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Minisum and minimax transfer point location problem with random demands points

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Abstract.

This paper is concerned with analyzing some models of the weighted transfer point location problem under the minisum and minimax criterions when demand points are randomly distributed over regions of the plane and the location of the service facility is known. In case of minisum objective with rectilinear distance, an iterative procedure was constructed for estimating the optimal transfer point location using the hyperbolic application procedure. Exact analytic solution was obtained when the random demand points follow uniform distributions. A unified analytic optimal solution was provided for all types of probability distributions of the random demand points when the distance is the squared Euclidean distance. For minimax objective with squared Euclidean distance, an iterative procedure based on Karush-Kuhn-Tucker conditions was developed to produce an approximate solution to the optimal solution. Illustrative numerical examples were provided.

Keywords: Transfer point location; Minisum facility location; Minimax facility; Probabilistic demands.

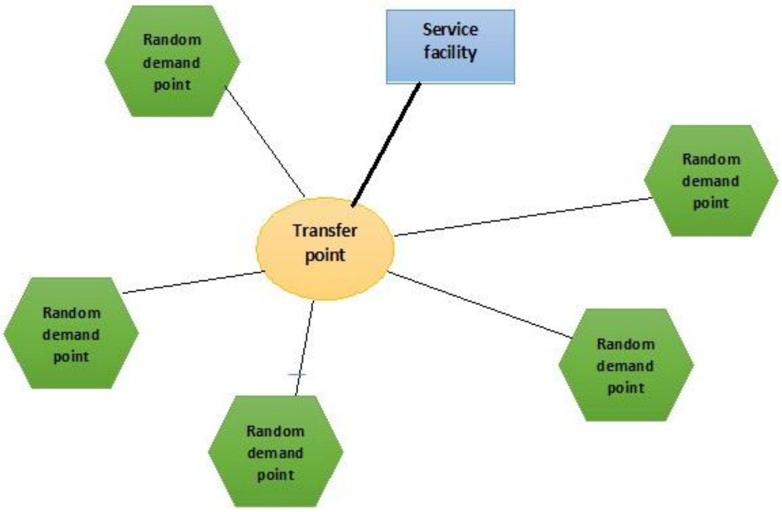
1. Introduction

Given the location of a service facility, the transfer point location problem (TPLP) is defined as the establishment of a new location called " transfer point " that connects all demand points to the service facility in such a way to fulfill some optimization criterion. The transfer point location problem was first studied by Berman et al. [6]. They introduced three models and proved properties of the solutions to the first model. Both planar and network variants, as well as the minisum and minimax objectives, were considered.In Berman et al. [4], He proposed heuristic approaches for the solution to the multiple TPLP and provided computational experiments. Later, Berman et al.[7] applied the results of a previous work to solve the multiple TPLP when the location of the facility is known. Both minisum and minimax versions of the models were investigated in the plane and on the network. Transfer point location problem is one of the latest related models to hub and spoke location models. According to O'Kelly [15], Hub refers to a central facility which connects a set of interacting points. Although the network hub location problem was first addressed by Goldman [12], the investigation on hub location started with pioneering researches of O'Kelly [16-17]. Continuous hub location problem is concerned with locating hub facilities on a plane rather than on the nodes of a network. For a comprehensive review of hub and spoke models see Campbell et al.[8] and Alumur and Kara [1]. All of the mentioned previous studies consider customer locations as being fixed. However, in several instances, the assumption of known fixed demands points does not hold. Consider for example, the location of a single fire station, which is to serve potential residents of a new residential community. In such a setting, one can assume that the location probabilistic distribution of each demand point on the plane is known but it is not known which particular one will request service. Randomness aspect in location theory has been studied by many researchers, see, e.g., [2-3, 5, 9, 11, 19]. Wheras the literature on stochastic models in facility location is important, research on stochastic models for the transfer point problem is rare. The first research on the stochastic transfer point problem was investigated by Hosseinnijou and Bashiri [13]. They analyzed two stochastic models of the weighted TPLP by considering the minimax criterion with random demand points whose coordinates follow uniform distributions. For the first model, they found analytical optimal solution and for the second one, they suggest a numerical scheme to find the optimal solution. Later, Youselfi et al. [20] developed a weighted transfer point location problem in which demand points have probabilistic coordinates. The proposed model is formulated as a probabilistic unconstrained nonlinear programming and the optimum values of decision variables are obtained in the form of probability distribution functions.

