870e-01	9.9550e-01	1.7130e-01	3.0510e-01	2.6870e+01	4.7875e+01
$m = 5$	5.2950e-01	9.4340e-01	6.8000e-02	1.2120e-01	5.5981e+0	9.9744e+0
$m = 10$	4.1520e-01	7.3990e-01	1.7700e-02	3.1600e-02	1.3647e+0	2.4316e+0
$m = 20$	2.6990e-01	4.8090e-01	4.5000e-03	8.0000e-03	4.9010e-01	8.7320e-01
$m = 40$	1.5530e-01	2.7670e-01	1.1000e-03	2.0000e-03	2.0930e-01	3.7290e-01
$m = 50$	1.2790e-01	2.2780e-01	7.1868e-04	1.3000e-03	1.6240e-01	2.8930e-01
$m = 100$	6.7800e-02	1.2080e-01	1.7975e-04	3.2026e-04	7.6400e-02	1.3610e-01
Ratio	1.8864	1.8858	3.9982	4.0592	2.1257	2.1256

Table 1: Computational results of θ-methods for (22)

	θ=0		θ=0.5		θ=1	
	AE	*RE*	*AE*	*RE*	*AE*	*RE*
$m = 3$	5.0650e-01	9.0710e-01	9.6300e-02	1.7250e-01	1.0353e+01	1.8542e+01
$m = 5$	4.2910e-01	7.6860e-01	3.2700e-02	5.8500e-02	2.3939e+0	4.2875e+0
$m = 10$	2.9450e-01	5.2750e-01	8.0000e-03	1.4300e-02	6.8300e-01	1.2233e+0
$m = 20$	1.7660e-01	3.1620e-01	2.0000e-03	3.5000e-03	2.6830e-01	4.8050e-01
$m = 40$	9.7300e-02	1.7430e-01	4.9404e-04	8.8484e-04	1.1990e-01	2.1470e-01
$m = 50$	7.9400e-02	1.4220e-01	3.1613e-04	5.6619e-04	9.3800e-02	1.6810e-01
$m = 100$	4.1300e-02	7.4100e-02	7.9013e-05	1.4151e-04	4.5000e-02	8.0500e-02
Ratio	1.9225	1.9190	4.0010	4.0011	2.0844	2.0882

Table 2: Computational results of θ-methods for (23)

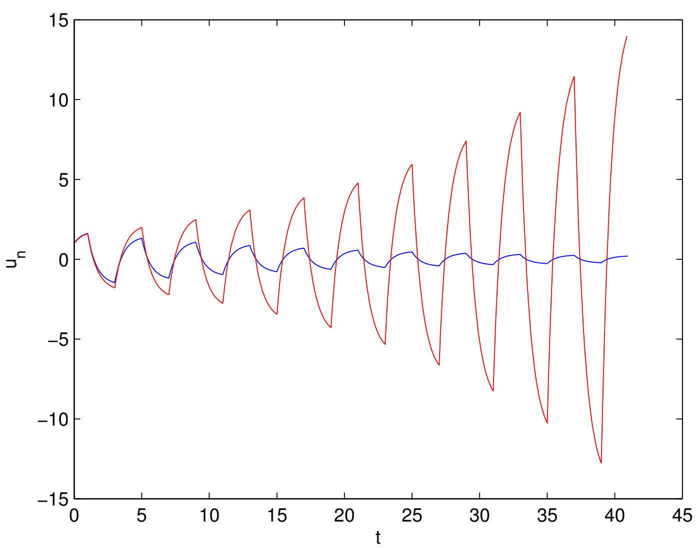

Figure 2: The numerical solution of (22) with $\theta = 0.4, m = 50$ (blue line) and $\theta = 1, m = 10$ (red line).

