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Quantum Algebra From Generalized Q-Algebra

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Abstract

The paper contains an investigation of the notion of Q-algebras. A brief introduction to quantum mechanics is given.
A brief introduction to $BCI/BCK/BCH$-algebra are given. A new generalization of Q-algebra has been introduced. The Q- quantum algebra has been studied.
Various examples have been given.
Keywords:

Q-algebras; $BCI/BCK/BCH$-algebra; quantum mechanics.

1. Introduction

The basic structure of quantum mechanics is quite different, the state of a system is given by a point in a space", It is can be thought of equivalently as the space of solutions of an equation of motion, or as the space of coordinates and momenta.
The quantities are just functions on this space. There is one distinguished observable, the energy or Hamiltonian. This functions determines how states evolve in time through Hamilton's equations.
BCI-algebra and BCK-algebra have been introduced by Y. Imai and K. Is$\acute{e}$ ([6, 8, 9]).
The former was raised in 1966 by Imai and Iseki.
Several generalizations of a BCI/BCK-algebra were introduced by many researchers, ([14, 15, 17, 18, 20, 21, 22, 24, 2]).
In the present paper we extend this work. We give a new generalization of Q-algebra and some there properties.

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2 Quantum mechanics

In quantum mechanics, stats refers to physical state of quantum system. The state of a quantum mechanical system is given by a nonzero vector in a complex vector space H with Hermitian inner product $\langle .,. \rangle$.
H may be finite or infinite dimensional, we may want to require H to be a Hilbert space, A Hilbert space H consists of a set of vectors and a set of scalars. We will use the notation introduced by Dirac for vectors in the state space H such a vector with a label ψ is denoted ψ.

The Hamiltonian H. Time evolution of states $\psi(t) \in H$ is given by the Schrodinger equation

$$\frac{d}{dt}\psi(t) = \frac{i}{h} H\psi(t)$$

The Hamiltonian observable H will have a physical interpretation in terms of energy, with the boundlessness condition necessary in order to assure the existence of a stable lowest energy state. h is a dimensional constant, called Planck's constant.

2.1 Group representations

The mathematical framework of quantum mechanics is closely related to what mathematicians describe as the theory of group representations.
A standard definition of a Lie group is as a smooth manifold, with group laws given by smooth maps.
Most of the finite dimensional Lie groups of interest are matrix Lie groups, which can be defined as closed subgroups of the group of invertible matrices of some fixed dimension.

A Lie algebra is a vector space g over a field F with an operation

$$[\cdot,\cdot] : g \times g \to g$$

which we call a Lie bracket, such that the following axioms are satisfied
⋄ It is bi linear.
⋄ It is skew symmetric $[x, x] = 0$ which implies $[x, y] = [y, x]$ for all $x, y \in g$.
⋄ It satisfies the Jacobi Identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

An action of a group G on a set M is given by a map

$$(g, x) \in G \times M \to g \cdot x \in M. \tag{1}$$

such that

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot (x)$$

and

$$e \cdot x = x$$

where e is the identity element of G

An action of the group $G_1 = R_3$ on R_3 by translations.

An action of the group $G_2 = O(3)$ of three dimensional orthogonal transformations of R_3. These are the rotations about the origin (possibly combined with a reflection).

Given a group action of G on M, functions on M come with an action of G by linear transformations, given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

where f is some function on M.
A group G is a set with an associative multiplication, such that the set contains an identity element, as well as the multiplicative inverse of each element.

3 Q-algebra

A Q-algebra [1] is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(Q_1) $x * x = 0$,

(Q_2) $0 * x = 0$,

(Q_3) $(x * y) * z = (x * z) * y$,

In Q-algebra X we can define a partial order by putting $x \leq y$ if and only if $x * y = 0$.

Let $X = 0, 1, 2, 3$be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a Q-algebra.

4 $BCH/BCI/BCK$-algebra

By a BCI-algebra we mean an algebra X with a constant 0 and a binary operation "$*$" satisfying the following axioms for all $x, y, z \in X$:

(BCI_1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCI_2) $x * 0 = x$,

(BCI_3) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

In BCI-algebra X we can define a partial order by putting $x \leq y$ if and only if $x * y = 0$.

Proposition 1. Let be a BCI-algebra. A subset S of X called sub algebra of X if the constant 0 of X in S, and $(S, *, 0)$ itself forms a BCI-algebra.
suppose that $(X, *, 0)$ is a BCI-algebra. Define a binary relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$ then $(x, \leq)$ is partially ordered set with 0 as a minimal element in the meaning that $x \leq 0$ implies $x = 0$ for any $x \in X$.

