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Applications of Probabilistic Methods on Some Stirling Sequences

Xin Wang[1],Wuyungaowa[2]

[1]School of Mathematical Sciences, Inner Mongolia University, Hohhot, China
Email: 1821670183@qq.com

[2]School of Mathematical Sciences, Inner Mongolia University, Hohhot, China
Email: wuyungw@163.com

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Abstract

In this paper, we investigate some combinatorial sequences based on the generalized Stirling numbers and the λ-analogues of r-Stirling numbers of the first kind, then derive their moment representations in use of probabilistic methods. We also provide identities related to r-Stirling numbers of the first kind, Stirling numbers and Daehee numbers.

Keywords:

Moment; Probabilistic method; Generating function; Generalized Stirling numbers; λ-analogues of r-Stirling numbers of the first kind.

1. Introduction and Preliminaries

Throughout this paper, we use the following notations:

$$\mathbb{N} = \{1, 2, 3, \cdots\}, \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \cdots\}.$$

Let $s(n, k; r)$ denote the generalized Stirling numbers of the first kind of order $k \in \mathbb{N}$, which are defined by the generating function[1, 2] to be

$$\sum_{n=k}^{\infty} s(n, k; r)\frac{t^n}{n!} = (1 + t)^{-r}\frac{(\ln(1 + t))^k}{k!}. \tag{1}$$

In special case, when $r = 0$, $s(n, k; r) = s(n, k)$ are called the Stirling numbers of the first kind.

Let $S(n, k; r)$ denote the generalized Stirling numbers of the second kind of order $k \in \mathbb{N}$, which are defined by the generating function[1, 2] to be

$$\sum_{n=k}^{\infty} S(n, k; r)\frac{t^n}{n!} = e^{rt}\frac{(e^t - 1)^k}{k!}. \tag{2}$$

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In special case, when $r=0$, $S(n,k;r)=S(n,k)$ are called the Stirling numbers of the second kind.

Let $S_1^{(r)}(n,k)$ denote the r-Stirling numbers of the first kind, which are defined by the generating function[3] to be

$$\sum_{n=k}^{\infty} S_1^{(r)}(n,k)\frac{t^n}{n!} = (1+t)^r \frac{(\ln(1+t))^k}{k!}. \tag{3}$$

Let $S_{1,\lambda}^{(r)}(n,k)$ denote the λ-analogues of r-Stirling numbers of the first kind, which are defined by the generating function[3] to be

$$\sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!} = (1+\lambda t)^{\frac{r}{\lambda}} \frac{(\frac{ln(1+\lambda t)}{\lambda})^k}{k!}. \tag{4}$$

When $\lim_{r\to 0} S_{1,\lambda}^{(r)}(n,k) = S_{1,\lambda}(n,k)$.

Remark 1. *[4] If f and g are exponential generating functions, and*

$$fg = (\sum_{r=0}^{\infty} \frac{a_r x^r}{r!})(\sum_{s=0}^{\infty} \frac{b_s x^s}{s!}),$$

then the coefficients of $\frac{x^n}{n!}$ in fg are given by

$$[\frac{x^n}{n!}](fg) = \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r}.$$

Remark 2. *Throughout this paper, symbol E denotes the expectation operator defined by*

$$Ef(X) = \int_{-\infty}^{+\infty} f(x)p(x)dx,$$

where random variable X is continuous, whose density function is $p(x)$. Specially, when $f(x)=x^n$, EX^n denotes n-order moment of random variable X.

1. *When r.v $u \sim U[0,1]$, $Eu^n = \frac{1}{n+1}$,*
2. *When r.v $X \sim \Gamma(1,1)$, $EX^n = n!$,*
3. *When r.v $Y \sim \Gamma(\alpha,\lambda)$,$\alpha$>0, λ>0, $EY^n = \lambda^{-n} < \alpha >_n$.*