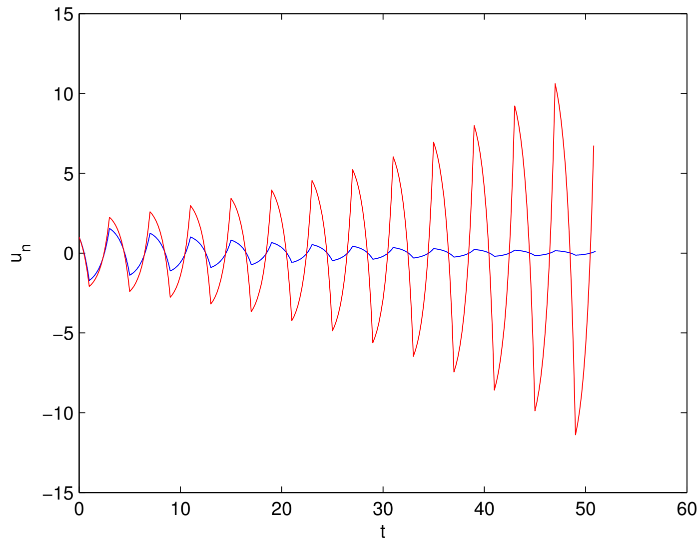

Figure 3: The numerical solution of (23) with $\theta = 0.35, m = 40$ (blue line) and $\theta = 0.8, m = 5$ (red line).

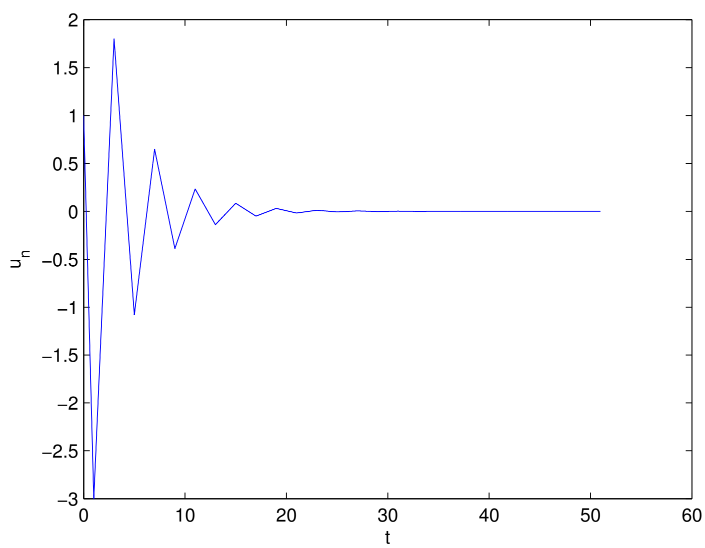

Figure 4: The numerical solution of (24) with $m = 80$.

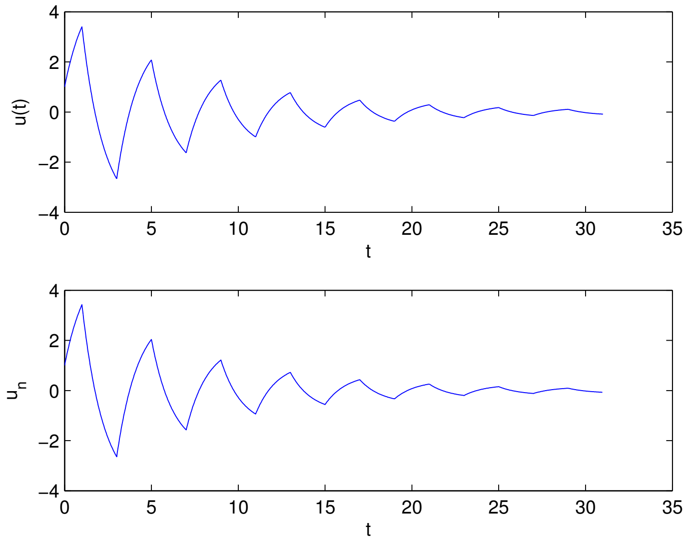

Figure 5: The analytic solution and the numerical solution of (25) with $\theta = 0.15$ and $m = 30$.

$$u'(t) = -1.5u(t) + 1.4u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{28}$$

$$u'(t) = 3.6u(t) - u(2[\frac{t+1}{2}]), \quad u(0) = 1, \tag{29}$$

$$u'(t) = -0.7u(2[\frac{t+1}{2}]), \quad u(0) = 1. \tag{30}$$

For (25)-(30), the analytic solutions of the first three problems are oscillatory; the analytic solutions of the last three problems are non-oscillatory by Theorem 3. In Figures 5-10, we plot the figures of the analytic solutions and the numerical solutions, respectively. From these figures, we can see that the numerical solutions of the first three problems are oscillatory; the numerical solutions of the last three problems are non-oscillatory, which are coincide with Theorem 6.