In this paper, we extend the investigation of stochastic transfer point models and propose new approaches to solve them. Our aim is to determine the optimal location of a transfer point under minisum and minimax criterions when coordinates of the demands points are random and the location of the service facility is known. The remaining of the paper is organized as follows. In Section 2, the Stochastic transfer point location models are formulated and main results are described. Illustrative examples are provided.

2. Analysis

Suppose there is a set of m demand points $\{Y_i = (U_i, V_i) : i = 1,2, \dots, m\}$ having random coordinates in the plane. Assume that U_i (resp.V_i) has probability density function $f_{U_i}(u)$ (resp. $f_{V_i}(v)$) and cumulative distribution function $F_{U_i}(u)$ (resp. $F_{V_i}(v)$). Let $X = (x, y)$ be the location of the transfer point, $S = (a, b)$ the service facility location, α factor by which travel to the transfer point from the service facility is multiplied ($0 < \alpha < 1$), and $w_i > 0$ be the cost per unit distance between the transfer point and random demand location i, $i = 1,2, \dots, m$. $d_R(.,.)$ and $d_{SE}(.,.)$denotes, respectively, the rectilinear and the squared euclidean distance. The following figure depicts a pictorial representation of the problem elements.



The problem we address in this section is of locating a transfer point by considering the following two problems:

- ***Minisum Problem:*** *Minimizes the weighted sum of the expected distances from random demands points to the service facility through the transfer point.*

- ***Minimax Problem:*** *Minimizes the maximum weighted expected distances from random demands points to the service facility through the transfer point.*

2.1. Minisum transfer point location problem

Let $X = (x, y)$ be the location of the transfer point, $S = (a, b)$ the location of the service facility, α factor by which travel to the transfer point from the service facility is multiplied ($0 < \alpha < 1$), and $w_i > 0$ be the cost per unit distance between the transfer point and random demand location i, $i = 1,2, \dots, m$.

The problem can be stated as

$$min_X\ F(X) = \sum_{i=1}^{m} w_i(E[d(X,Y_i)] + \alpha d(X,S)) \tag{1}$$

$d(.,.)$ denotes either the rectilinear or the squared Euclidean distance.

2.1.1. Case of rectilinear distance

The rectilinear distance between the demand point $Y_i = (U_i, V_i)$ and the transfer point $X = (x, y)$ is $d_R(X, Y_i) = |x - U_i| + |y - V_i|$ and between the the transfer point $X = (x, y)$ and service facility $S = (a, b)$ is $d_R(X, S) = |x - a| + |y - b|$.

Problem (1) becomes

$$min_{(x,y)}\ F(x,y) = \sum_{i=1}^{m} w_i(E[|x - U_i| + |y - V_i|] + \alpha(|x - a| + |y - b|)) \tag{2}$$

where $E[\dots]$ is the expected value of some random variable.

We have $E[d_R(X, Y_i)] = E[|x - U_i|] + E[|y - V_i|]$,

$E[|x - U_i|] = E(U_i) - x\left[1 - 2F_{U_i}(x)\right] - 2\int_{-\infty}^{x} u f_{U_i}(u)du$,

and $E[|y - V_i|] = E(V_i) - y\left[1 - 2F_{V_i}(y)\right] - 2\int_{-\infty}^{y} v f_{V_i}(v)dv$ (by definition of the expected value), so the objective function $F(x,y)$ of problem (1) can be written as $F(x,y) = f_1(x) + f_2(y)$

where $f_1(x) = \sum_{i=1}^{m} w_i(E[|x - U_i|] + \alpha|x - a|) = \sum_{i=1}^{m} w_i\left(E(U_i) - x\left[1 - 2F_{U_i}(x)\right] -\right.$
$2{-}\infty xufUiudu + awx{-}a$

and $f_2(y) = \sum_{i=1}^{m} w_i(E[|y - V_i|] + \alpha|y - b|) = \sum_{i=1}^{m} w_i\left(E(V_i) - y\left[1 - 2F_{V_i}(y)\right] -\right.$
$2{-}\infty yvfVivdv + awy{-}b$

