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Cohomology Groups, Currents and D_X – Schemes on ∂ – Cohomology

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Abstract.

We consider some cohomology groups lemmas as given by Poincaré and Dolbeault-Grothendieck, to establish the De Rham and Dolbeault theorems through currents, and after to be applied to define currents on Dolbeault cohomology. One advantage of this application of currents is the commutation between differential operator and current, which will be demonstrated to a complex holomorphic manifold whose co-cycles under a current are complex domains conformed by holomorphic hyperplanes. In the paper are explained wifely these versions and are applied some D_X –schemes to study of complex holomorphic manifolds and its tomography in cycles of co-dimensions 1, and $n-q$.
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Keywords: ∂ – cohomology; Čech cohomology; Cohomology Groups; Currents; D_X – Schemes; Radon Transform; Resolution of Acyclic Sheaves.

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1. Introduction

Classes of cohomology are obtained to the representation of basic properties of smooth manifolds. These results a useful tool to the study of differential operators that compose the different field equations from the differential forms, which can be represented these differential operators. However, we want cohomology groups of sheaves whose germs are these differential operators and that considering to a complex Riemannian manifold as the best adequate to model the space-time including the microscopic effects and singularities, can be worked generalizations of curvature and other observables. However, in the invariant studies between homology and curvature [1], is very important the relations that can be given to start of a good integrals theory to explore a complex Riemannian manifold on the base of the ∂ – cohomology and the holomorphic functions that can be used to reconstruct this ∂ – cohomology.

Then images of integral transforms are geometrical invariants to reconstruct the Riemannian manifold, considering that cohomology classes determine integral kernels on the complex spaces as $\mathbb{C}^n$, or even $\mathbb{CP}^n$. But the inversion formula, that is to say, the reconstruction problem of the cohomology class from the integral transform establishes the problem of that many forms represent the same cohomology class. In this situation the natural solution for this

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last problem is consider an integral transform by a differential operator, considering the pertinent parity of space dimension where the inversion has place.

In the sheaves context, where the differential operators are their germs, there are many ways to define cohomology groups for a sheaf and such groups are eventually all isomorphic.
Likewise, is punctually important consider some facts on de Rham cohomology and its extension in the complex domain through Dolbeault cohomology.
In a de Rham complex, we consider the exterior differential forms on some smooth manifold M, with the exterior derivative d, as the differential, then this complex stays written as:

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \ldots, \tag{1}$$

where $\Omega^k(M)$, is the differential $k-$form space. The de Rham cohomology will help to classify these different types of closed forms on a manifold M [1].
However, if these object spaces are sheaf spaces, as we want establish to give a direct classification of the differential operators seen, then as germs of these sheaves satisfy to a de Rham cohomology that:

$$0 \to \mathbb{R} \to D^0 \xrightarrow{d} D^1 \xrightarrow{d} D^2 \xrightarrow{d} \ldots, \tag{2}$$

In addition, we can write to the Dolbeault cohomology[2] that:

$$0 \to \Omega_X^p \to D^{p,0} \xrightarrow{\bar{\partial}} D^{p,1} \xrightarrow{\bar{\partial}} D^{p,2} \xrightarrow{\bar{\partial}} \ldots, \tag{3}$$

from which de Rham cohomology group and Dolbeault group can be defined.
Then their cochain complex to de Rham cohomology is:

$$0 \to \mathbb{R} \to D'^0 \xrightarrow{d} D'^1 \xrightarrow{d} D'^2 \xrightarrow{d} \ldots, \tag{4}$$

and the corresponding to Dolbeault cohomology:

$$0 \to \Omega_X^p \to D'^{p,0} \xrightarrow{\bar{\partial}} D'^{p,1} \xrightarrow{\bar{\partial}} D'^{p,2} \xrightarrow{\bar{\partial}} \ldots, \tag{5}$$

We want to prove that the new cohomology groups are isomorphic to the old ones.
From (2)-(5), if we regard $\mathbb{R}$, as a sheaf group of constant real-valued function; it gives different resolutions[3]. The question is whether these different resolutions give the same cohomology group.

There are many ways to define cohomology groups for a sheaf and such groups are eventually all isomorphic.

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[1] This classification is realized through two closed forms, for example ξ, and $\zeta \in \Omega^k(M)$, which are cohomologous if these differ for an exact form. Likewise, if $\xi - \zeta$, is exact. Then is defined the $k-\mathrm{th}$ de Rham cohomology group $H_{dR}^k(M)$, to be the set of equivalence classes, that is to say, the set of closed forms in $\Omega^k(M)$, modulo the exact forms.
Note that, for any manifold M, with n, connected components $H_{dR}^0(M) \cong \mathbb{R}^n$.
This follows from the fact that any smooth function on M, with zero derivative (i.e. locally constant) is constant on each one the connected components of M.

[2] This cohomology is the complex analogous of the de Rham cohomology, that is to say, is the corresponding analogous of the de Rham cohomology to complex manifolds.
[3] Recall that given a sheaf $\mathfrak{F}$, over a manifolds X, a resolution of $\mathfrak{F}$, is a complex $0 \to \mathfrak{F}^0 \to \mathfrak{F}^1 \to \ldots$, together with a homomorphism $\mathfrak{F}^0 \to \mathfrak{F}^1$, such that

$$0 \to \mathfrak{F}^0 \to \mathfrak{F}^1 \to \mathfrak{F}^2 \ldots,$$

is an exact complex of sheaves.