Definition 1. *The characteristic function of random variable X is defined as*

$$\varphi(t) = Ee^{itX}, i^2 = -1, -\infty < t < \infty. \tag{5}$$

When the moments of all orders of r.v. X exist, the following relation expression holds true,

$$EX^n = [\frac{(it)^n}{n!}]\varphi(t), \quad i^2 = -1. \tag{6}$$

Remark 3. *[5] If random variable X is distributed as $\Gamma(\alpha,\lambda)$, where $\alpha,\lambda>0$, its characteristic function is*

$$\varphi(t) = Ee^{itX} = (1-\frac{it}{\lambda})^{-\alpha}. \tag{7}$$

Remark 4. *X and Y are two random variables, when Cov(X,Y)=0, we have E(XY)=EX·EY, where*

$$Cov(X,Y) = E(XY) - E(X)E(Y) \tag{8}$$

Then we give five lemmas to introduce moment representations of some special combinatorial sequences.

Lemma 1. *[4] Assume that r.v $u_1, u_2 \sim U[0,1]$,then Harmonic numbers $H_n = \sum_{n=0}^{\infty} \frac{1}{k}$ have the following moment representation,*

$$H_n = nE(1-u_1u_2)^{n-1}, \quad n \geqslant 1. \tag{9}$$

Lemma 2. *[4] Suppose that r.v.s $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, r.v.s $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, r.v u_i and Γ_j are independent respectively for all i, j. When $n, k \geq 1$, Stirling numbers of the first kind $s(n,k)$ satisfy*

$$\begin{aligned} s(n,k) =& (-1)^{n-k}\binom{n}{k}E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k} \\ =& (-1)^{n-k}\binom{n-1}{k-1}E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_{k-1}\Gamma_{k-1} + \Gamma_k)^{n-k}, \end{aligned} \tag{10}$$

It is demanded that s(n,0)=s(0,k)=0, s(0,0)=1.

Lemma 3. *[4] Suppose that r.v.s $u_1, u_2, \cdots, i.i.d \sim U[0,1]$, r.v.s $\Gamma_1, \Gamma_2, \cdots, i.i.d \sim \Gamma(1,1)$, r.v u_i and Γ_j are independent respectively for all i, j. When $n, k \geq 1$, Stirling numbers of the second kind $S(n,k)$ satisfy*

$$S(n,k) = \binom{n}{k}E(u_1 + u_2 + \cdots + u_k)^{n-k} = \frac{1}{(n-k)!}E(\Gamma_1 + 2\Gamma_2 + \cdots + k\Gamma_k)^{n-k}, \tag{11}$$

It is demanded that S(n,0)=S(0,k)=0, s(0,0)=1.

Lemma 4. *[6] Assume r.v.s $u_1, u_2, ..., i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, ..., i.i.d \sim \Gamma(1,1)$, and for all i, j, r.v u_i and Γ_j are independent. When $n, m-k \in \mathbb{Z}_{\geqslant 0}$, $k \in \mathbb{N}$, we have the following moment representation,*

$$D_{n,\xi}^{(k)} = (-\xi)^n E(u_1\Gamma_1 + \cdots + u_k\Gamma_k)^n, \quad n \geqslant 0. \tag{12}$$

Lemma 5. *[6] Assume r.v.s $u_1, u_2, ..., i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, ..., i.i.d \sim \Gamma(1,1)$, r.v.$X \sim \Gamma(r,2)$,r>0,and for all i, j, r.v u_i and Γ_j are independent.we have the following moment representation,*

$$CG_n^{(r)} = \sum_{k=r}^{n}(-1)^{n-r}\frac{(n)_k}{(k-r)!}E(u_1\Gamma_1 + \cdots + u_r\Gamma_r)^{k-r}EX^{n-k}, \quad n \geqslant r. \tag{13}$$

2. Moment Representations of the generalized Stirling numbers

In this section, we use probabilistic method to derive moment representations about the generalized Stirling numbers of the first kind,the generalized Stirling numbers of the second kind,r-Stirling numbers of the first kind and the λ-analogues of r-Stirling numbers of the first kind.

