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On the vanishing cohomology theory of some operator algebras

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Abstract.

We concerned with the vanishing of the dihedral and reflexive cohomology groups of stable C^*-algebra. Wodzicki has proved that the cyclic cohomology of stable C^*-algebra is vanished. We extend this fact to prove that the reflexive and dihedral cohomology of this class are also vanish.

Key words: *Dihedral homology – Stable algebras - C^*-algebra -cohomology.*
Mathematics Subject Classification*: 55Q05, 57Q10*

1- Introduction.

The vanishing cohomology group of operator algebras has been studied a lot. Consider the a unital semi-group algebra $l^1(Z_+)$of$N$, then the third cohomology group$H^3(l^1(Z_+), l^1(Z_+)`) = 0$[15], and fornon-unital Banach algebra$I = l^1(Z_+)$, then$HC^3(I, I`) = 0$[15]. If $A$is biflat algebra and n is odd and $\varepsilon = \pm 1$, then ${}^{\varepsilon}HD^n(A) = 0, n \in N$[4]. For an algebra A and A-bimodule M, the class of algebra can defined as Amenable algebras if the continuous derivation from A into M are inner [8]. Both of Dihedral and Hochschild cohomology groups vanish, in the event that A will be a C*-algebra or a nuclear C^*-algebra ([12],[13],[15]).

Here, the vanishing of Reflexive and Dihedral cohomology groups of C^*-algebra will be studied with given examples of non-trivial dihedral cohomology groups of a commutative Banach algebra.

1- Dihedral (Co)homology of operator algebra

We recall the definition properties of Banach algebra and its homology from[1],[3] and [11]. For acommutative ring $k = \mathbb{C}$ and the unital Banach algebra$A$, the complex $C(A) = (C^*(A), b_*)$is the boundary operator

$$b_n(a_* \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i \; a_0 \otimes .. \otimes a_i \, \mathrm{a}_{\mathrm{i}-1} \otimes .. \otimes \mathrm{a}_{\mathrm{n}-1},$$

where $C_n(A) = A \otimes \dots \otimes A$ is tensor product of algebras$(n+1$ times$)$and,$b_*: C_n(A) \to C_{n-1}(A)$.

It is well known that $b_{n-1}b_n = 0$, and hence $kerb_n \supset Imb_{n+1}$.

$$H_n(A) = H\big(C(A)\big) = \frac{kerb_n}{Imb_{n-1}} \qquad (1)$$

is Hochschild homology of $A$with involutive and denote by $\big(HH_*(A)\big)$.

If A is an unital Banach algebras, the cyclic group of order $(n+1)$ by the operator $t_n: C_n(A) \to C_n(A)$:

$$t_n(a_0 \otimes \dots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \dots \otimes a_{n-1}.$$

The quotient complex $CC_n(A) = \frac{C_n(A)}{Im(1-t_n)} \subset CC_*(A)$.

For the Connes-Tsygan bicomplex $CC_*(A)$ and the chain complex$CC_\bullet(A) = \big(CH_\bullet(A), b\grave{}_\bullet\big)$(see [5]), then the subcomplex$\big(\ker(1-t_\bullet), b\grave{}_\bullet\big) \subset \big(CH_\bullet(A), b\grave{}_\bullet\big)$has homology as the complex$(CC_\bullet(A), b_\bullet)$ as:

$$H_\bullet\big(CC_\bullet(A)\big) = H_\bullet(CH_\bullet(A), b_\bullet)/im(1-t_\bullet) = H_\bullet(CH_\bullet(A), b_\bullet)/\ker N = H_\bullet(im\, N, b\grave{}_\bullet)$$
$$= H_\bullet\big(\ker(1-t_\bullet), b\grave{}_\bullet\big)(2)$$

where

$$CH_n(A) = A^{\otimes n+1} = A \otimes \dots \otimes A \;\; (n+1\; times),$$
$$b_n, b\grave{}_n: CH_n(A) \to CH_{n-1}(A),$$