We consider the Čech cohomology [4] and the resolution of flasque sheaves [2] [5] then any sheaf $\mathfrak{F}$, on X, admits a resolution $0 \to \mathfrak{F}^0 \to \mathfrak{F}^1 \to \ldots$, such that all schemes $\mathfrak{F}^i, i \geq 0$, are flasque. If $\mathfrak{F}$, is a sheaf of Abelian groups over a manifold X, then $\mathfrak{F}$, admits a resolution of flasque sheaves $\mathfrak{F} \to \mathfrak{F}^\bullet$, so that the $j\text{th}-$cohomology group $H^j(X,\mathfrak{F})$, of the sheaf $\mathfrak{F}$, is the $j\text{th}-$cohomology of the complex which is defined by:

$$H^j(X,\mathfrak{F}) := \frac{\ker(\phi^j : \mathfrak{F}^j(X) \to \mathfrak{F}^{j+1}(X))}{\operatorname{Im}(\phi^{j-1} : \mathfrak{F}^{j-1}(X) \to \mathfrak{F}^j(X))}, \tag{6}$$

This definition of cohomology is independent of the chosen flasque resolution. If the manifold X, is para-compact[6], then the above two groups are isomorphic, that is to say, $\check{H}^q(X,\mathfrak{F}) \cong H^q(X,\mathfrak{F})$.
Remember that the Flasque sheaves are soft and acyclic. Thus, we will be interested in their resolution.

2. Considering Resolutions of Acyclic Sheaves

Let $\mathfrak{F}^\bullet$, be a resolution of a sheaf $\mathfrak{F}$, for sheaves $\mathfrak{F}^\bullet$. We say that the resolution $\mathfrak{F}^\bullet$, is acyclic on X, if $H^s(X,\mathfrak{F}^q) = 0,\ \forall q \geq 0,$ and $s \geq 0$. The following theorem said that resolution of "flasque" sheaves can be replaced by acyclic sheaves.

Theorem 2. 1 (de Rham-Weil isomorphism theorem). If $\mathfrak{F}^\bullet$, is a resolution of a sheaf $\mathfrak{F}$, by sheaves $\mathfrak{F}^\bullet$, such that $\mathfrak{F}^\bullet$, is acyclic on X, then there is a functorial isomorphism

$$H^p(\mathfrak{F}^\bullet(X)) \xrightarrow{\cong} H^p(X,\mathfrak{F},d), \tag{7}$$

Proof. [2]. ◆

To determine which sheaf is acyclic we need the following definition.

Def. 2. 1. A sheaf $\mathfrak{F}$, is called soft if the restriction

$$\Gamma(X,\mathfrak{F}) \to \Gamma(K,\mathfrak{F}), \tag{8}$$

[4] For any sheaf $\mathfrak{F}$, of an Abelian group over a topological space, the Čech cohomology group is the cohomological space $\check{H}^p(X,\mathfrak{F})$. This cohomology is a cohomological theory based on the intersection properties of open covers of a topological space. The Čech cohomology of $\mathcal{U}$, (let $\mathcal{U}$, be an open cover of X) with the values in $\mathfrak{F}$, is defined to be the cohomology of the cochain complex $(C^\bullet(\mathcal{U},\mathfrak{F}),\delta)$. Thus the $q\text{th}-$Čech cohomology is given by

$$\check{H}^q(X,\mathfrak{F}) := H^q((C^\bullet(\mathcal{U},\mathfrak{F}),\delta)) = Z^q(\mathcal{U},\mathfrak{F})/B^q(\mathcal{U},\mathfrak{F}),$$

The Čech cohomology of X, is defined by considering refinements of open covers. If $\mathcal{V}$, is a refinement of $\mathcal{U}$, then there is a mapping in cohomology

$$\check{H}^\bullet(\mathcal{U},\mathfrak{F}) \to \check{H}^\bullet(\mathcal{V},\mathfrak{F}),$$

The open covers of X, form a directed set under refinement, so the above mapping leads to a direct system of Abelian groups. The Čech cohomology of *X* with values in $\mathfrak{F}$, is defined as the direct limit

$$\check{H}(X,\mathfrak{F}) := \varinjlim_{\mathcal{U}} \check{H}^\bullet(\mathcal{V},\mathfrak{F}),$$

of this system.

[5] A flasque sheaf (also called a flabby sheaf) is a sheaf $\mathfrak{F}$, with the following property: if X, is the base topological space on which the sheaf is defined and

$$U \subset V \subset X,$$

Are open subsets, then the restriction mapping

$$\tau_{U\subset V} : \Gamma(V,\mathfrak{F}) \to \Gamma(U,\mathfrak{F}),$$

Is surjective as a mapping of groups (rings, modules, etc).

[6] A para-compact space is a topological space in which every open cover admits an open locally finite refinement.

is surjective for any closed subset $K \subset X$, that is to say, every section of $\mathfrak{F}$, on a closed subset K, can be extended to X. We consider the following proposition.

Proposition 2. 1. Soft sheaves are acyclic. Any sheaf of modules over a soft is soft and hence acyclic.

Proof. [3]. ◆

This is frequently applied to the sheaf of continuous (or differentiable) functions on a manifold, which is easily shown to be soft. Notice that the sheaf of holomorphic functions on a complex manifold is not soft.
We consider the following example.

$$H^j(X,\mathfrak{F}) := \frac{\ker(\phi^j : \mathfrak{F}^j(X) \to \mathfrak{F}^{j+1}(X))}{\operatorname{Im}(\phi^{j-1} : \mathfrak{F}^{j-1}(X) \to \mathfrak{F}^j(X)))}, \tag{6}$$

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We consider the following example.