So that $min_{(x,y)}\ F(x,y) = \min_x f_1(x) + \min_y f_2(y)$

Since $f_1(x)$ and $f_2(y)$ have the same form, then any procedure developed for minimizing $f_1(x)$ will apply to $f_2(y)$.

Consider the problem

$$\min_x f_1(x) \tag{3}$$

$$\text{where } f_1(x) = \sum_{i=1}^{m} w_i \left(E(U_i) - x\left[1 - 2F_{U_i}(x)\right] - 2\int_{-\infty}^{x} u f_{U_i}(u) du \right) + \alpha\bar{w}|x-a|$$

, and $\bar{w} = \sum_{i=1}^{m} w_i$.

Now, we show that $f_1(x)$ is convex.

Let $f_{11}(x) = \sum_{i=1}^{m} w_i \left(E(U_i) - x\left[1 - 2F_{U_i}(x)\right] - 2\int_{-\infty}^{x} u f_{U_i}(u) du \right)$

and $f_{12}(x) = \alpha\bar{w}|x-a|$, so that $f_1(x) = f_{11}(x) + f_{12}(x)$

The second derivative of $f_{11}(x)$ is given by $f_{11}^{''}(x) = 2\sum_{i=1}^{m} w_i f_{U_i}(x)$

Since $f_{U_i}(x)$ is a probability density function, therefore $f_{11}^{''}(x) \geq 0$, so that $f_{11}(x)$ is convex.

Clearly $f_{12}(x) = \alpha\bar{w}|x-a|$ is a strictly convex function. Therefore $f_1(x)$ is strictly convex and hence $f_1(x)$ has a unique global minimum.

To find this global minimum, we use an approach called Hyperbolic Approximation Procedure or HAP (Eyster et al. [10]). The non-differentiable function part of $f_1(x)$ (resp.$f_2(y)$), namely $\alpha\bar{w}|x-a|$ (resp. $\alpha\bar{w}|y-b|$) is perturbed by adding a positive-valued constant ε to obtain the smooth functions $\hat{f}_1(x)$ (resp. $\hat{f}_2(y)$called hyperbolic approximations given by :

$\hat{f}_1(x) = \sum_{i=1}^{m} w_i \left(E(U_i) - x\left[1 - 2F_{U_i}(x)\right] - 2\int_{-\infty}^{x} u f_{U_i}(u) du \right) + \alpha\bar{w}[(x-a)^2 + \varepsilon]^{1/2}$ and

$$\hat{f}_2(y) = \sum_{i=1}^{m} w_i \left(E(V_i) - y\left[1 - 2F_{V_i}(y)\right] - 2\int_{-\infty}^{y} v f_{V_i}(v) dv \right) + \alpha\bar{w}[(y-b)^2 + \varepsilon]^{1/2}$$

where ε is a small positive number.

We first note that $\hat{f}_1(x)$ (resp. $\hat{f}_2(y)$) is strictly convex and therefore has a unique global minimum.

Problem (3) is approximated by:

$$\min_{x} \hat{f}_1(x) \tag{4}$$

According to Rosen and Xue [18], the optimal solution to problem (4) can be made as close as possible to the optimal solution of problem (3) by choosing small value of ε. Therefore any solution to problem (4) can be considered as an approximate solution to problem (3).

