Journal of Progressive Research in Mathematics
Vol 18,2 - 2021 — E-ISSN: 2395-0218
http://www.scitecresearch.com/journals/index.php/jprm

Special Properties of the Zeros of the Analytic Representations of Finite Quantum Systems

Muna Tabuni

Department of Mathematics, University of Tripoli, Libya
Email: *mona_bashat@yahoo.co.uk*
Email: M.Tabuni@uot.edu.ly

Received: May 27, 2021; Accepted: June 18, 2021; Published: July 20, 2021

Abstract

The paper contains an investigation on the special properties of the zeros of the analytic Representations of Finite Quantum Systems.
These zeros and its paths define the completely the finite quantum system.
The present paper study Construction of the analytic representation from its zeros.
A brief introduction to analytic representation of finite quantum systems is given. The zeros of this function and there evolution time are considered.
The analytic function $f(z)$ have exactly d zeros.
The analytic function have been constructed from its zeros.

Keywords:

Construction, analytic, representation, zeros.

1. Introduction

This Paper is devoted to discuses some problem related to the zeros in analytic representation of finite quantum systems on a torus. Analytic functions are considered from [1, 2, 3] and used in various places in physic sciences.
Recently, there has been a lot of work on quantum systems where the position and momentum take values in $\mathbb{Z}_d$ (the integers modulo d). Ref [15] has used the Zak transform [16, 17] to introduce an analytic representation.

Ref [12, 13] has considered analytic representations of finite quantum systems on a torus. The analytic function has exactly $\mathfrak{N}$ zeros which zeros define uniquely the quantum state.
Ref [13] has been studied the motion of the zeros.In some cases $\mathfrak{N}$ of the zeros follow the same path and in other cases each zero follow its own path.

How to cite this paper: M.Tabuni (2021) Special Properties of the Zeros of the Analytic Representations of Finite Quantum Systems. Journal of Progressive Research in Mathematics, 18(2), 69-83. Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2063

Ref. [15, 14] have constructed the function $f(z)$ from its zeros.
In The present work we discuss the constructino of this function from its zeros.
We suppose that d zeros ζ_n in the cell S are given, and that they satisfy the constraint of Eq. (46). In other words, $d-1$ zeros are given and the last is found through the constraint of Eq. (46).
We provr that the analytic representation have Constructed from its zeros.
We interduced Position and Momentum States.
We discussed briefly Displacement Operators. We considred the Zak transform.
several examples are considered.

2 Finite Quantum Systems

We consider a quantum system with a $d-$dimensional Hilbert space $\mathcal{H}$, where position and momentum take values in $\mathbb{Z}_d$.
We use the notation $|f\rangle$ for the states in this particular Hilbert space $L^2(\mathbb{Z}_d)$.

2.1 Position and Momentum States

An orthonormal basis in this system consists of the position states $|\mathcal{X};m\rangle$ and momentum states $|\mathcal{P};m\rangle$, where $m \in \mathbb{Z}_d$.

$$\langle \mathcal{X};m|\mathcal{X};n\rangle = \delta_{mn}, \quad \langle \mathcal{P};m|\mathcal{P};n\rangle = \delta_{mn}, \tag{1}$$

$$\sum_{m=0}^{d-1} |\mathcal{X};m\rangle\langle \mathcal{X};m| = \sum_{m=0}^{d-1} |\mathcal{P};m\rangle\langle \mathcal{P};m| = \mathbf{1}. \tag{2}$$

δ_{mn} in Eq. 1 is the Kronecker Delta satisfying

$$\delta_{mn} = \begin{cases} 0 \quad if \quad m \neq n; \\ \\ 1 \quad if \quad m = n. \end{cases} \tag{3}$$

The Fourier operator is a unitary operator defined as

$$\mathcal{F} = d^{-1/2} \sum_{m,n=0}^{d-1} \omega(mn)|\mathcal{X};m\rangle\langle \mathcal{X};n| = \sum_{m=0}^{d-1} \omega(mn)|\mathcal{P};m\rangle\langle \mathcal{X};m|. \tag{4}$$

where

$$\omega(\alpha) = exp(\frac{2\pi i \alpha}{d}); \quad \mathcal{F}^4 = \mathbf{1}. \tag{5}$$