Example 2. 1. Let X, be a real smooth manifold of dimension n. We consider the resolution of sheaves

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \ldots \to \mathcal{E}^q \to \mathcal{E}^{q+1} \xrightarrow{d} \ldots \to \mathcal{E}^n \to 0 \qquad ,(9)$$

[7] A para-compact space is a topological space in which every open cover admits an open locally finite refinement.

Since $\mathcal{E}^0 = \mathcal{E}_X = C^\infty(X),$ is soft, all sheaves $\mathcal{E}^p,$ are $\mathcal{E}_X-$modules so that $\mathcal{E}^p,$ are acyclic. Then de Rham cohomology groups of $X,$ are precisely

$$H^p_{DR}(X,\mathbb{R}) \cong H^p(X,\mathcal{E}^\bullet,d), \tag{10}$$

As was mentioned in the applications of finite refinements over sheaves, we need define also a fine sheaf, as the object created over $X,$ with "partitions of unity". Likewise, for any open cover of the space $X,$ we can find a family of homeomorphisms from the sheaf to itself with sum $1,$ such that each homomorphism is $0,$ outside of some element of the open cover.

Fine sheaves are usually and uniquely used over paracompact Hausdorff spaces $X.$ Typical examples is the sheaf of continuous real functions over such space, or smooth functions over a smooth (para-compact Hausdorff) manifold, or modules over these sheaves of rings [4].

Fine sheaves on para-compact Hausdorff spaces are soft and acyclic.

3. Some on D_X- Shemes [4, 5]

A D_X-scheme is a $\mathcal{E}_X-$scheme, since their connection of the D_X-scheme is defined over smooth scheme. The specific term as D_X-scheme is because we consider their connection.

Fix a base field k, and a smooth scheme X, over k. A D_X-scheme is a scheme equipped with a flat connection over X. For an affine scheme, this is equivalent to being the spectrum of an D_X-algebra. For example, affine D_X-schemes of finite type have the form:

$$\mathrm{Spec}((\mathrm{Sym}\, D_x \otimes_{\mathcal{O}_X} \mathfrak{F})/\mathcal{I}), \tag{11}$$

for some coherent $\mathcal{O}_X-$sheaf $\mathfrak{F},$ and some D_X-ideal sheaf $\mathcal{I}$. Throughout this research, we will often pass freely from D_X-algebras to affine D_X-schemes and vice-versa (the two categories are opposite in the usual sense) The integral transforms arise as a solution in geometrical analysis, if is the case, for example analyzing vector holomorphic bundles.

A very important example of an affine D_X-scheme is $\mathrm{Spec}(\mathrm{Sym}\ \mathcal{M}),$ for any D_X-module $\mathcal{M}$. This suggests that D_X-algebras are generalizations of D_X-modules, which is supported by the following fact: D_X-modules parameterize solutions of linear differential equations, while D_X-algebras parameterize solutions of nonlinear differential equations. The difference implies the use of the quotient corresponding algebra, related with (11). More precisely, suppose we take the D_X- chemes$(\mathrm{Sym}D^n_X)$, where the ideal $\mathcal{I}$, is generated (locally) by "polynomials" $\mathrm{P}_1,\ldots,\mathrm{P}_k \in \mathrm{Sym}D^n_X$.

Then giving a map of D_X-modules:

$$(\mathrm{Sym}D^n_X)/\mathcal{I} \to \mathcal{O}_x, \tag{12}$$

is the same as to give a collection of functions $f_1,\ldots,f_n,$ which satisfy the system of nonlinear differential equations:

$$\mathrm{P}_1(f_1,\ldots,f_n)=0, \tag{13}$$

A map of D_X-schemes is a morphism of D_X-algebras at the level of coordinate rings. A more involved notion is the following:

Def. 3. 1. Given a morphism of D_X-schemes $\mathcal{Y}\to\mathcal{Z},$ the functor of horizontal sections $\mathrm{HorHom}(\mathcal{Z},\mathcal{Y}),$ is given by:

$$S \in \mathrm{Sch} \to \mathrm{HorHom}(\mathcal{Z}\times S, \mathcal{Y}), \tag{14}$$

$\mathrm{HorHom},$ consists of horizontal morphisms, i.e. morphisms of D_X-schemes.

The above definition is completely analogous to that of the functor Sect, replacing $\mathcal{O}_X -$ with D_X . Note that for a morphism of $\mathcal{O}_X -$ −algebras to be a morphism of $D_X -$algebras is a closed condition. Since the functor of sections is representable, it follows that the functor of horizontal sections is also representable.
Moreover $\mathrm{HorSect}\,(\mathcal{Z},\mathcal{Y})\to \mathrm{Sect}(\mathcal{Z},\mathcal{Y})$, is a closed embedding.