Theorem 1. *Assume that r.v.s $u_1, u_2, ..., i.i.d \sim U[0,1]$, $\Gamma_1, \Gamma_2, ..., i.i.d \sim \Gamma(1,1)$, r.v.$X \sim \Gamma(r,1)$,r>0,and for all i, j, r.v u_i and Γ_j are independent. When $n \geq k$, $k \in \mathbb{N}$, we have*

$$s(n,k;r) = (-1)^{n-k}\binom{n}{k}E(X + u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k}. \tag{14}$$

Proof. The generating function of the generalized Stirling numbers of the first kind is known as

$$\sum_{n=k}^{\infty} s(n,k;r)\frac{t^n}{n!} = (1+t)^{-r}\frac{(\ln(1+t))^k}{k!}, \tag{15}$$

Taking the coefficients of t^n in the left-hand side of Eq.(15), we get

$$\frac{k!}{n!}s(n,k;r) = [t^n]\ln(1+t)^k(1+t)^{-r} = [(-it)^n]\ln(1-it)^k(1-it)^{-r}$$
$$= [(-it)^n]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}(\ln(1-it))^k$$
$$= [(-it)^{n-k}]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}(\frac{ln(1-it)}{-it})^k$$
$$= [(-it)^n]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}(\sum_{m\geq 1}\frac{(it)^{m-1}}{m})^k$$
$$= [(-it)^n]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}(\sum_{m\geq 0} Eu^m(it)^m)^k$$
$$= [(-it)^n]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}\sum_{n=0}^{\infty}\sum_{m_1+\cdots+m_k=n}(E(u_1)^{m_1})\cdots(E(u_k)^{m_k})(it)^n$$
$$= [(-it)^n]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}\sum_{n=0}^{\infty}\sum_{m_1+\cdots+m_k=n}\binom{n}{m_1,m_2,\cdots m_k}(E(u_1)^{m_1})\cdots(E(u_k)^{m_k})(m_1!)\cdots(m_k!)\frac{(it)^n}{n!}$$
$$= [(-it)^n]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}\sum_{n=0}^{\infty}\sum_{m_1+\cdots+m_k=n}\binom{n}{m_1,m_2,\cdots m_k}(E(u_1\Gamma_1)^{m_1})\cdots(E(u_k\Gamma_k)^{m_k})\frac{(it)^n}{n!}$$
$$= [(-it)^n]\sum_{n\geq 0} EX^n\frac{(it)^n}{n!}\sum_{n=0}^{\infty}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^n\frac{(it)^n}{n!}$$
$$= [(-it)^n]\sum_{n\geq 0}(\sum_{k=0}^{n}\binom{n}{k}EX^{n-k}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^k)\frac{(it)^n}{n!}$$
$$= [(-it)^n]\sum_{n\geq 0}E(X+u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^n\frac{(it)^n}{n!} \tag{16}$$

From Eq.(16), we can see that

$$\frac{k!}{n!}s(n,k;r) = (-1)^{n-k}\sum_{k=0}^{n}E(X+u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{n-k}\frac{1}{(n-k)!},$$

thus we have

$$s(n,k;r) = (-1)^{n-k}\binom{n}{k}E(X+u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{n-k}.$$

□

Corollary 1. *In theorem 1, when $k=n$, we obtain the equation:*

$$s(n,n;r) = 1. \tag{17}$$

In theorem 1, when $k=1$, we obtain the equation:

$$s(n,1;r) = (-1)^{n-1}\sum_{k=0}^{n-1}k!\binom{n}{k+1} < r >_{n-k-1}. \tag{18}$$

Proof.

$$\begin{aligned} s(n,1;r) =&(-1)^{n-1}nE(X+u\Gamma)^{n-1} \\ =&(-1)^{n-1}nE(\sum_{k=0}^{n-1}\binom{n-1}{k}(X)^{n-k-1}(u\Gamma)^{k}) \\ =&(-1)^{n-1}n\sum_{k=0}^{n-1}\frac{(n-1)!}{(n-k-1)!}E(X)^{n-k-1}E(u\Gamma)^{k} \\ =&(-1)^{n-1}\sum_{k=0}^{n-1}\frac{1}{k+1}\frac{n!}{(n-k-1)!}<r>_{n-k-1} \\ =&(-1)^{n-1}\sum_{k=0}^{n-1}k!\binom{n}{k+1}<r>_{n-k-1}. \end{aligned}$$