Such that:

$$b\grave{}_n(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n),$$
$$b_n(a_0 \otimes \dots \otimes a_n\;) = b\grave{}_n + (-1)^n (a_n a \otimes \dots \otimes a_{n-1}),$$
$$t_n: CH_n(A) \to CH_n(A),$$

such that

$$t_n(a_0 \otimes \dots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes \dots \otimes a_{n-1}) \quad \text{and} \quad N_n = 1 = t_n^1 + \dots + t_n^n.$$

The complexes $(ker(1-t_\bullet), b\grave{}_\bullet)$ and $(CC_\bullet(A), b_\bullet)$ are isomorphism which given by as an operator $N_\bullet: CC_\bullet(A) \to \big(\ker(1-t_\bullet), b\grave{}_\bullet\big)$. The action of the group$\mathbb{Z}/2$ on the complex$CC_\bullet(A)$, by the operator ${}^{\varepsilon}h$and on the complex$\big(\ker(1-t_\bullet), b\grave{}_\bullet\big)$ by the operator

$${}^{\varepsilon}r: a_0 \otimes a_1 \otimes \dots \otimes a_n \to (-1)^{\frac{n(n+1)}{2}} \varepsilon a_n^* \otimes a^*{}_{n-1} \otimes \dots \otimes a_0^*,$$

are equal, where a^* is the image $a \in A$ under involution $*: A \to A,\ \varepsilon = \pm 1$, such that ${}^{\varepsilon}h_{\bullet}.t_{\bullet} = t_{\bullet}{}^{-1}\,{}^{\varepsilon}h_{\bullet}$ then we have $N_{\bullet}({}^{\varepsilon}h_{\bullet}) = ({}^{\varepsilon}h_{\bullet})N_{\bullet}$.

Since ${}^{\varepsilon}r_{\bullet} = t_{\bullet}\,{}^{\varepsilon}h_{\bullet}$, then

$${}^{\varepsilon}h_{\bullet}N_{\bullet} = N_{\bullet}\,{}^{\varepsilon}h_{\bullet} = (N_{\bullet}t_{\bullet})\,{}^{\varepsilon}h_{\bullet} = N_{\bullet}(t_{\bullet}{}^{\varepsilon}h_{\bullet}) = N_{\bullet}\quad {}^{\varepsilon}r_{\bullet}.$$

then the dihedral homology of A is:

$$\varepsilon HD_{\bullet}(A) = H_{\bullet}(ker(1 - t_{\bullet})/(Im(1 - {}^{\varepsilon}h_{\bullet}) \cap ker(1 - t_{\bullet}))). \qquad (3)$$

For a commutative unital Banach algebra A. We denote by $C^n(A)(n = 0,1, \dots)$ the Banach space of continuous $(n + 1)$-linear functionals on A*;* and we call it n-dimensional co-chains. Let $t_n: C^n(A) \to C^n(A), (n = 1,2, \dots)$ be the operator

$$t_n f(a_0, a_1, \dots, a_n) = (-1)^n = (-1)^n f(a_1, \dots, a_n, a_0),$$

if $t_0 = I$. We write $t = t_n$. An operator f satisfying $tf = f$ and called cyclic. If $CC^n(A)$ denotes closed subspace of $C^n(A)$ which formed as the cyclic co-chains. $(CC^0(A) = C^0(A) = A^*$ since A^* is the dual Banach space for A).

by proposition (4) in [4], $Im(1 - t_n)$ is closed in $C^n(A)$ and $CC^n(A) = C^n(A)/Im(1 - t_n)$. The induce operator $dc_n: CC^{n+1}(A) \to CC^n(A)$ in the respective quotient spaces. Then, the quotient complex $CC^{\bullet}(A)$ of $CC(A)$ was obtained. The cohomology $CH^{\bullet}(A)$ of $CC^{\bullet}(A)$ is n-dimensional Banach cyclic cohomology group of A. If $r_n: C_n(A) \to C_n(A), n = 0,1, \dots$ is an operator on the formula

$$r_n(a_0 \otimes \dots \otimes a_n) = (-1)^{\frac{n(n+1)}{2}} \in a_0^* \otimes a_n^* \otimes \dots \otimes a_1^*, \in = \pm 1,$$

where $*$ is an involution on A.