What is more, in Figure 5, let $\theta = 0.15, m = 30$, we can compute that $A_1 \approx 1.4528, A_2 \approx -0.6528, A_3 \approx 3.1708, A_1(m) \approx 1.4482, A_2(m) \approx -0.6482$ and $A_3(m) \approx 3.1468$. Obviously, $b = 4.3 \in (A_3, +\infty)$ and $b = 4.3 \in (A_3(m), +\infty)$. So the analytic solution and the numerical solution of (25) are both oscillatory and asymptotically stable. That is, the relationships between stability and oscillation are in agreement with Theorems 12 and 13. For (22)-(24), (26)-(30), we can verify them analogously (see Figures 2-4, 6-10).

Conflicts of interest

There are no conflicts to declare.

Acknowledgments

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References

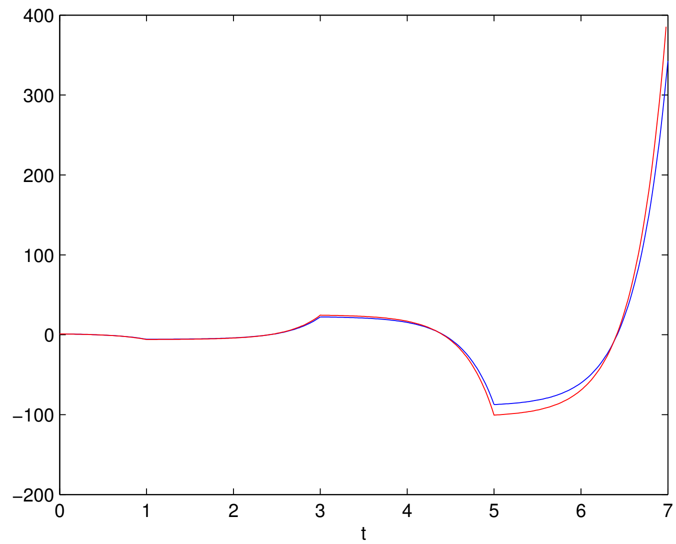

Figure 6: The analytic solution (blue line) and the numerical solution (red line) of (26) with $\theta = 0.75$ and $m = 50$.

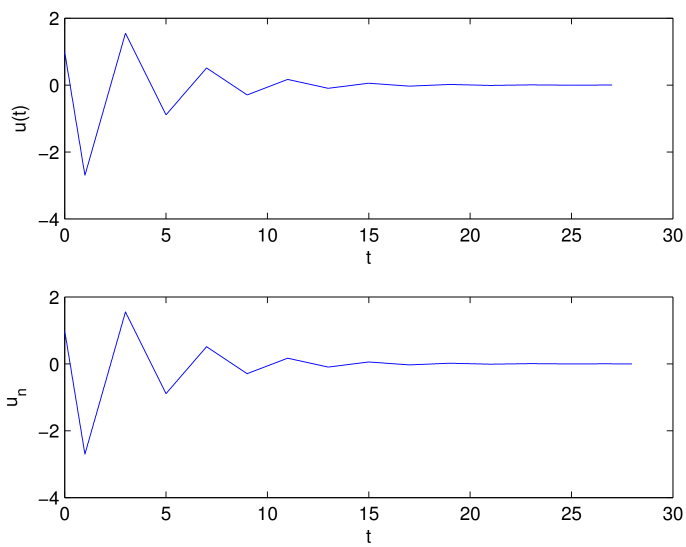

Figure 7: The analytic solution and the numerical solution of (27) with $m = 60$.