The HAP iterations for problem (4) are given by:

$$x^{(t+1)} = \frac{1}{2\alpha\bar{w}} \sum_{i=1}^{m} w_{ij} \left[1 - 2F_{U_i}\left(x^{(t)}\right)\right]\left[\left(x^{(t)} - a\right)^2 + \varepsilon\right]^{1/2} + a$$

Algorithm HAP

STEP_0. Choose $x^{(0)} = \sum_{i=1}^{m} w_i \alpha_i \Big/ \sum_{i=1}^{m} w_i$ as initial point (*)

Let $\varepsilon > 0$, and $\delta > 0$. Set $t = 0$ and go to step_1.

STEP_1. $x^{(t+1)} = \frac{1}{2\alpha\bar{w}} \sum_{i=1}^{m} w_{ij} \left[1 - 2F_{U_i}\left(x^{(t)}\right)\right]\left[\left(x^{(t)} - a\right)^2 + \varepsilon\right]^{1/2} + a$

STEP_2. If $\left|x^{(t+1)} - x^{(t)}\right| \leq \delta$, then stop.

Otherwise, replace t with t +1 and go to step_1.

Rosen and Xue [18] prove that the HAP is a descent algorithm and that it always converges to the minimum of the objective function from any initial point.

(*)Although the algorithm allows choosing an arbitrary initial point, it is preferred that the initial location of the transfer point lie in the convex hull of the demand points (considering their expected values). One possible choice is the following center of gravity rule (Rosen and Xue [18]).

$x^{(0)} = \sum_{i=1}^{m} w_i \mu_i \Big/ \sum_{i=1}^{m} w_i$ and $y^{(0)} = \sum_{i=1}^{m} w_i \bar{\mu}_i \Big/ \sum_{i=1}^{m} w_i$ (for the problem $min_y \hat{f}_2(y)$)

where $\mu_i = E[U_i]$ and $\bar{\mu}_i = E[V_i]$.

Example 1. Suppose we have three demand points $Y_i = (U_i, V_i)$, $i = 1,2,3$,distributed according to a bivariate uniform distribution over the rectangular region $[a_i, b_i]$x$[c_i, d_i]$, $i = 1,2,3$.The service facility is located at $S = (5,4)$ and $\alpha = 0.4$.Table 1 gives the data for this example. We solve this problem using a matlab code devised for the proposed algorithm with $\varepsilon = 0.01$ and, $\delta = 0.01$ with the starting point $X^{(0)} = \left(x^{(0)}, y^{(0)}\right) = (3.37, 3.62)$. The optimal location of the transfer point is $X^* = (x^*, y^*) = (4.93, 4.0)$.

Table 1

I	w_i	(a_i, b_i)	(c_i, d_i)
1	2	(1,4)	(3,11)
2	2	(2,10)	(4,9)
3	3	(7,12)	(1,4)

In what follows, we present another method to solve problem (3) for the special case where U_i and V_i follow the uniform distributions over the interval $[a_i, b_i]$ and $[c_i, d_i]$, $i = 1,2.., m$, respectively.

The probability density functions are given by:

$$f_{V_i}(v)=\begin{cases}\dfrac{1}{c_i-d_i}, c_i \le v \le d_i \\ 0, otherwise\end{cases} \quad f_{U_i}(u)=\begin{cases}\dfrac{1}{b_i-a_i}, a_i \le u \le b_i \\ 0, otherwise\end{cases}$$

and their corresponding probability distribution functions by:

$$F_{U_i}(u)=\frac{u-a_i}{b_i-a_i}, a_i \le u \le b_i \; F_{V_i}(v)=\frac{v-c_i}{d_i-c_i}, c_i \le v \le d_i$$

Problem (3) can be expressed as

$$\min_x f_1(x)=\sum_{i=1}^{m} w_i^{'}[(x-a_i)^2+(x-b_i)^2]+\alpha\bar{w}|x-a|$$

where $\bar{w}=\sum_{i=1}^{m} w_i$, and $w_i^{'}=\frac{w_i}{2(b_i-a_i)}$

$f_1(x)$ can also be written as

$$f_1(x)=\begin{cases}\displaystyle\sum_{i=1}^{m} w_i^{'}[(x-a_i)^2+(x-b_i)^2]+\alpha\bar{w}(a-x), & x<a \\ \displaystyle\sum_{i=1}^{m} w_i^{'}[(x-a_i)^2+(x-b_i)^2]+\alpha\bar{w}(x-a), & x\ge a\end{cases}$$

and its piecewise derivative

$$f_1^{'}(x)=\begin{cases}\displaystyle\sum_{i=1}^{m} w_i\left(2\frac{x-a_i}{b_i-a_i}-1\right)-\alpha\bar{w}, & x<a \\ \displaystyle\sum_{i=1}^{m} w_i\left(2\frac{x-a_i}{b_i-a_i}-1\right)+\alpha\bar{w}, & x\ge a\end{cases}$$