□

In theorem1, when $k=1, r=1, n\geq 1$, we obtain the equation:

$$s(n,1;1) = (-1)^{n-1}(n!)nE(1-u_1u_2)^{n-1}. \tag{19}$$

Proof.

$$\begin{aligned} s(n,1;1) =&(-1)^{n-1}\sum_{k=0}^{n-1}\frac{1}{k+1}\frac{n!}{(n-k-1)!}(n-k-1)! \\ &= (-1)^{n-1}(n!)\sum_{k=0}^{n-1}\frac{1}{k+1} \\ &= (-1)^{n-1}(n!)H_n \\ &= (-1)^{n-1}(n!)nE(1-u_1u_2)^{n-1}. \end{aligned}$$

□

In theorem1, when $k=n-1, r=1$, we obtain the equation:

$$s(n,n-1;1) == -\frac{n(n+1)}{2}. \tag{20}$$

Proof.

$$\begin{aligned} s(n,n-1;1) &= -nE(X+u_1\Gamma_1+u_2\Gamma_2+\cdots+u_{n-1}\Gamma_{n-1}) \\ &= -n(1+\frac{1}{2}+\cdots+\frac{1}{2}) \\ &= -n(1+\frac{n-1}{2}) \\ &= -\frac{n(n+1)}{2}. \end{aligned}$$

□

Theorem 2. *Assume that r.v.s $u_1, u_2, \ldots,$ i.i.d $\sim U[0,1]$, $\Gamma_1, \Gamma_2, \ldots,$ i.i.d $\sim \Gamma(1,1)$, r.v.$X \sim \Gamma(r,1)$,r>0,and for all i, j, r.v u_i and Γ_j are independent. When $n \geq k$, $k \in \mathbb{N}$, we have the second moment representation of the generalized Stirling numbers of the first kind*

$$s(n,k;r) = \sum_{m=k}^{n}(-1)^{n-k}\frac{(n)_m}{k!(m-k)!}EX^{n-m}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{m-k}. \tag{21}$$

Proof.

$$\sum_{n=k}^{\infty} s(n,k;r)\frac{t^n}{n!} = (1+t)^{-r}\frac{(\ln(1+t))^k}{k!}$$
$$= \sum_{n=0}^{\infty}(-1)^n < r >_n \frac{t^n}{n!}\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}(-1)^n EX^n\frac{t^n}{n!}\sum_{n=k}^{\infty}(-1)^{n-k}\binom{n}{k}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{n-k}\frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty}(\sum_{m=k}^{n}\binom{n}{m}(-1)^{n-m}EX^{n-m}\binom{m}{k}(-1)^{m-k}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{m-k})\frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty}(\sum_{m=k}^{n}(-1)^{n-k}\frac{(n)_m}{k!(m-k)!}EX^{n-m}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{m-k})\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, theorem 2 is proved. □

Corollary 2. *In theorem 2, taking $r=2$, we obtain the equation:*

$$s(n,k;2) = \frac{1}{k!}CG_n^k. \tag{22}$$

Corollary 3. *From corollary 2, when $k=1, n\geqslant 1$,the following relationship holds true,*

$$s(n,1;2) = CG_n = \sum_{m=1}^{n}\binom{n}{m}mD_{m-1}Ch_{n-m}, \quad n\geqslant 1.[6] \tag{23}$$

Theorem 3. *Suppose that r.v.s $u_1, u_2, \ldots$, i.i.d $\sim U[0,1]$, $\Gamma_1, \Gamma_2, \ldots$, i.i.d $\sim \Gamma(1,1)$, then the generalized Stirling numbers of the second kind $S(n,k;r)$ satisfy*

$$S(n,k;r) = \sum_{m=k}^{n}\frac{(n)_m}{(m-k)!k!}r^{n-m}E(u_1+u_2+\cdots+u_k)^{m-k} \tag{24}$$
$$= \sum_{m=k}^{n}\binom{n}{m}\frac{r^{n-m}}{(m-k)!}E(\Gamma_1+2\Gamma_2+\cdots+k\Gamma_k)^{m-k}. \tag{25}$$

Proof.