4. de Rham and Dolbeault Theorems for Currents

A basic observation is that the Poincaré and Dolbeault-Grothedieck lemmas still hold for currents. These open big possibilities on the application of currents as scrutiny (as test function) and classes element determination in a resolution cohomology. If $(\mathcal{D}^q, d)$, and $(\mathcal{D}'^{p,q}, d)$, denote the complex of schemes of degree q- currents (respectively of (p,q)-currents we still have de Rham and Dolbeault sheaf resolutions

$$Cch_{deRham}: 0 \to \tilde{\mathbb{R}} \to \mathcal{D}'^0 \xrightarrow{d} \mathcal{D}'^1 \xrightarrow{d} \mathcal{D}'^2 \xrightarrow{d} \dots, \tag{15}$$

and

$$Cch_{Dolbeault}: 0 \to \Omega_X^p \to \mathcal{D}'^{p,0} \xrightarrow{\bar\partial} \mathcal{D}'^{p,1} \xrightarrow{\bar\partial} \mathcal{D}'^{p,2} \xrightarrow{\bar\partial} \dots, \tag{16}$$

Since the sheaves $\mathcal{D}'^p$, are all $\mathcal{E}_X$ – modules, they are acyclic so that we have the canonical isomorphisms

$$H_{DR}^q(X,\mathbb{R}) \cong H^q(X, \mathcal{D}'^\bullet, d) = \frac{\ker(d: \mathcal{D}'^q(X) \to \mathcal{D}'^{q+1}(X))}{\mathrm{Im}(d: \mathcal{D}'^{q-1}(X) \to \mathcal{D}'^q(X)))}, \tag{17}$$

and the Dolbeault corresponding

$$H^{p,q}(X,\mathbb{R}) \cong H^q(X, \mathcal{D}'^{p\bullet}, \bar\partial) = \frac{\ker(\bar\partial: \mathcal{D}'^{p,q}(X) \to \mathcal{D}'^{p,q+1}(X))}{\mathrm{Im}(\bar\partial: \mathcal{D}'^{p,q-1}(X) \to \mathcal{D}'^{p,q}(X))}, \tag{18}$$

In other words, we can attach a cohomology class $\{\Theta\} \in H_{DR}^q(M,\mathbb{R})$, to any closed current Θ, of degree q, or respectively a cohomology class $\{\Theta\} \in H^{p,q}(M)$, to any $\bar\partial$ – closed current of bi-degree (p,q).
In general, a current $T \in D'_1(M)$, is called a distribution or generalized function (e.g. we take test functions instead of test forms).

5. Cohomologies by *D*$_X$-Schemes [4, 5]

The *Verdier duality* implies the following natural bijection for D_X – modules:

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes_k V) \cong \mathrm{Hom}(\mathrm{H}^n_{\mathrm{dR}}(X,\mathcal{M}), V), \tag{19}$$

for any D_X – module $\mathcal{M}$, and any vector space V . By definition, $\mathrm{H}^\bullet_{\mathrm{dR}}(X,\mathcal{M})$, are the cohomology groups of the complex of sheaves of k - vector spaces:

$$\dots \to \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^i T^* X \to \mathcal{M} \otimes_{\mathcal{O}_X} \Lambda^{i+1} T^* X \to \dots, \tag{20}$$

These cohomology groups coincide with $R^\bullet \pi_*(\mathcal{M})$, where $\pi: X \to pt$, is the projection to a point. Note that (20) implies that

$$\mathrm{H}_\nabla(X, \mathrm{Sym}\mathcal{M}) = \mathrm{Sym}\mathrm{H}^n_{\mathrm{dR}}(X,\mathcal{M}), \tag{21}$$

The shift by n happens when we pass from $\mathcal{D}_X$ – modules to quasi-coherent $\mathcal{O}_x$ −modules, as we will be doing now. This induces a long exact sequence on cohomology:

$$\ldots \to H_{\mathrm{dR}}^{n-1}(X - x, \mathcal{M}) \xrightarrow{\phi} \mathcal{M}_x \to H_{\mathrm{dR}}^{n-1}(X, \mathcal{M}) \to H_{\mathrm{dR}}^{n}(X - x, \mathcal{M}), \tag{22}$$

We affirm that the last group is 0. To see this, recall that Lichtenbaum's theorem says that the Čech cohomological dimension of $X - x$, is at most $n-1$, that is to say, $H^n(X - x, \mathcal{F}) = 0$, for any quasi-coherent $\mathcal{F}$. As the $\mathcal{D}_X$ – module $\mathcal{M}$, is a quotient of the form:

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{M}, \tag{23}$$

for some quasi-coherent $\mathcal{F}$, and

$$\mathrm{H}_{\mathrm{dR}}^{\mathrm{n}}(X - x, \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) = H^n(X - x, \mathcal{F}) = 0, \tag{24}$$

it also follows that $\mathrm{H}_{\mathrm{dR}}^{\mathrm{n}}(X - x, \mathcal{M}) = 0$. Therefore, (21) and (22) imply:

$$\mathrm{H}_{\nabla}(X, \mathrm{Sym}\mathcal{M}) = \mathrm{Sym}(\mathcal{M}_x / \mathrm{Im}\phi), \tag{25}$$

6. Results: Integral Geometry

We consider the before tools. Then we have the following results with applications in geometrical analysis [7].
Let M, be a complex holomorphic manifold (or complex Riemannian manifold [8]). We consider its corresponding reductive homogeneous space determined by the flag manifold $\mathrm{F} = G_{\mathbb{C}} / P$, with P, a parabolic subgroup of $G_{\mathbb{C}}$. We consider an open orbit given by the Stein manifold $\mathrm{F}_D = G / H$, with H, a compact subgroup of the real form G, of $G_{\mathbb{C}}$.
In general, let $T \in D'_1(M)$, be the current in a complex holomorphic manifold M. Likewise, let be M, and N, two complex holomorphic manifolds, then the holomorphic mapping