The solution to equation $f_1^{'}(x)=0$, denoted x_0, is given by

$$x_0=\begin{cases}\dfrac{\sum_{i=1}^{m} w_i\left(\frac{a_i+b_i}{b_i-a_i}\right)+\alpha\bar{w}}{2\sum_{i=1}^{m}\frac{w_i}{b_i-a_i}}, & \text{if} x<a \\ \dfrac{\sum_{i=1}^{m} w_i\left(\frac{a_i+b_i}{b_i-a_i}\right)-\alpha\bar{w}}{2\sum_{i=1}^{m}\frac{w_i}{b_i-a_i}}, & \text{if} x\ge a\end{cases}$$

Since $f_1(x)$ is strictly convex, therefore the unique optimal solution $x^*$to problem (3)

$$\min_x f_1(x)$$

is given by $$x^* = \begin{cases} \dfrac{\sum_{i=1}^m w_i\left(\frac{a_i+b_i}{b_i-a_i}\right)+\alpha\bar{w}}{2\sum_{i=1}^m \frac{w_i}{b_i-a_i}}, & \text{if} \dfrac{\sum_{i=1}^m w_i\left(\frac{a_i+b_i}{b_i-a_i}\right)+\alpha\bar{w}}{2\sum_{i=1}^m \frac{w_i}{b_i-a_i}} < a \\ \dfrac{\sum_{i=1}^m w_i\left(\frac{a_i+b_i}{b_i-a_i}\right)-\alpha\bar{w}}{2\sum_{i=1}^m \frac{w_i}{b_i-a_i}}, & \text{if} \dfrac{\sum_{i=1}^m w_i\left(\frac{a_i+b_i}{b_i-a_i}\right)-\alpha\bar{w}}{2\sum_{i=1}^m \frac{w_i}{b_i-a_i}} \geq a \\ a & , \quad \text{otherwise} \end{cases}$$

2.1.2. Case of squared euclidean distance

The squared euclidean distance between the demand point $Y_i = (U_i, V_i)$ and the transfer point $= (x, y)$ is $d_{SE}(X, Y_i) = (x - U_i)^2 + (y - V_i)^2$ and between the service facility $S = (a, b)$ and the transfer point $= (x, y)$ is $d_{SE}(X, S) = (x - a)^2 + ((y - b)^2$. Sustituting into problem (1) gives

$$\min_{(x,y)} F(x,y) = \sum_{i=1}^m w_i(E[(x - U_i)^2 + (y - V_i)^2] + \alpha[(x - a)^2 + (y - b)^2])$$

$E[d_S(X, Y_i)] = E[(x - U_i)^2] + E[(y - V_i)^2]$ and by definition of expected value, we have

$$E[(x - U_i)^2] = (x - E[U_i])^2 + Var[U_i]$$

$$E[(y - V_i)^2] = (y - E[V_i])^2 + Var[V_i]$$

$Var[U_i] = \sigma^2{}_i$(resp. $Var[V_i] = \bar{\sigma}^2{}_i$) is the variance of the random variable U_i(resp.V_i).