$$\sum_{n=k}^{\infty} S(n,k;r)\frac{t^n}{n!} = e^{rt}\frac{(e^t-1)^k}{k!}$$
$$= \sum_{n=0}^{\infty}\frac{(rt)^n}{n!}\sum_{n=k}^{\infty}S(n,k)\frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty}(\sum_{m=k}^{n}\binom{n}{m}r^{n-m}S(m,k))\frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty}(\sum_{m=k}^{n}\frac{n!r^{n-m}}{m!(n-m)!}\binom{m}{k}E(u_1+u_2+\cdots+u_k)^{m-k})\frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty}(\sum_{m=k}^{n}\frac{(n)_m}{k!(m-k)!}r^{n-m}E(u_1+u_2+\cdots+u_k)^{m-k})\frac{t^n}{n!}.$$

$S(n,k) = \frac{1}{(n-k)!}E(\Gamma_1 + 2\Gamma_2 + \cdots + k\Gamma_k)^{n-k}$, therefore

$$\sum_{n=k}^{\infty} S(n,k;r)\frac{t^n}{n!} = \sum_{n=k}^{\infty}(\sum_{m=k}^{n}\binom{n}{m} r^{n-m}\frac{1}{(m-k)!}E(\Gamma_1 + 2\Gamma_2 + \cdots + k\Gamma_k)^{m-k})\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain theorem 3.

□

Corollary 4. *In theorem 3, taking $k = n, r = 1$, we obtain the equation:*

$$S(n,n;1) = 1. \tag{26}$$

In theorem 3, taking $k = 1, r = 1$, we obtain the equation:

$$S(n,1,1) = \sum_{m=1}^{n} \frac{(n)_m}{(m-1)!}E(u_1)^{m-1} = \sum_{m=1}^{n}\binom{n}{m} = 2^n - 1. \tag{27}$$

In theorem 3, taking $k = n-1, r = 1$, we obtain the equation:

$$S(n,n-1,1) = \frac{n(n+1)}{2}. \tag{28}$$

Proof.

$$\begin{aligned} S(n,n-1,1) &= \sum_{m=n-1}^{n} \frac{(n)_m}{(m-n+1)!(n-1)!}E(u_1 + u_2 + \cdots + u_{n-1})^{m-n+1} \\ &= n + nE(u_1 + u_2 + \cdots + u_{n-1}) \\ &= n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \end{aligned}$$

□

In theorem 3, taking $k = 2, r = 1$, we obtain the equation:

$$S(n,2;1) = \frac{3^n + 1}{2} - 2^n. \tag{29}$$

Proof.

$$
\begin{aligned}
S(n,2;1) &= \sum_{m=2}^{n} \frac{(n)_m}{(m-2)!2!} E(u_1+u_2)^{m-2} \\
&= \sum_{m=2}^{n} \frac{(n)_m}{(m-2)!2!} E \sum_{k=0}^{m-2} \binom{m-2}{n} u_1^k u_2^{m-2-k} \\
&= \frac{1}{2} \sum_{m=2}^{n} \sum_{k=0}^{m-2} \frac{n!}{(n-m)!} \frac{1}{(m-2-k)!k!} \frac{1}{k+1} \frac{1}{m-k-1} \\
&= \frac{1}{2} \sum_{m=2}^{n} \binom{n}{m} \sum_{k=0}^{m-2} \binom{m}{k+1} \\
&= \frac{1}{2} \sum_{m=2}^{n} \binom{n}{m} (2^m - 2) \\
&= \frac{1}{2} (\sum_{m=2}^{n} \binom{n}{m} 2^m - \sum_{m=2}^{n} 2\binom{n}{m}) \\
&= \frac{1}{2} [3^n - 2n - 1 - 2(2^n - n - 1)] \\
&= \frac{1}{2} (3^n - 2^{n+1} + 1).
\end{aligned}
$$

□

Theorem 4. *Assume that r.v.s $u_1, u_2, \dots$, i.i.d $\sim U[0,1]$, $\Gamma_1, \Gamma_2, \dots$, i.i.d $\sim \Gamma(1,1)$, r.v.$X \sim \Gamma(r-n+1,1)$,r-n+1>0,and for all i, j, r.v u_i , Γ_j and X are independent. When $n \geq k$, $k \in \mathbb{N}$, we have the moment representation of r-Stirling numbers of the first kind*

$$
S_1^{(r)}(n,k) = (-1)^{n-k} \binom{n}{k} E(-X + u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k}. \tag{30}
$$