$$T_{\bar{\partial}} : D'^{p,q}(M) \to D'^{p,q}(N),$$

defines a current where the rules of complex differential calculus can be easily extended to this case.
Let $M \cong \mathbb{C}^n$, and we consider linearly concave domains[8] (or more yet $\mathbb{CP}^n$). D, has structure of complex vector space. Let $D_1 = \mathbb{C}^n / \bar{D}$, be a holomorphic linearly convex domain conformed for holomorphic hyperplanes $\pi_i(D)$, with $i = 1, 2, \ldots$ in D_1. Let $H(D_1)$, the holomorphic complex functions space defined on D_1 [9]. Let $D \to M$, a fibered vector bundle seated in the complex holomorphic manifold M. Let $A^{p,q}(D)$, the space of (p,q) – forms on M, with values in D (that is to say, the space of global sections of the fibered tangent bundle $\wedge^{p,q} T^*(M) \otimes D$).
By this way, the bi-graded algebra is the space

$$A(D) = \bigoplus_{n+m=p} A^{p,q}(D),$$

Theorem. 4. 1 (F. Bulnes). We consider the current

$$T_{\bar{\partial}} : H(D) \to L(D), \tag{26}$$

where $H(D)$, is a space of holomorphic complex functions, $L(D)$, is a co-cycles space (in this case linearly concave domains). Then (26) on a Dolbeault cohomology is a Dolbeault cohomology of the current, that is to say

$$T(\bar{\partial} D) = \bar{\partial} T(D), \tag{27}$$

Note. Here the current can be viewed as generalized function of cycles in a complex vector space *D*.

[8] A domain D, in $\mathbb{C}^n$, (or $\mathbb{CP}^n$) is linearly concave if $\bigcup_i \pi_i(D) \subset D$, with $i = 1, 2, \ldots$ (holomorphic planes).

Proof. We consider the diagrams of complexes[9]

$$
\begin{array}{ccccc}
D & \xrightarrow{f} & H(D) & \xrightarrow{T_{\bar{\partial}}} & L(D) \\
\downarrow \text{Functional} & & \downarrow \text{Functional} & & \downarrow \text{Functional} \\
D^* & \xrightarrow{\phi} & H(D^*) & \xrightarrow{T_{\bar{\partial}}^*} & L(D^*)
\end{array}
\tag{28}
$$

and the corresponding to the bi-graded algebra

$$
\begin{array}{ccccccc}
 & & A^{0,0}(D) & \xrightarrow{\bar{\partial}'} & A^{0,1}(D) & \xrightarrow{T_{\bar{\partial}}} & A^{0,1}(\pi(D)) \\
 & & \downarrow ? & & \downarrow T & & \downarrow \tilde{T} \\
0 & \longrightarrow & B^{0,0}(D^*) & \xrightarrow{\bar{\partial}} & B^{0,1}(D^*) & \xrightarrow{T_{\bar{\partial}}^*} & B^{0,1}(\pi D^*),
\end{array}
\tag{29}
$$

where $\pi(D)$, is a subspace of co-dimension 1 in D[10]. We fix the satellites $\bar{\partial}' T$, and $\bar{\partial} T^{-1}$, from (29) and we compose the diagram (29) with the diagram (28). Indeed, due to that T, is injective in $H(D)$, then

$$
0 \to D \to H(D) \to L(D) \to 0,
\tag{30}
$$

is an exact succession then

$$
0 \to B^{0,0}(D^*) \to B^{0,1}(D^*) \to B^{0,1}(\pi(D^*)) \to 0,
\tag{31}
$$

is exact for each $i = 0,1.$ Then to the composition $\bar{\partial}' T(f),\ \forall f \in H(D),$ we have that

$$
A^{0,0}(D^*) \to A^{0,1}(D^*) \to A^{0,1}(\pi(D^*)),
\tag{32}
$$

is an epimorphism $(\bar{\partial}' T(f) = 0).$
Of this result, we have an unique homomorphism $T_{\bar{\partial}} : A^{0,0}(D) \to B^{0,0}(D^*),$ such that inserted in the diagram (29) leaves commutative to (28). Let $\Phi_2 : H(D) \to H(D^*),$ and we consider (22), the exact succession to define $T_{\bar{\partial}}(L(D)).$ Then we have that

$$
\begin{array}{ccccc}
D & \xrightarrow{f} & H(D) & \xrightarrow{T_{\bar{\partial}}} & L(D) \\
\downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 \\
D^* & \xrightarrow{\phi} & H(D^*) & \xrightarrow{T_{\bar{\partial}}^*} & L(D^*)
\end{array}
\tag{33}
$$

where through the composition (29) we have:

$$
\begin{aligned}
\bar{\partial} B^{0,0}(\Phi_3) T_0(L(D)) &= B^{0,0}(D^*)(\Phi_2)\bar{\partial}_1 T_0(L(D)) \\
&= B^{0,0}(D^*)(\Phi_2) T(H(D))\bar{\partial}_1 \\
&= T(H(D^*)) A^{0,1}(\Phi_2)\bar{\partial}_1 \\
&= T(H(D^*))\bar{\partial}' A^{0,,0}(\Phi_3) \\
&= \bar{\partial} T_0(L(D^*)) A^{0,0}(\Phi_3),
\end{aligned}
$$

Since $\bar{\partial} : B^{0,0}(D^*) \to B^{0,1}(D^*),$ has null kernel, we obtain:

$$
B^{0,0}(\Phi_3) T_0(L(D)) = T_0(L(D^*)) A^{0,0}(\Phi_3),
$$

Then T, is independent of the choosing of the auxiliary succession (30) to define $T(L(D)).$ To demonstrate that T, commutes with the connected homomorphisms, we consider the exact succession