The objective function $F(x, y)$ of problem (1) can then be written as

$$F(x, y) = f_1(x) + f_2(y)$$

Where $f_1(x) = \sum_{i=1}^m w_i((x - E[U_i])^2 + Var[U_i]) + \alpha(x - a)^2) = \sum_{i=1}^m w_i((x - \mu_i)^2 + \sigma^2{}_i) + \alpha x{-}a2)$

and $f_2(y) = \sum_{i=1}^m w_i((y - E[V_i])^2 + Var[V_i]) + \alpha(y - b)^2) = \sum_{i=1}^m w_i((y - \bar{\mu}_i)^2 + \bar{\sigma}^2{}_i) + \alpha y{-}b2)$

so that

$$\min_{(x,y)} F(x, y) = \min_x f_1(x) + \min_y f_2(y) \ (5)$$

Then the optimal solution to problem (5) may be obtained by solving the independent problems $\min_x f_1(x)$ and$\min_y f_2(y)$.

Consider the problem $\min_x f_1(x)$

where $\quad f_1(x) = \sum_{i=1}^{m} w_i([(x-\mu_i)^2 + \sigma^2{}_i] + \alpha[(x-a)^2])$

The second derivative of $f_1(x), f_1^{''}(x) = 2(1+\alpha)\sum_{i=1}^{m} w_i$, is strictly positive. This implies that $f_1(x)$ has unique global minimum. Setting the first derivative to zero gives the following unique solution

$$x^* = \frac{\alpha\overline{w}a + \sum_{i=1}^{m} w_i\mu_i}{\overline{w}(1+\alpha)}$$

Similarly, the unique global minimum of problem $\min_y f_2(y)$ is given by

$$y^* = \frac{\alpha\overline{w}b + \sum_{i=1}^{m} w_i\bar{\mu}_i}{\overline{w}(1+\alpha)}$$

We conclude that the unique global minimum of problem (5) is

$$x^* = \frac{\alpha\overline{w}a + \sum_{i=1}^{m} w_i\mu_i}{\overline{w}(1+\alpha)}$$

$$y^* = \frac{\alpha\overline{w}b + \sum_{i=1}^{m} w_i\bar{\mu}_i}{\overline{w}(1+\alpha)}$$

Remark 1. The above optimal solution of problem (5) has a unique expression for all types of bivariate distribution of the random demand points. Moreover, the optimal solution depends only on the expected values of the random variable U_i and V_i.

Example 2. Suppose we have three demand points $Y_i = (U_i, V_i)$, $i = 1,2,3$ distributed according to a bivariate normal distribution, U_i □ $N(\mu_i, \sigma_i^2)$ and V_i □ $N(\bar{\mu}_i, \bar{\sigma}_i^2)$. The service facility is located at $S = (3,5)$ and the value of $\alpha = 0.3$.Tables 2 gives the data for this example. Simple calculation gives $X^* = (x^*, y^*) = (2.871, 2.435)$as the optimal location of the transfer point.

Table 2

I	w_i	(μ_i, σ_i)	$(\bar{\mu}_i, \bar{\sigma}_i)$
1	1	(3,1)	(3,5)
2	3	(2,2)	(4,3)
3	2	(4,1)	(2,2)

2.2. Minimax transfer point location problem

Let $X = (x, y)$ be the location of the transfer point, $S = (a, b)$ the location of the service facility, α factor by which travel to the transfer point from the service facility is multiplied ($0 < \alpha < 1$), and $w_i > 0$ be the cost per unit distance between the transfer point and random demand location i, $i = 1,2, \dots, m$.

The problem can be stated as

$$\min_X F(X) = \max_{1\le i\le m}\{w_i(E[d(X, Y_i)] + \alpha d(X, S))\} \tag{6}$$

$d(.,.)$ denotes either the rectilinear or the squared Euclidean distance.

2.2.1. Case of squared euclidean distance

The squared euclidean distance between the demand point $Y_i = (U_i , V_i)$ and the transfer point $X = (x, y)$ is $d_{SE}(X , Y_i) = (x - U_i)^2 + (y - V_i)^2$ and between the the transfer point $X = (x, y)$ and service facility $S = (a, b)$ is $d_{SE}(X , S) = (x - a)^2 + (y - b)^2$.