Note that: $Im(id_{t_n(A)} = 1 - t_n)$ is closed in $C^{\bullet}(A)$.

The quotient complex,

$$CD^n(A) = \frac{C^n(A)}{Im(1 - t_n) + Im(1 - r_n)}$$

of a complex $C^{\bullet}(A)$. $HD^n(A)$ is an n-dimensional cohomology of $CD^n(A)$ and called n-dimensional dihedral cohomology group of a unital Banach algebra A.

similarly, we can get the reflexive cohomology $HR^n(A)$.

2- Main result

In this part we prove the main theorem of our study. We prove the vanishing state of C^*-algebra

Definition3.1:

If C^*-algebra A isomorphic to the tensor product algebra $(K \otimes A)$, then it is called stable, for an algebra K which is compact operators on a separable infinite-dimensional Hilbert space.

In ([2], [6]) we find the definitions of the simplicial, cyclic, reflexive and dihedral cohomology of operator algebra. Following [10] the relations between Hochschild, cyclic, reflexive and dihedral cohomology are given by the following commutate diagram $\mathfrak{C}(A)$:

$$\begin{array}{ccccccccc}
\ldots \to & {}^{-\alpha}\mathrm{HR}^{n+1}(A) & \to & {}^{-\alpha}\mathrm{HD}^{n+1}(A) & \to & {}^{\alpha}\mathrm{HD}^{n+3}(A) & \to & {}^{-\alpha}\mathrm{HR}^{n+2}(A) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{\alpha}\mathrm{HR}^{n-1}(A) & \to & {}^{\alpha}\mathrm{HD}^{n-1}(A) & \to & {}^{-\alpha}\mathrm{HD}^{n+1}(A) & \to & {}^{\alpha}\mathrm{HR}^{n}(A) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & H^{n-1}(A) & \to & HC^{n-1}(A) & \to & HC^{n+1}(A) & \to & H^{n}(A) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{-\alpha}\mathrm{HR}^{n-1}(A) & \to & {}^{-\alpha}\mathrm{HD}^{n-1}(A) & \to & {}^{\alpha}\mathrm{HD}^{n+1}(A) & \to & {}^{-\alpha}\mathrm{HR}^{n}(A) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{\alpha}\mathrm{HR}^{n-3}(A) & \to & {}^{\alpha}\mathrm{HD}^{n-+3}(A) & \to & {}^{-\alpha}\mathrm{HD}^{n-1}(A) & \to & {}^{\alpha}\mathrm{HR}^{n}(A) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

Suppose that M_m is the algebra of matrices of ordered m with m coefficients in algebra A over ring k with identity. Then the natural isomorphism $HH^*(M_m(A)) \approx HH^*(A)$ holds [7]. It is called a Morita equivalence. Following [14] the cyclic cohomology is Morita equivalence. If A be involutive algebra with identity, the following assertion holds [see [9]].

Proposition 3.2:

There exists an isomorphism;

$$\mathrm{Tr}_*: \quad {}^{\alpha}\mathrm{HD}^*(M_m(A)) \to \quad {}^{\alpha}\mathrm{HD}^*(A)$$

for all and $m > 1$ and $n > 0$.

We shall denote by the $B^*(A)$ the reflexive or dihedral cohomology $\left({}^{\alpha}\mathrm{HR}^*(A) \text{ or } {}^{\alpha}\mathrm{HD}^*(A)\right)$ of algebra A.

Our aim now is to prove the following assertion [14].