The position and momentum states are related to each other through the finite Fourier transformation

$$|\mathcal{P};n\rangle = d^{-1/2} \sum_{m=0}^{d-1} \omega(mn)|\mathcal{X};m\rangle. \tag{6}$$

$|\mathcal{X};m\rangle$ and $|\mathcal{P};m\rangle$ are the eigenstates of the position and momentum operators, which are given by

$$\hat{x} = \sum_{n=0}^{d-1} m|\mathcal{X};m\rangle\langle \mathcal{X};m|; \qquad \hat{p} = \sum_{n=0}^{d-1} m|\mathcal{P};m\rangle\langle \mathcal{P};m| \tag{7}$$

where

$$\hat{F}\hat{x}\hat{F}^{\dagger} = \hat{p}; \quad \hat{F}\hat{p}\hat{F}^{\dagger} = -\hat{x}. \tag{8}$$

2.2 Displacement Operators

In the finite quantum system, the position and momentum are both integers modulo d, therefore, the phase space is the toroidal lattice $\mathbb{Z}_d \times \mathbb{Z}_d$.

The displacement operators in this particular phase space are defined as

$$\hat{Z} = exp(i\frac{2\pi}{d}\hat{x}); \quad \hat{X} = exp(-i\frac{2\pi}{d}\hat{p}). \tag{9}$$

They are unitary operators and perform displacements along the P and X axes in the $\mathbb{Z}_d \times \mathbb{Z}_d$ phase space, obeying the relations

$$\hat{X}^d = \hat{Z}^d = \mathbf{1}; \quad \hat{X}^\beta \hat{Z}^\alpha = \hat{Z}^\alpha \hat{X}^\beta \omega(-\alpha\beta). \tag{10}$$

where α, β are integers in $\mathbb{Z}_d$.

$$\hat{Z}^\alpha |P;m\rangle = |P;m+\alpha\rangle; \quad \hat{Z}^\alpha |X;m\rangle = \omega(\alpha m)|X;m\rangle; \tag{11}$$

$$\hat{X}^\beta |P;m\rangle = \omega(-m\beta)|P;m\rangle; \quad \hat{X}^\beta |X;m\rangle = |X;m+\beta\rangle. \tag{12}$$

The general displacement operators are defined as

$$\hat{D}(\alpha,\beta) = \hat{Z}^\alpha \hat{X}^\beta \omega(-2^{-1}\alpha\beta) = \hat{X}^\beta \hat{Z}^\alpha \omega(2^{-1}\alpha\beta). \tag{13}$$

with

$$\hat{D^\dagger} = \hat{D}(-\alpha,-\beta). \tag{14}$$

It is easy to see that

$$\hat{D}(\alpha,\beta)|X;m\rangle = \omega(2^{-1}\alpha\beta + \alpha m)|X;m+\beta\rangle; \tag{15}$$

$$\hat{D}(\alpha,\beta)|P;m\rangle = \omega(-2^{-1}\alpha\beta - \beta m)|P;m+\alpha\rangle. \tag{16}$$

3 Zak Transform

We introduce a map between states in the infinite dimensional harmonic oscillator Hilbert space H and the d-dimensional Hilbert space $\mathcal{H}$.

Let $|g\rangle$ be a state in H with (normalized) wavefunction in the x-representation $g(x) = \langle x|g\rangle$.

Using the map we define the corresponding state $|f\rangle$ in $\mathcal{H}$ as

$$f_m = \mathcal{N}^{-1/2} \sum_{\omega=-\infty}^{\infty} g[(\frac{2\pi}{d})^{1/2}(m+d\omega)]; \tag{17}$$

$$\tilde{f}_m = \mathcal{N}^{-1/2} \sum_{\omega=-\infty}^{\infty} \tilde{g}[(\frac{2\pi}{d})^{1/2}(m+d\omega)], \tag{18}$$

where $m \in \mathbb{Z}_d$. $\mathcal{N}$ is a normalization factor given by

$$\mathcal{N} = \sum_{m=0}^{d-1}\{\sum_{\omega=-\infty}^{\infty} g^*[(\frac{2\pi}{d})^{1/2}(m+d\omega)]\} \{\sum_{\omega'=-\infty}^{\infty} g[(\frac{2\pi}{d})^{1/2}(m+d\omega')]\}. \tag{19}$$

This map is a special case of the Zak transform [16, 17]. Through this map a state $|f\rangle$ in $L^2(\mathbb{Z}_d)$ is constructed from the state $|g\rangle$ in $L^2(\mathbb{R})$.