Problem (6) becomes

$$\min_{(x,y)} F(x, y) = \max_{1\le i\le m}\{w_i(E[(x - U_i)^2 + (y - V_i)^2] + \alpha[(x - a)^2 + (y - b)^2])\}$$

or $min_{(x,y)} F(x, y) = \max_{1\le i\le m}\{w_i((x - \mu_i)^2 + (y - \bar{\mu}_i)^2 + \sigma_i^2 + \bar{\sigma}_i^2 + \alpha[(x - a)^2 +$
$y{-}b2]$ (7)

Let $r = \max_{1\le i\le m}\{w_i((x - \mu_i)^2 + (y - \bar{\mu}_i)^2 + \sigma_i^2 + \bar{\sigma}_i^2 + \alpha[(x - a)^2 + (y - b)^2])\}$.

Then problem (7) is equivalent to the following non-linear program

$$min(r)$$
$$s.t. \;\; w_i((x - \mu_i)^2 + (y - \bar{\mu}_i)^2 + \sigma_i^2 + \bar{\sigma}_i^2 + \alpha[(x - a)^2 + (y - b)^2) \le r,$$
$i = 1,2, . . m$ (8)

Remark 2. Note that the nonlinear program (8) has a unique expression for all types of bivariate distribution of the random demand points.

We now use a convenient iterative procedure to solve problem (8). The approach produces successively improved approximation to a solution to the Karush-Kuhn-Tucker conditions that are sufficient for this problem. These conditions are:

1. $\sum_{i=1}^m \lambda_i w_i(x - \mu_i) = 0$, $i = 1,2, \dots, m$

2. $\sum_{i=1}^m \lambda_i = 1$

3. $\lambda_i[w_i((x - \mu_i)^2 + (y - \bar{\mu}_i)^2 + \sigma_i^2 + \bar{\sigma}_i^2 + \alpha[(x - a)^2 + (y - b)^2]) - r] = 0$

4. $\lambda_i \ge 0$, $i = 1,2, \dots, m$and λ_i are the Lagrange multipliers.

Iterative procedure for estimating the optimal transfer point location X^*

The present algorithm is a variant of Lawson-Charalambous algorithm (Love *et al* [14]). Before describing the algorithm, we define the following auxiliary problem:
For given values $\bar{\lambda}_i$ of λ_i, $i = 1,2,\dots,m$, we will need to solve the following nonlinear program:

$$\min_{(x,y)}\left(g(X,\bar{\lambda}) = \sum_{i=1}^{m} \bar{\lambda}_i w_i((x-\mu_i)^2 + (y-\bar{\mu}_i)^2 + \alpha[(x-a)^2 + (y-b)^2] + \gamma_i{}^2\right), \text{ where } \gamma_i{}^2 = \sigma_i^2 + \bar{\sigma}_i^2 \qquad (9)$$

Problem (9) is a minisum facility location problem with squared Euclidean distance (see Section 2.1.2).

Algorithm

STEP_0. Let $t = 0$ andsome $\varepsilon > 0$. Set $\lambda_i^{(t)} = 1$, $i = 1,2,..,m$.

STEP_1. Find a minimizer $X^{(t)} = (x^{(t)}, y^{(t)})$ of $\left(g(X^{(t)}, \lambda^{(t)})\right)$

Then set: $\lambda_i^{(t+1)} = \lambda_i^{(t)} \dfrac{w_i\left(\left(x^{(t)}-\mu_i\right)^2+\left(y^{(t)}-\bar{\mu}_i\right)^2+\alpha\left[\left(x^{(t)}-a\right)^2+\left(y^{(t)}-b\right)^2\right]+\gamma_i{}^2\right)}{S}$

where $S = \sum_{i=1}^{m} \lambda_i^{(t)} w_i \left(\left(x^{(t)} - \mu_i\right)^2 + \left(y^{(t)} - \bar{\mu}_i\right)^2 + \alpha\left[\left(x^{(t)} - a\right)^2 + \right.\right.$

$y(t)-b2+\gamma i2$

STEP_2. Calculate $r_1 = F\left(X^{(t)}\right)$

$$\text{Let} r_0 = \sum_{i=1}^{m} \lambda_i^{(t+1)} w_i \left(\left(x^{(t+1)} - \mu_i\right)^2 + \left(y^{(t+1)} - \bar{\mu}_i\right)^2 + \alpha\left[\left(x^{(t+1)} - a\right)^2 + \left(y^{(t+1)} - b\right)^2\right] + \gamma_i{}^2\right)$$

If $\frac{|r_1 - r_0|}{r_0} < \varepsilon$, then stop. Otherwise let $t = t + 1$ and return to $\boldsymbol{STEP\_2}$.