Proof. The generating function of the r-Stirling numbers of the first kind is known as

$$
\sum_{n=k}^{\infty} S_1^{(r)}(n,k) \frac{t^n}{n!} = (1+t)^r \frac{(\ln(1+t))^k}{k!}, \tag{31}
$$

Taking the coefficients of t^n in the left-hand side of Eq.(31), we get

$$
\begin{aligned}
\frac{k!}{n!} S_1^{(r)}(n,k) &= [t^n](\ln(1+t))^k (1+t)^r \\
&= [(-t)^{n-k}](\frac{1}{t} \ln(\frac{1}{1-t}))^k (1-t)^r \\
&= [(-t)^{n-k}](\sum_{i \geqslant 1} \frac{t^i - 1}{i})^k \sum_{n \geqslant 0} (r)_n \frac{(-t)^n}{n!} \\
&= [(-t)^{n-k}] \sum_{n \geqslant 0} E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^n \frac{t^n}{n!} \sum_{n \geqslant 0} < r-n+1 >_n \frac{(-t)^n}{n!} \\
&= [(-t)^{n-k}] \sum_{n \geqslant 0} E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^n \frac{t^n}{n!} \sum_{n \geqslant 0} E(-X)^n \frac{t^n}{n!} \\
&= [(-t)^{n-k}] \sum_{n \geqslant 0} \binom{n}{m} E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-m} E(-X)^m \frac{t^n}{n!} \\
&= [(-t)^{n-k}] \sum_{n \geqslant 0} E(-X + u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^n \frac{t^n}{n!}.
\end{aligned}
$$

we can see that

$$\frac{k!}{n!}S_1^{(r)}(n,k) = (-1)^{n-k}\sum_{k=0}^{n} E(-X + u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k}\frac{1}{(n-k)!},$$

thus we have

$$S_1^{(r)}(n,k) = (-1)^{n-k}\binom{n}{k}E(-X + u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k}.$$

□

The generating function of the λ-analogues of r-Stirling numbers of the first kind is known as

$$\sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!} = (1+\lambda t)^{\frac{r}{\lambda}}(\frac{\ln(1+\lambda t)}{\lambda t})^k\frac{t^k}{k!},$$

thus

$$\lim_{r\to 0} S_{1,\lambda}^{(r)}(n,k) = S_{1,\lambda}(n,k), \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} = \frac{1}{k!}(\frac{\ln(1+\lambda t)}{\lambda})^k. \tag{32}$$

Theorem 5. *Suppose that r.v.s $u_1, u_2, \ldots,$ i.i.d $\sim U[0,1]$, $\Gamma_1, \Gamma_2, \ldots,$ i.i.d $\sim \Gamma(1,1)$, and r.v u_i, Γ_j are independent for all i, j, When $n \geq k$, $k \in \mathbb{N}$, we have the moment representation of $S_{1,\lambda}(n,k)$,*

$$S_{1,\lambda}(n,k) = (-\lambda)^{n-k}\binom{n}{k}E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k}. \tag{33}$$

Proof.

$$\sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} = \frac{1}{k!}(\frac{\ln(1+\lambda t)}{\lambda})^k = \frac{t^k}{k!}(\frac{\ln(1+\lambda t)}{\lambda t})^k, \tag{34}$$

$$\sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^{n-k}}{n!} = \frac{1}{k!}(\frac{\ln(1+\lambda t)}{\lambda t})^k, \tag{35}$$