Theorem 3.3:

For a stable C^*-algebra A, we get thatthe reflexive and dihedral cohomology of A are vanishing, i.e

$${}^{\alpha}\mathrm{HR}^*(A) = 0, \qquad {}^{\alpha}\mathrm{HD}^*(A) = 0, \qquad \alpha = \pm 1.$$

Firstly, we need the following facts:

Lemma 3.4: [4]

For a C^*-algebra A without unit, and $i: A \to M_k(A)$ where M_k is thematrices of C^*-*algebra* A, $k > 0$ such that,

$$a \to \begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

then i is a quasi-isomorphism.

Proof:

If A is a C^*-algebra without unit. If $\overline{A} = A \oplus \mathbb{C}$ since $\overline{A}$ is the algebra A with unity, and for the short exact sequence

$$0 \to A \to \overline{A} \to \mathbb{C} \to 0 \quad (1)$$

Then the corresponding inclusion of algebra extensions

$$\begin{array}{ccc} A & \to \overline{A} \to & \mathbb{C} \\ \downarrow & \downarrow & \downarrow \\ M_k(A) & \to M_k(\overline{A}) \to & M_k(\mathbb{C}) \end{array} \quad (2)$$

In [13] and [14], for C^*-*algebra* $\mathrm{M_k}(\mathrm{A})$, we find that it is excision in Hochschild and cyclic homology. Also, we find that it is extended to reflexive and dihedral cohomology,

$$\begin{array}{ccc} 0 \to \mathrm{B}_*(\mathrm{A}) & \to \mathrm{B}_*(\overline{\mathrm{A}}) & \to \mathrm{B}_*(\mathbb{C}) \to 0 \\ \downarrow & \downarrow & \downarrow \\ 0 \to \mathrm{B}_*\big(\mathrm{M_k}(\mathrm{A})\big) & \to \mathrm{B}_*\big(\mathrm{M_k}(\overline{\mathrm{A}})\big) & \to \mathrm{B}_*\big(\mathrm{M_k}(\mathbb{C})\big) \to 0 \end{array} \quad (3)$$

Where for Morita invariance indihedral and reflexive cohomology we find that $\mathrm{B}_*\big(\mathrm{M_k}(\mathbb{C})\big) \to B_*(\mathbb{C})$ and $\mathrm{B}_*(\overline{\mathrm{A}}) \to \mathrm{B}_*\big(\mathrm{M_k}(A)\big)$ are isomorphisms, then $\mathrm{B}^*(\mathrm{A}) \xrightarrow{\sim} \mathrm{B}^*\mathrm{M_k}(\mathrm{A})$.

Proposition 3.5:

Consider the C^*-*algebra* A and an algebra $\mathrm{q_n}$ of continuous functions on the n-sphere, then $\mathrm{B}^{n+1}(\mathrm{K}\otimes\mathrm{A}) \approx \mathrm{B}^{n}\big(\mathrm{K}\otimes\mathrm{q}_1\otimes\mathrm{A}\big)$ *is isomorphism and exists, where* $\mathrm{q_n}$, $\mathrm{n} = 0,1, \ldots$vanishes at the Northern pole.

Proof:

For an algebra of continuous functionsJ which defined on the unit interval [0,1], then the exact sequence

$$0 \to \mathrm{q}_1 \to \mathrm{J} \xrightarrow{\mathrm{p}} \mathbb{C} \to 0 \quad (4)$$

vanish at the left end, $\ker \mathrm{p} = \mathrm{q}_1$.