If a BCI-algebra X satisfies $0 * x = 0$, for all $x \in X$, then we say that X is a BCK-algebra.
Every BCI-algebra satisfying $0 * x = 0$ for all $x \in X$ is a BCK-algebra.
Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:
(1) $(x * y) * z = (x * z) * y$
(2) $((x * z) * (y * z)) * (x * y) = 0$
(3) $x * 0 = x$
(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

A nonempty subset I of X is called an ideal of X if it satisfies
(I) $0 \in I$ *and*
(II) $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.
A non-empty subset I of X is said to be an H-ideal of X if it satisfies (I) and
(II) $x * (y * z) \in I$ *and* $y \in I$ *imply* $x * z \in I$, *for all* $x, y, z \in X$.
A BCI-algebra is said to be associative if $(x * y) * z = x * (y * z)$, for all $x, y, z \in X$.

[16] Let $(X; *, 0)$ be a BCI-algebra. Then the following hold:

(i) $x * x = 0$,

(ii) $0 * (0 * x) = x$,

(iii) $(x * y) * z = (x * z) * y$,

for all $x, y, z \in X$.
BCH-algebra [1] is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(i) $x * x = 0$,

(ii) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

(iii) $(x * y) * z = (x * z) * y$,

for all $x, y, z \in X$.
Every BCI-algebra is a BCH-algebra.

Let $X = 0, 1, 2, 3$be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X, *, 0)$ is not a $BCH/BCI/BCK$-algebra.

Theorem 2. [1] Every Q-algebra $(X, *, 0)$ satisfying the associative law is a group under the operation \\$*$".

Theorem 3. [1] Every BCH-algebra X is a Q-algebra.
Every Q-algebra X satisfying $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$ is a BCH-algebra.

Theorem 4. [1] Every Q-algebra satisfying condition $(x * y) * (x * z) = z$ and $x * y = 0$ and $y * x = 0$ imply that $x = y$ for all $x, y, z \in X$ is a BCI-algebra.

Theorem 5. [1] Every Q-algebra X satisfying conditions $(x * y) * z = x * (z * (0))$, $x * y = 0$and $y * x = 0$ imply that $x = y$ and $(x * y) * x = 0$ for all $x, y, z \in X$ is a BCK-algebra.

5 Q- quantum algebra

By a Q- quantum algebra we mean a vector space V over a field F with an operation "$*$" satisfying the following axioms for all $u, v, w \in V$:

(Q_1) $u * u = 0$,

(Q_2) $0 * u = 0$,

(Q_3) $(u * v) * w = (u * v) * w$,

. *for all* $u, v, w \in V$.

consider the flowing table

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

Let $V = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$.
And will defined

$$AB = \begin{bmatrix} A_{11}B_{11} & A_{12}B_{12} \\ A_{21}B_{21} & A_{22}B_{22} \end{bmatrix}$$

Then $(V, *, 0)$ satisfies BCI_1,

$$\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Also $(V, *, 0)$ satisfies BCI_2,

$$= \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \left(\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$$

Also $(V, *, 0)$ satisfies BCI_3,

$$\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

0r

$$\left(\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

this means $xy = 0$ or $yx = 0$ from the table $x = y$.
Then $(V, *, 0)$ is BCI- algebra , now we have that every BCI-algebra is a BCH-algebra, hence $(V, *, 0)$ is BCH- algebra and by theorem 3 $(V, *, 0)$ is Q- algebra.

5.1 Representations and quantum mechanics

A representation (π, V) of a group G is a homomorphism

$$\pi \in G \to \pi(g) \in GL(V)$$

where GL(V) is the group of invertible linear maps $V \to V$, with V a vector space.
Saying the map π is a homomorphism means

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2)$$

for all $g_1, g_2 \in G$.
We call a map $f : (X_1; *_1, S_1) \longrightarrow (X_2; *_2, S_2)$ between two BCK-algebras an homomorphism, if f is for any $x, y \in X_1$

$$f(x * y) = f(x) * f(y).$$

If the mapping f is onto and one- to-one then f called isomorphism.

Consider the flowing table

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

; $V = 0, 1, a, b, c$

Here $(V, *, 0)$ is Q- algebra
We defined

$$\pi(x) = e^A, A = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

for all $x \in V$.
The map π is a homomorphism

$$\pi(x_1)\pi(x_2) = \pi(x_1 x_2)$$

for all $x_1, x_2 \in V$.
Also the mapping f is onto and one- to-one, hence π is isomorphism.

6 Discussion

We have introduced the quantum mechanics. The basic structure of quantum mechanics is quite different, the state of a system is given by a point in a space",It is can be thought of equivalently as the space of solutions of an equation of motion, or as the space of coordinates and momenta.The Group representations has been , the definition of Lie algebra is given. The Representations and quantum mechanics have been discussed.
We have introduced the Q-algebra. A brief introduction to $BCI/BCK/BCH$-algebra are given.A new generalization of Q-algebra has been introduced.
The Q- quantum algebra has been defined. Various examples have been given.

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