Taking the coefficients of t^{n-k} in the left-hand side of Eq.(35), we get

$$\begin{aligned}
&S_{1,\lambda}(n,k)\frac{k!}{n!} = [t^{n-k}](\frac{\ln(1+\lambda t)}{\lambda t})^k \\
&= [t^{n-k}](\sum_{m\geq 1}(-1)^{m-1}\frac{(\lambda t)^m}{m}\frac{1}{\lambda t})^k \\
&= [t^{n-k}](\sum_{m\geq 0}(-\lambda t)^m Eu)^k \\
&= [t^{n-k}](-\lambda)^n \sum_{m_1+\cdots+m_k=n}\binom{n}{m_1, m_2, \cdots m_k}E(u_1)^{m_1}\cdots E(u_k)^{m_k} m_1!\cdots m_k!\frac{t^n}{n!} \\
&= [t^{n-k}](-\lambda)^n E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^n\frac{t^n}{n!} \\
&= (-\lambda)^{n-k}E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k}\frac{1}{(n-k)!}.
\end{aligned}$$

thus we have

$$S_{1,\lambda}(n,k) = (-\lambda)^{n-k}\binom{n}{k}E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k}.$$

□

Corollary 5. *Under the circumstance of theorem 6,according to the moment representation of k-order Twisted Daehee numbers,we have* $S_{1,\lambda}(n,k) = \binom{n}{k} D^k_{n-k,\lambda}$*, taking* $k = 1$*, we obtain the equation*

$$S_{1,\lambda}(n,1) = nD_{n-1,\lambda}, \tag{36}$$

taking $\lambda = 1$*, we obtain the equation*

$$S_{1,1}(n,k) = s(n,k) = \binom{n}{k} D^k_{n-k}. \tag{37}$$

Theorem 6. *Suppose that r.v.s* $u_1, u_2, \ldots,$ *i.i.d* $\sim U[0,1]$, $\Gamma_1, \Gamma_2, \ldots,$ *i.i.d* $\sim \Gamma(1,1)$*,and r.v* u_i, Γ_j *are independent for all* i, j*, When* $n \geq k$, $k \in \mathbb{N}$*, we have the moment representation of* λ*-analogues of r-Stirling numbers of the first kind,*

$$S^{(r)}_{1,\lambda}(n,k) = \sum_{m=0}^{n-k} \binom{n}{k} \binom{n-k}{m} (-\lambda)^{n-k-m} E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-k-m} (r)_{m,\lambda}. \tag{38}$$

Proof. The generating function of λ-analogues of r-Stirling numbers of the first kind is known as

$$\sum_{n=k}^{\infty} S^{(r)}_{1,\lambda}(n,k) \frac{t^n}{n!} = (1+\lambda t)^{\frac{r}{\lambda}} \left(\frac{\ln(1+\lambda t)}{\lambda t}\right)^k \frac{t^k}{k!}, \tag{39}$$

Taking the coefficients of t^{n-k} in the left-hand side of Eq.(39), we get

$$\begin{aligned} S^{(r)}_{1,\lambda}(n,k) \frac{k!}{n!} &= [t^{n-k}] \sum_{n=0}^{\infty} (-\lambda)^n E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} (r)_{n,\lambda} \frac{t^n}{n!} \\ &= [t^{n-k}] \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} (-\lambda)^{n-m} E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-m} (r)_{m,\lambda} \right) \frac{t^n}{n!} \\ &= \sum_{m=0}^{n-k} \binom{n-k}{m} (-\lambda)^{n-m-k} E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k)^{n-m-k} (r)_{m,\lambda}. \end{aligned}$$

□

3. Identities of the generalized Stirling numbers and Special Combinatorial Sequences

In this section, we use moment forms of special combinatorial sequences, characteristic function and generating function method to investigate the relationships among the generalized Stirling numbers of the first kind $s(n,k;r)$,the λ-analogues of r-Stirling numbers of the first kind $S^{(r)}_{1,\lambda}(n,k)$,the Daehee numbers and the Stirling numbers of the first kind, then we obtain combinatorial identities about them.