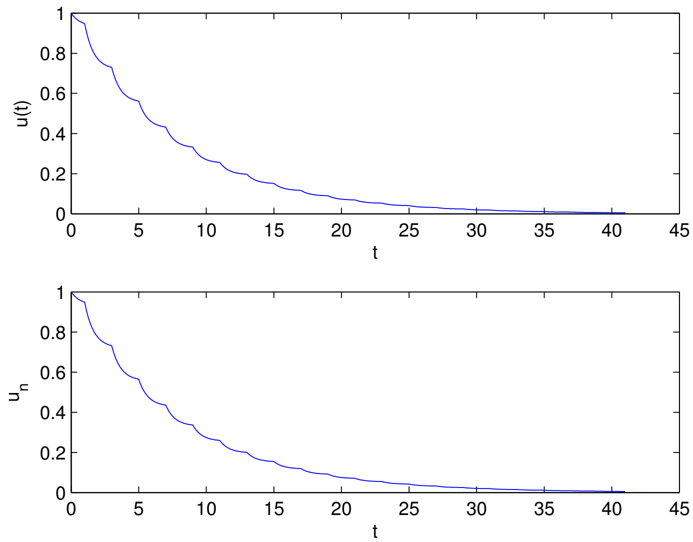

Figure 8: The analytic solution and the numerical solution of (28) with $\theta = 0.7$ and $m = 40$.

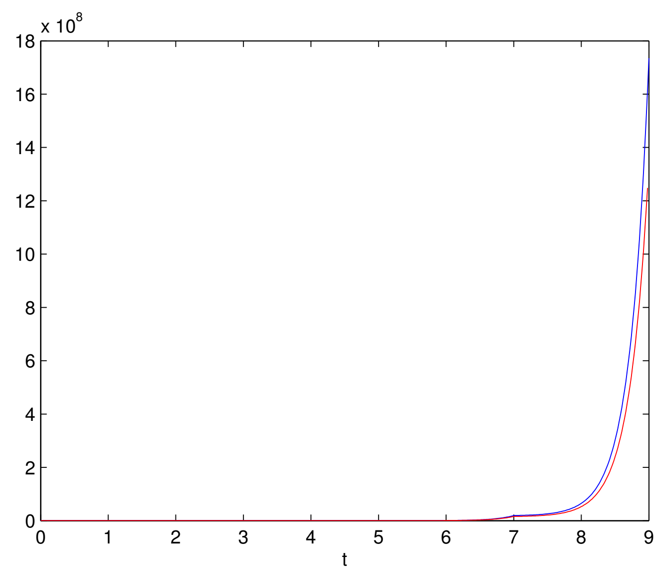

Figure 9: The analytic solution (blue line) and the numerical solution (red line) of (29) with $\theta = 0.3$ and $m = 50$.

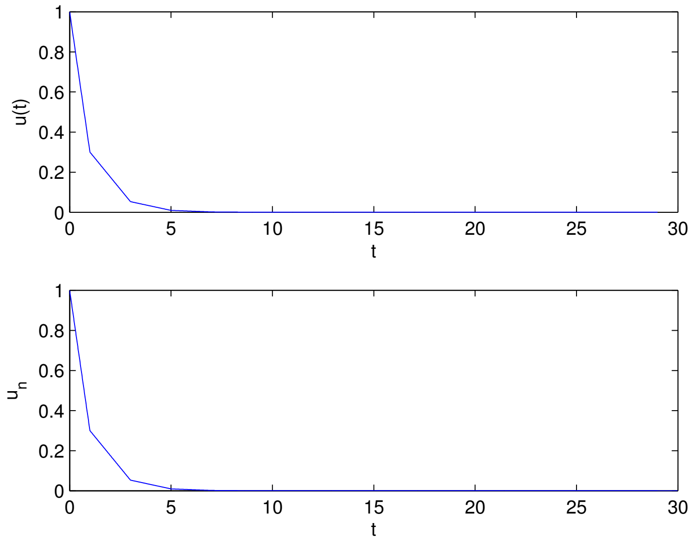

Figure 10: The analytic solution and the numerical solution of (30) with $m = 70$.

[1] Busenberg, S. and Cooke, K. (1993). Vertically Transmitted Diseases: Models and Dynamics. Springer, Berlin.

[2] Chiu, K.S. (2021) Global exponential stability of bidirectional associative memory neural networks model with piecewise alternately advanced and retarded argument. Comput. Appl. Math., 40(8).

[3] Cooke, K.L. and Wiener, J. (1984). Retarded differential equations with piecewise constant delays. J. Math. Anal. Appl., 99(1): 265-297.