3.1 Example

We consider the number eigenstates $|n\rangle$ whose wave function is

$$g(x,n) = \langle x|n\rangle = (\frac{1}{\sqrt{\pi}2^2 n!})^{1/2} exp(\frac{-x^2}{2}) H_n(x), \tag{20}$$

following ref. [15, 14], we use Eq. 17 to find

$$f_m(n) = \mathcal{N}(n)^{-1/2} \sum_{\omega=-\infty}^{\infty} g[(\frac{2\pi}{d})^{1/2}(m + d\omega), n]; \tag{21}$$

$$\tilde{f}_m(n) = \mathcal{N}(n)^{-1/2} \sum_{\omega=-\infty}^{\infty} \tilde{g}[(\frac{2\pi}{d})^{1/2}(m + d\omega), n], \tag{22}$$

3.2 Example

We consider the coherent states $|\alpha\rangle$ whose wave function is

$$g(x,\alpha) = \langle x|\alpha\rangle = \pi^{-1/4} exp(-\frac{1}{2}x^2 + \alpha x - \frac{1}{2}\alpha_R\alpha), \tag{23}$$

where $\alpha = \alpha_R + i\alpha_I$.
Following ref. [15, 14], we use Eq. 17 to find

$$f_m(\alpha) = \frac{1}{\sqrt{\mathcal{N}(\alpha) d\sqrt{\pi}}} exp(\frac{i}{2}\alpha_I\alpha)\vartheta_3[\frac{\pi m}{d} - \alpha(\sqrt{\frac{\pi}{2d}}); \frac{i}{d}], \tag{24}$$

where ϑ_3 is the Jacobi Theta functions which is defined as

$$\vartheta(u;v) = \sum_{n=-\infty}^{\infty} exp(i\pi\tau n^2 + i2nu), \tag{25}$$

and τ is a complex number with positive imaginary parts.

4 Analytic Representation

4.1 Definition of an Analytic Representation

We consider an arbitrary pure normalised state $|F\rangle$

$$|F\rangle = \sum_{m=0}^{d-1} F_m|X;m\rangle \qquad \sum_{m=0}^{d-1} |F_m|^2 = 1. \tag{26}$$

We will use the notation

$$|F^*\rangle = \sum_{m=0}^{d-1} F_m^*|X;m\rangle; \quad \langle F| = \sum_{m=0}^{d-1} F_m^*\langle X;m|$$
$$|F^*\rangle = \sum_{m=0}^{d-1} F_m\langle X;m|. \tag{27}$$

Following ref. [15], we define the analytic representation of $|F\rangle$ as:

$$f(z) = \pi^{-1/4} \sum_{m=0}^{d-1} F_m\vartheta_3[\frac{\pi m}{d} - z(\frac{\pi}{2d})^{1/2}; \frac{i}{d}] \tag{28}$$

The function $f(z)$ satisfies

$$f[z + (2\pi d)^{1/2}] = f(z), \tag{29}$$

$$f[z + i(2\pi d)^{1/2}] = f(z) exp[\pi d - i(2\pi d)^{1/2} z]. \tag{30}$$

$f(z)$ is defined on a square area S on the complex plane

$$S = [a, a + (2\pi d)^{1/2}) \times [b, b + (2\pi d)^{1/2}); \quad a, b \in \mathbb{R}. \tag{31}$$

The scalar product is given by

$$\langle F^*|G\rangle = (2\pi)^{-1/2} d^{-3/2} \int_S d^2 z exp(-z_I^2) f(z) g(z^*); \tag{32}$$

where

$$z = z_R + iz_I; \quad z = dz_R dz_I. \tag{33}$$

The orthogonality relation

$$\frac{1}{\pi\sqrt{2d^3}} \int_S d^2 z \exp(-z_I^2) \Theta_3[\frac{n\pi}{d} - z\sqrt{\frac{\pi}{2d}}; id^{-1}] \times \Theta_3[\frac{m\pi}{d} - z^* \pi^{1/2} (2d)^{-1/2}; id^{-1}] = \delta(m, n), \tag{34}$$

where $m, n \in \mathbb{Z}_d$.