Example 3. Suppose we have three demand points $Y_i = (U_i, V_i)$, $i = 1,2,3$ distributed according to a bivariate normal distribution, $U_i \square\ N(\mu_i, \sigma_i^2)$ and $V_i \square\ N(\bar{\mu}_i, \bar{\sigma}_i^2)$. The service facility is located at $S = (10,20)$ and the value of $\alpha = 05$.Table 3 gives the data for this example. We solved this problem using a matlab code devised for the proposed algorithm with $\varepsilon = 0.001$. The optimal location of the transfer point is at $X^* = (x^*, y^*) = (8.269,\ 17.261)$

Table 3

i	w_i	(μ_i, σ_i)	$(\bar{\mu}_i, \bar{\sigma}_i)$
1	1	(3,1)	(20,1)
2	4	(10,3)	(25,6)
3	2	(15,4)	(10,2)

2.2.2. Case of rectilinear distance.

Consider problem (6) with rectilinear distance

$$\min_X F(X) = \max_{1\le i\le m}\{w_i(E[d_R(X,Y_i)] + \alpha d_R(X,S))\} \qquad (10)$$

Substituting $d_R(X\,,Y_i) = |x - U_i| + |y - V_i|$ and $d_R(X\,,S\,) = |x - a| + |y - b|$ into problem(10) gives

$$min_{(x,y)}\, F(x,y) = \max_{1\le i\le m}\{w_i(E[|x - U_i| + |y - V_i|] + \alpha(|x - a| + |y - b|))\} \; (11)$$

By definition of expected value, we have

$$E[|x - U_i|] = E(U_i) - \; x\left(1 - \; 2F_{U_i}(x)\right) - \; 2\int_{-\infty}^{x} uf_{U_i}(u)du$$

$$E[|y - V_i|] = E(V_i) - \; y\left(1 - \; 2F_{V_i}(y)\right) - \; 2\int_{-\infty}^{y} vf_{V_i}(v)dv$$

Let $r = \max_{1\le i\le m}\{w_i(E[|x - U_i| + |y - V_i|] + \alpha(|x - a| + |y - b|))\}$. Then problem

Problem (11) is equivalent to the following nonlinear program

$$min\, r$$
$$s.t.\;\; w_i(E[|x - U_i| + |y - V_i|] + \alpha(|x - a| + |y - b|)) \le r,$$
$$i = 1,2,..m$$

or$min(r)$

$$s.t.\ \ w_i\left(E(U_i) - x\left(1 - 2F_{U_i}(x)\right) - 2\int_{-\infty}^{x} uf_{U_i}(u)du + E(V_i) - y\left(1 - 2F_{V_i}(y)\right) - 2\int_{-\infty}^{y} vf_{V_i}(v)dv + \alpha(|x-a| + |y-b|)\right) \leq r,$$

$$i = 1,2,..m$$

The form of the above problem will vary according to the random distributions of the demand points (U_i , V_i). Therefore finding closed form for the optimal solution is infeasible. This problem is a nonlinear convex program (see Section 2.1.1) and hence many numerical methods exist which solve it efficiently.

3 Conclusions

This paper dealt with determining the optimal location of a transfer point in the plane under minisum and minimax criterions when coordinates of the demands points are random and the location of the service facility is known. New approaches have been develop to investigate both minisum and minimax versions under rectilinear and squared Euclidean distances. Some results were described and illustrative examples were provided. Possible extension of this work would be the study of the obnoxious TPLP and the multi-TPLP with random demands points.

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