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[9] $H(D)$, is a space of holomorphic complex functions, and $H(D^*)$, is their analogous in current images. $L(D)$, is a co-cycles space under current.
[10] The hyperplane of equation in D, is of the form $h(z,\zeta) = z_1\zeta_1 + z_2\zeta_2 + \ldots + z_n\zeta_n, \quad \forall \zeta, z \in \mathbb{C}.$

$$0 \to L(D^*)' \to H(D^*) \to L(D^*)'' \to 0, \tag{34}$$

and let,

$$0 \to D^{*}'' \to H(D^*)'' \to L(D^*)'' \to 0, \tag{35}$$

exact with the space $H(D)''$, as projectivized by T. Then exist mappings

$$\xi : H(D^*)'' \to H(D^*), \qquad \zeta : D^{*}'' \to L(D^*)', \tag{36}$$

such that the following diagram

$$\begin{array}{ccccccccc} 0 \to & D'' & \xrightarrow{f} & H(D)'' & \xrightarrow{T_{\bar{\partial}}} & L(D)'' & \to 0 \\ & \zeta \downarrow & & \xi \downarrow & & & \\ 0 \to & L(D^*)' & \xrightarrow{\phi} & L(D^*) & \xrightarrow{T_{\bar{\partial}}^{*}} & L(D^*)'' & \to 0 \end{array} \tag{37}$$

is commutative. This gives the commutative diagram:

$$\begin{array}{ccccccc} & A^{0,0}(D) & \xrightarrow{\bar{\partial}'} & A^{0,1}(D) & \xrightarrow{T_{\bar{\partial}}} & A^{0,1}(\pi(D)) \\ & \downarrow \tilde{T}'' & & \downarrow T'' & & \downarrow \tilde{T}' \\ 0 \longrightarrow & B^{0,0}(D^*) & \xrightarrow{\bar{\partial}} & B^{0,1}(D^*) & \xrightarrow{T_{\bar{\partial}}^{*}} & B^{0,1}(\pi D^*), \end{array} \tag{38}$$

then is implied the identity (27). ◆

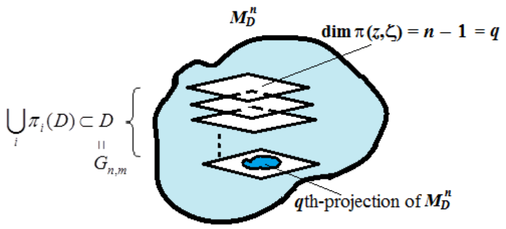

Fig 1: Tomography of a $n-$dimensional Stein manifold.

Lemma 4. 1 (F. Bulnes). $T_{\bar{\partial}}$, is the Radon transform of the $\bar{\partial}-$cohomology in a $q^{\text{th}}-$projection $(n-1)-$dimensional of the complex Riemannian manifold.

Proof. The Radon transform can be viewed as the cohomological spaces mapping or mapping of cohomology classes:

$$H^{p,n}(D,V) \to H^{p,n-1}(D,V), \tag{39}$$

Thus only is necessary to demonstrate that $T_{\bar{\partial}}(H^{p,n}(D,V))$, is the qth$-$projection $(n-1)-$dimensional[11] of $H^{p,n}(D,V)$, in $H^{0,q}(D,V)$, that is $H^{0,n}(D,V)$. Indeed, the Radon transform in the complex context D, is the analytic and continuous mapping

$$H(D) \to L(D),$$

because, remember that the Radon transform is a generalized function in a distribution space [10], even exist other extensions of the Radon transform to the Boehmians, which are defined as sequences of convolution quotiens and

[11] Cycles or $(n-1-q)-$linearly concave domains, which in a particular case can be $(n-1-q)-$hyperplanes (for example when $D^* = \mathfrak{L}(H(D), \mathbb{C})$).

include Schwartz distributions and other regular operators [11]. Then in the complex case, and to complex coordinates systems $\{z_i\}$ and $\{\zeta_i\}$ we have:

$$f^{\wedge}(\zeta_1,\dots,\zeta_n,p)=\left(\frac{i}{2}\right)^{n-1}\int_M f(z_1,\dots,z_n)\delta[p-[(\zeta_1,\dots,\zeta_n),(z_1,\dots,z_n)]\bullet<dz_1,\dots dz_n,dz_1,\dots dz_n>, \tag{40}$$

$\forall f\in H(D)$. Let $\bar{\partial}$, the complex scalar co-boundary operator

$$\bar{\partial}:\mathcal{E}(\wedge^{p,q}T^*(M)\otimes V_{\mathbb{C}})\to\mathcal{E}(\wedge^{p,q+1}T^*(M)\otimes V_{\mathbb{C}}), \tag{41}$$

Let $D^*=\mathcal{L}(H(D),\mathbb{C})$, the set of corresponding hyperplanes to D. Let the mapping:

$$ev_f:\mathcal{L}(H(D),\mathbb{C})\to\mathbb{C}, \tag{42}$$

the evaluation of $f\in H(D)$, in the complex hyperplane $\pi(z)$, of D, with rule of correspondence

$$\pi(z)f=<\pi(z),f>, \tag{43}$$

By the *theorem 4. 1*, we have $Tf(\bar{\partial})=\bar{\partial}(Tf),\ \forall f^{\wedge}\in L(D)$, then we have:

$$\bar{\partial}(f^{\wedge}(z))=\bar{\partial}(f^{\wedge})\otimes z+f^{\wedge}\bar{\partial}(z)=\bar{\partial}(z)\otimes T(f)\in\wedge^{p,q+1}, \tag{44}$$

where in particular the exterior algebra $\mathcal{A}^{0,q}(T^*M\otimes V_{\mathbb{C}})$, is generated by elements of the form $\pi(z)\wedge\bar{\partial}(T(f)(z))$. Then

$$T_{\bar{\partial}}(f)\otimes ev_f=<\pi(z)f,\bar{\partial}(Tf)>, \tag{45}$$

Therefore $p=0$. Thus $T_{\bar{\partial}}(f)\otimes ev_f\in A^{0,q}(V)$. Then

$$H^{0,q}(\mathbb{C}^n/D,V)\to H^{0,q}(D_1,V), \tag{46}$$

which is a Dolbeault cohomology. ◆

Now we establish a generalizing of the relations established in the *proposition 4. 1*, to the derived category level.