Theorem 7. *Under the circumstance of theorem 1,the generalized Stirling numbers of the first kind* $s(n,k;r)$ *and the Stirling numbers of the first kind* $s(n,k)$ *have the following equation:*

$$s(n,k;r) = \sum_{l=0}^{n-k} (-1)^l \binom{n}{l} < r >_l s(n-l,k). \tag{40}$$

Proof.

$$S_k = u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_k\Gamma_k \tag{41}$$

$$
\begin{aligned}
s(n,k;r) &= (-1)^{n-k}\binom{n}{k}E(X+S_k)^{n-k} \\
&= (-1)^{n-k}\binom{n}{k}E\sum_{l=0}^{n-k}\binom{n-k}{l}X^l S_k^{n-k-l} \\
&= (-1)^l\frac{n!}{k!l!(n-k-l)!}\sum_{l=0}^{n-k}EX^l\frac{k!(n-k-l)!}{(n-l)!}(-1)^{n-k-l}\binom{n-l}{k}ES_k^{n-k-l} \\
&= \sum_{l=0}^{n-k}(-1)^l\binom{n}{l} < r >_l s(n-l,k).
\end{aligned}
$$

□

Corollary 6. *Under the circumstance of theorem 7,the higher-order Daehee numbers* $D_n^{(k)} = (-1)^n E(u_1\Gamma_1 + \cdots + u_k\Gamma_k)^n = (-1)^n ES_k^n$ *we obtain the equation*

$$
\begin{aligned}
s(n,k;r) &= (-1)^{n-k}\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}EX^{n-k-l}ES_k^l \\
&= \sum_{l=0}^{n-k}(-1)^{n-k+l}\binom{n}{k}\binom{n-k}{l} < r >_{n-k-l} D_l^{(k)}. \qquad (42)
\end{aligned}
$$

when $k=1, r=1$*, we obtain the equation:*

$$
s(n,1;1) = \sum_{l=0}^{n-1}(-1)^{n+l-1}\frac{n!}{l!}D_l. \qquad (43)
$$

Theorem 8. *Under the circumstance of theorem 6,we obtain the equation:*

$$
S_{1,\lambda}^{(r)}(n,k) = \sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}D_{n-k-m,\lambda}^{(k)}(r)_{m,\lambda}, \qquad (44)
$$

$$
= \sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}\lambda^{n-k-m}D_{n-k-m}^{(k)}(r)_{m,\lambda}, \qquad (45)
$$

$$
= \sum_{m=0}^{n-k}\binom{k+m}{m}\lambda^{n-k-m}s(n,k+m)(r)_{m,\lambda}. \qquad (46)
$$

(45)(46)is obviously true,next prove(47),

Proof.

$$
\begin{aligned}
S_{1,\lambda}^{(r)}(n,k) &= \sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}(-\lambda)^{n-k-m}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{n-k-m}(r)_{m,\lambda} \\
&= \sum_{m=0}^{n-k}\frac{1}{\binom{n}{k+m}}(\lambda)^{n-k-m}\binom{n}{k}\binom{n-k}{m}(-1)^{n-k-m}\binom{n}{k+m}E(u_1\Gamma_1+u_2\Gamma_2+\cdots+u_k\Gamma_k)^{n-k-m}(r)_{m,\lambda} \\
&= \sum_{m=0}^{n-k}\frac{(k+m)!}{m!k!}(\lambda)^{n-k-m}s(n,k+m)(r)_{m,\lambda}.
\end{aligned}
$$

□

Corollary 7. *Under the circumstance of theorem 5, we obtain the equation:*

$$S_{1,\lambda}(n,k) = \lambda^{n-k} s(n,k). \tag{47}$$

because $s(n,k) = (-1)^{n-k}\binom{n-1}{k-1}E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_{k-1}\Gamma_{k-1} + \Gamma_k)^{n-k}$,*thus we have the second moment representation of* $S_{1,\lambda}(n,k)$,

$$S_{1,\lambda}(n,k) = (-\lambda)^{n-k}\binom{n-1}{k-1}E(u_1\Gamma_1 + u_2\Gamma_2 + \cdots + u_{k-1}\Gamma_{k-1} + \Gamma_k)^{n-k}. \tag{48}$$

Corollary 8. *Under the circumstance of theorem 5 and theorem 6,the* λ*-analogues of r-Stirling numbers of the first kind and* $S_{1,\lambda}(n,k)$ *have the following equation:*

$$S^{(r)}_{1,\lambda}(n,k) = \sum_{m=0}^{n-k}(-1)^l\binom{n}{k}S_{1,\lambda}(n-k,m)(r)_{m,\lambda}. \tag{49}$$

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