If the sequence (4) was tensored by $(\mathrm{K}\otimes\mathrm{A})$, then we get the split exact sequence

$$0 \to (\mathrm{K}\otimes\mathrm{q}_1\otimes\mathrm{A}) \to (\mathrm{K}\otimes\mathrm{J}\otimes\mathrm{A}) \to (\mathrm{K}\otimes\mathrm{A}) \to 0 \quad (5)$$

from (5) we get the long exact sequence inreflexive and dihedral cohomology (see [9]).

$$\ldots \to \mathrm{B}^{n+1}(\mathrm{K}\otimes\mathrm{J}\otimes\mathrm{A}) \to \mathrm{B}^{n+1}(\mathrm{K}\otimes\mathrm{A}) \xrightarrow{\partial} \mathrm{B}^{n}(\mathrm{K}\otimes\mathrm{q}_1\otimes\mathrm{A}) \to \mathrm{B}^{n}(\mathrm{K}\otimes\mathrm{J}\otimes\mathrm{A}) \to \cdots \quad (6)$$

where the connecting homomorphism ∂ is commute with the canonical maps: $\mathrm{HR}^n \xrightarrow{\mathrm{I}} \mathrm{HD}^n$, $\mathrm{HR}^n \to \mathrm{HR}^n$, and $\mathrm{HD}^n \to \mathrm{HD}^n$. If A is C^*-*algebra* and F is split-exact of the split C^*-extensions and stable, then $F(A) = F(K\otimes A)$ is functor between category of graded complex vector spaces to a category of C^*-*algebra* (see [8]). Any stable and split-exact functor is homotopy invariant. Since the zero and identity endomorphisms of $(J \otimes A)$ are homotopic, then $F(J \otimes A) = B^*(K \otimes J \otimes A) = 0$. using this result and sequence (6) we can easily deduce $B^{n+1}(K \otimes A) \approx B^n(K \otimes q_1 \otimes A)$.

Proof theorem 3.3:

From the above proposition, the following commutative diagram is obtained as,

$$\begin{array}{ccc} {}^{\alpha}HR^n(K \otimes A) & \xrightarrow{I} & {}^{\alpha}HD^n(K \otimes A) \\ \downarrow & & \downarrow \\ {}^{\alpha}HR^0\left(K \otimes q_n \otimes A\right) & = & {}^{\alpha}HD^0\left(K \otimes q_n \otimes A\right) \end{array} \quad (7)$$

then we obtain the isomorphism:

$$I: \quad {}^{\alpha}HR^*(K \otimes A) \xrightarrow{I} \quad {}^{\alpha}HD^*(K \otimes A).$$

The following Connes long exact sequence obtain the relation between reflexive and dihedral cohomology,

$$\begin{aligned} \dots \to \quad & {}^{\alpha}HR^1(K \otimes A) \to \quad {}^{\alpha}HD^0(K \otimes A) \to \quad {}^{-\alpha}HD^2(K \otimes A) \to \quad {}^{\alpha}HR^2(K \otimes A) \\ & \to \quad {}^{\alpha}HD^1(K \otimes A) \xrightarrow{S} \quad {}^{-\alpha}HD^3(K \otimes A) \to \cdots \to \quad {}^{\alpha}HR^n(K \otimes A) \\ & \to \quad {}^{\alpha}HD^{n-1}(K \otimes A) \xrightarrow{S} \quad {}^{-\alpha}HD^{n+1}(K \otimes A) \\ & \to \cdots \end{aligned} \quad (8)$$

for the periodic operators. From (7) and (8) we have;

$$^{\alpha}HD^*\left(K \otimes A\right) = \quad ^{\alpha}HR^*\left(K \otimes A\right) = 0, \qquad \alpha = \pm 1$$

Example 3.6:

Let $u = \mathcal{F}(H)/k$ be the Calkin algebra then,

$$^{\alpha}HR^*\left(u\right) = \quad ^{\alpha}HD^*\left(u\right) = 0.$$

Example 3.7:

Let H be the Hilbert space with infinite dimensional and $\mathcal{F}$(H)be the algebra of bounded operators on H.Then

$$^{\alpha}HR^*\left(\mathcal{F}(H)\right) = 0 and \quad ^{\alpha}HD^*\left(\mathcal{F}(H)\right) = 0.$$

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