Theorem 4. 2. A D_X-scheme to the geometrical problem of complex cycle decomposition by $T_{\bar{\partial}}$, (using the *theorem of Appendix A*,) is:

$$\mathrm{Hom}_{\mathrm{Moduli}_n}(X,\mathrm{Spec}(\mathcal{D}^{'\mathrm{p,q}}(D)))\cong\mathrm{Hom}_{\mathrm{CAlg(Sp)}}(\mathcal{D}^{'\mathrm{p,q}}(D),\mathcal{G}), \tag{47}$$

We consider a *Koszul duality application* and the relative details on the inverse limits to obtain Spf, in the context of "CRings", $\mathrm{CAlg(Sp)}$, [12]then we obtain the validity of the identity inside of the space $\mathrm{CAlg(Sp)}$ in the scheme (A. 1). In particular in the D_X-scheme to geometrical problem subjacent that is tomography the manifold *M*, through complex vector spaces, $\mathcal{G}$, is a space of co-cycles obtained by Radon transform on *M*. However, these co-cycles could represent loops or contours as certain residues of the tomography, which are consider through Koppelman's formula in $\bar{\partial}-$cohomology.

Likewise, by Koszul duality we can consider the functor between geometrical aspects in moduli problems (in this case cycles of manifold tomography) and the D_X-algebra that generalize the $D-$modules in D_X- schemes (in this case, $k-$ modules) and consider the scheme of CRings,

$$\mathrm{Hom}_{D_X}(J\mathcal{A},\mathcal{B})\cong\mathrm{Hom}_{\mathrm{Alg}_k}(X,\mathrm{Spec}J\mathcal{A}), \tag{48}$$

where $J\mathcal{A}$, is a D_X-algebra generated by $\mathcal{A}$.

[12] $\mathrm{CAlg(Sp)}$, is an enlargement of the ordinary category of commutative rings. In one it is defined the identity (27).

Studies of Gindikin and Henkin [9], showed how one can construct kernels on a complex manifold given a holomorphic connection on a complex vector bundle over the manifold. However, appear certain terms that are called "parasitary terms", which depend on the curvature of the connection. This could be related with other frameworks of the Radon transform and its spectrum. However, is very interesting in curvature studies and generalizations of curvature where the curvature is obtained as tempered distributions[13] [7, 12, 13] and could be used to obtain in the $\bar{\partial}-$cohomology context a generalization of curvature. In this sense was obtained an important generalization in the local context of the complex Riemannian manifold using versions of Radon transform and orbital integrals on homogeneous spaces considering the $G-$structure $K-$invariant of the complex Riemannian manifold [14].
A particular case in certain sense, in complex analysis, could be a factorization of the cycle of a bounded complex of vector bundles in terms of certain associated differential forms (in this case complex forms) and residue currents. The residue currents are the Cauchy integrals values of the contours (projections obtained in the ∂M).
For example, we consider a cousin Dolbeault cohomology to arrive to that the filtration of $M = G/B,$ by $B,$ orbits of co-dimension major that $k,$ that carry to an exact sequence (newly the differentials of the Cousin complex arise as the connecting homomorphism) is a sophisticated tomography of the Radon transform as the Penrose transform [15].
Indeed, we consider the $\mathcal{E}_X$ $-$modules exact sequence:

$$0 \to \mathbb{E}^{n,\bullet}_{[X_{k+1}]}(M,\mathcal{L}_\lambda) \to \mathbb{E}^{n,\bullet}_{[X_k]}(M,\mathcal{L}_\lambda) \to \mathbb{E}^{n,\bullet}_{[X_K/X_{k+1}]}(M,\mathcal{L}_\lambda) \to 0, \tag{49}$$

If $X_{k+1},$ is defined in $X_k,$ by the vanishing of a single function then the connecting homomorphism is once again defined by the residue theorem. We consider $\mathcal{L}_\lambda$, as the homogeneous line bundle determined by the character $\lambda,$ of $B.$ Then in the cohomological context the corresponding integration is the given by the field geometrical integration

$$H^{n,m}_{[X_k/\infty]}(M,\mathcal{L}_\lambda) \to H^{n,m+1}_{[\infty]}(M,\mathcal{L}_\lambda), \tag{50}$$

Indeed, to calculate the cohomology of $\mathcal{L}_\lambda$, is necessary and sufficient that the Cousin complex $\mathbb{E}^{n,\bullet}_{[X_K/X_{k+1}]}(M,\mathcal{L}_\lambda),$ has cohomology only in degree $k,$ and this cohomology forms the k^{th} $-$term of the Cousin complex[14]. Thus we have the following exact sequence using the cycles of single point about $k,$ (these, characterized in the *flatness* and *conformaly* in the orbits language of $G/B,$ comes given by the points $wB,$ of $B,$ that are characterization of invariants proper in the category $\mathrm{Mod}_{\mathrm{Coh}}(D\mathcal{L}),$ [5] to the $D_{G/H}$ - modules $G-$invariant):

$$0 \to H^{n,\bullet}_{[X_{k+1}]}(M,\mathcal{L}_\lambda) \to H^{n,\bullet}_{[X_k]}(M,\mathcal{L}_\lambda) \to H^{n,\bullet}_{[X_K/X_{k+1}]}(M,\mathcal{L}_\lambda) \to 0, \tag{51}$$

---

[13] Theorem on generalization of curvature as tempered distribution on a differentiable manifold obtained by Dr. Francisco Bulnes.