[4] Cooke, K.L. and Wiener, J. (1987). An equation alternately of retarded and advanced type. Proc. Amer. Math. Soc., 99: 726-732.

[5] Dhama, S., Abbas, S., and Sakthivel, R. (2021). Stability and approximation of almost automorphic solutions on time scales for the stochastic Nicholson's blowflies model . J. Integral Equ. Appl., 33(1): 31-51.

[6] Jayasree, K.N. and Deo, S.G. (1991). Variation of parameters formula for the equation of Cooke and Wiener. Proc. Amer. Math. Soc., 112(1): 75-80.

[7] Kocic, V.L. and Ladas, G. (1993). Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic Publishers, Dordrecht.

[8] Küpper, T. and Yuang, R. (2002). On quasi-periodic solutions of differential equations with piecewise constant argument. J. Math. Anal. Appl., 267(1): 173-193.

[9] Liang, H., Liu, M.Z., and Yang, Z.W. (2014) Stability analysis of Runge-Kutta methods for systems $u'(t) = Lu(t) + Mu([t])$. Appl. Math. Comput., 228: 463-476.

[10] Liu, M.Z., Gao, J.F., and Yang, Z.W. (2007) Oscillation analysis of numerical solution in the θ-methods for equation $x'(t) + ax(t) + a_1x([t-1]) = 0$. Appl. Math. Comput., 186(1): 566-578.

[11] Liu, M.Z., Gao, J.F., and Yang, Z.W. (2009). Preservation of oscillations of the Runge-Kutta method for equation $x'(t) + ax(t) + a_1x([t-1]) = 0$. Comput. Math. Appl., 58(6): 1113-1125.

[12] Liu, P.Z. and Gopalsamy, K. (1999). Global stability and chaos in a population model with piecewise constant arguments. Appl. Math. Comput., 101(1): 63-88.

[13] Muroya, Y. (2008). New contractivity condition in a population model with piecewise constant arguments. J. Math. Anal. Appl., 346(1): 65-81.

[14] Shah, S.M. and Wiener, J. (1983). Advanced differential equations with piecewise constant argument deviations. Int. J. Math. Math. Sci., 6(4): 671-703.

[15] Song, M.H. and Liu, M.Z. (2012). Numerical stability and oscillation of the Runge-Kutta methods for the differential equations with piecewise continuous arguments alternately of retarded and advanced type. J. Inequal. Appl., 2012: 1-13.

[16] Song, M.H., Yang, Z.W., and Liu, M.Z. (2005). Stability of θ-methods for advanced differential equations with piecewise continuous arguments. Comput. Math. Appl., 49(9-10): 1295-1301.

[17] Wang, G.Q. (2007). Periodic solutions of a neutral differential equation with piecewise constant arguments. J. Math. Anal. Appl., 326(1): 736-747.

[18] Wang, Q., Zhu, Q.Y., and Liu, M.Z. (2011). Stability and oscillations of numerical solutions for differential equations with piecewise continuous arguments of alternately advanced and retarded type. J. Comput. Appl. Math., 235(5): 1542-1552.

[19] Wang, W.S. (2013). Stability of solutions of nonlinear neutral differential equations with piecewise constant delay and their discretizations. Appl. Math. Comput., 219(9): 4590-4600.

[20] Wiener, J. and Aftabizadeh, A.R. (1988). Differential equations alternately of retarded and advanced type. J. Math. Anal. Appl., 129(1): 243-255.

[21] Wiener, J. and Cooke, K.L. (1989). Oscillations in systems of differential equations with piecewise constant argument. J. Math. Anal. Appl., 137(1): 221-239.

[22] Wiener, J. (1993). Generalized Solutions of Functional Differential Equations. World Scientific, Singapore.

[23] Yang, Z.W., Liu, M.Z., and Nieto, J.J. (2009). Runge-Kutta methods for first-order periodic boundary value differential equations with piecewise constant arguments. Comput. Math. Appl., 233(4): 990-1004.

[24] Yin, H.F. and Wang, Q. (2021). Dynamic behavior of Euler-Maclaurin methods for differential equations with piecewise constant arguments of advanced and retarded type. Fund. J. Math. Appl., 4(3): 165-179.