[14] Given a filtration of a complex manifold M, by closed analytic sub-varieties and a line bundle $\mathcal{L},$ we filter the Dolbeault complex of currents with values in $\mathcal{L},$ by their algebraic supports in these manifolds. Let us consider the filtered complex

$$\mathbb{E}^{n,\bullet} \otimes \mathcal{L}(M) \supset \mathbb{E}^{n,\bullet} \otimes \mathcal{L}_{[X_1]}(M) \supset \cdots \supset \mathbb{E}^{n,\bullet} \otimes \mathcal{L}_{[X_n]}(M) \to 0,$$

We will call the Cousin double complex associated to this filtered complex the (meromorphic) Cousin Dolbeault complex. Consecutive terms of the filtration give rise to the short exact sequences

$$\mathbb{E}^{n,\bullet} \otimes \mathcal{L}_{[X_{k+1}]}(M) \xrightarrow{i_k} \mathbb{E}^{n,\bullet} \otimes \mathcal{L}_{[X_k]}(M) \longrightarrow$$
$$\mathbb{E}^{n,\bullet} \otimes \mathcal{L}_{[X_k/X_{k+1}]}(M) \to 0,$$

The vertical complexes of the Cousin Dolbeault complex are the cones mapping of the injections i_k, shifted down by k, degrees. Their cohomology is the cohomology of the complexes $\mathbb{E}^{n,\bullet} \otimes \mathcal{L}_{[X_k/X_{k+1}]}(M)[k].$

Then the co-cycles of (51) are the images under the corresponding integrals, consequence of compute (considering the changing to the coordinate $w=\frac{1}{x}$, centrated on ∞, in the section t):

$$\bar{\partial}T(\phi)=-\int_{C_r} f\phi dxs=\int_{C_r} f\left(\frac{1}{w}\right)\phi(w)w^{n-2}dwt, \tag{52}$$ [15]

$\forall t\in T, w\in W.$ Then in the cohomological context the corresponding integration is the given by the field geometrical integration (50). Then we can give the following cohomology and connecting homomorphism (50) taking an element $fdzs$, from $H^{n,m}_{[X_k/\infty]}(M,\mathcal{L}_\lambda)$, and lift in back to an element T, in $\mathbb{E}^{n,m}(M,\mathcal{L}_\lambda)$. Then we have the Penrose transform [15]

$$\mathbb{E}^{n,m}_{[Cw/\partial Cw]}(M,\mathcal{L}_\lambda)\xrightarrow{\cong}\mathbb{E}^{n,m}_{[\infty]}(M,\mathcal{L}_\lambda)\overset{T}{\leftrightarrows}$$
$$H^{n,m}_{[X_K/\infty]}(M,\mathcal{L}_\lambda)\to 0, \tag{53}$$

with the correspondence rules $\leftrightarrows$, given by the current

$$T(\phi dz)=\int_{\mathbb{P}^1} f\phi dz\wedge dzs, \tag{54}$$

where $H^{n,m}_{[Cw/\partial Cw]}(M,\mathcal{L}_\lambda)$, is a Verma module depending of the weight of λ. For example, consider the points of Lie algebra (differential operators) $\mathfrak{sl}(2,\mathbb{C})$. Their Verma modules to the weight $-n$, of λ, are the cohomological spaces $H^{1,0}_{[\mathbb{P}^1/\infty]}(\mathbb{P}^1,\mathcal{O}(n))$, and $H^{1,1}_{[\infty]}(\mathbb{P}^1,\mathcal{O}(n))$, where $\mathcal{O}(n)$, is the corresponding sheaf of homogeneous line bundle of degree n. The current determined in (54) is $T_{\bar{\partial}}$.

Appendix

Theorem (F. Bulnes, I. Verkelov) A. 1. Considering the functors Φ, Ψ, with the properties given in [3], we have the following scheme

$$\mathrm{Hom}_{\mathrm{Modulin}}(X,\mathrm{Spec}(B))\cong\mathrm{Hom}_{\mathrm{CAlg(Sp)}}(B,\mathbb{S}), \tag{A.1}$$

Proof. [5]. ◆

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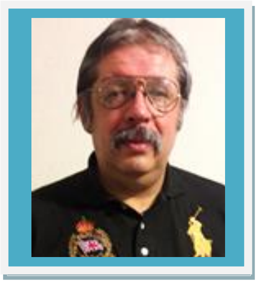
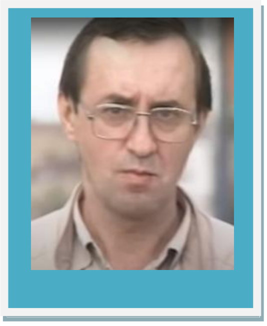
Authors' Biography



Prof. Dr. Francisco Bulnes, ORCID: 0000-0002-2007-7801, is an international and very famous mathematician in East Europe, Middle East and Asia; specialized in integration on infinite dimensional spaces. He is the leader of international research group. IINAMEI Director. Many awards and Doctorates in Honoris Causa given from Universities and ONG likewise OG's.



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