Follows from the theory presented in ref. [15]. We provide direct proof:

4.2 Proof

Using the definition of Theta functions:

$$\Theta_3(u;\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2 + i2nu), \tag{35}$$

we rewrite the right-hand side R of Eq. (34) as:

$$R = \frac{1}{\pi\sqrt{2d^3}} \sum_{k,\ell} \exp[\frac{i2\pi(km + \ell n)}{d}] R_1 \times \int_0^{\sqrt{2\pi d}} dz_I \exp[-z_I^2 + \sqrt{\frac{2\pi}{d}}(k - \ell) z_I - (\frac{\pi}{d})(k^2 + \ell^2)]; \tag{36}$$

where

$$R_1 = \int_0^{\sqrt{2\pi d}} dz_R \exp[-i\sqrt{\frac{2\pi}{d}}(k + \ell) z_R] = (2\pi d)^{1/2} \delta(k, -\ell) \tag{37}$$

Inserting Eq. (37) into Eq. (36), we get:

$$R = \pi^{-1/2} d^{-1} \sum_{k=-\infty}^{\infty} \exp[\frac{i2\pi k(m-n)}{d}] \times \int_0^{\sqrt{2\pi d}} dz_I \exp\{-[z_I - \sqrt{\frac{2\pi}{d}} k]^2\} \tag{38}$$

We can rewrite k as $k = k_0 + dN$, where $0 \leq k_0 \leq d - 1$ and N is an integer. Then

$$R = \pi^{-1/2} d^{-1} \sum_{k=0}^{d-1} \exp[\frac{i2\pi k_0(m-n)}{d}] \times \sum_{N=-\infty}^{\infty} \int_0^{\sqrt{2\pi d}} dz_I \exp -[z_I - \sqrt{\frac{2\pi}{d}}(Nd + k_0)]^2. \tag{39}$$

However

$$\sum_{N=-\infty}^{\infty}\int_0^{\sqrt{2\pi d}} dz_I \exp-[z_I-\sqrt{\frac{2\pi}{d}}(Nd+k_0)]^2$$

$$= \int_{-\infty}^{\infty} dz_I \exp-[z_I-\sqrt{\frac{2\pi}{d}}k_0]^2\} = \pi^{1/2}. \tag{40}$$

Therefore

$$R = d^{-1}\sum_{k=0}^{d-1}\exp[\frac{i2\pi k_0(m-n)}{d}] = \delta(m,n).$$

Using this, we can prove that

$$F_m = \frac{1}{\pi\sqrt{2d^3}}\int_S d^2 zexp(-z_I^2)\vartheta_3[\frac{\pi m}{d} - z(\frac{\pi}{\sqrt{2d}}); \frac{i}{d}]f(z^*). \tag{41}$$

Following ref. [15], we derive the analytic representations of position eigenstates $|\mathcal{X};m\rangle$ and momentum eigenstates $|\mathcal{P};m\rangle$ are

$$f(z) = \pi^{-1/4}\vartheta_3[\frac{\pi m}{d} - z(\frac{\pi}{2d})^{1/2}; \frac{i}{d}] \tag{42}$$

$$f(z) = \pi^{-1/4}exp(-\frac{1}{2}z^2)\vartheta_3[\frac{\pi m}{d} - zi(\frac{\pi}{2d})^{1/2}; \frac{i}{d}] \tag{43}$$

correspondingly.

5 Zeros of the functions $f(z)$

We shall now denote as ζ_n the zeros of $f(z)$, i.e. $f(\zeta_n) = 0$.
Let $f(z)$ be an analytic function. We consider the integrals

$$J_0 = \oint_\ell \frac{dz}{2\pi i}\frac{\partial_z f(z)}{f(z)}; \quad J_1 = \oint_\ell \frac{dz}{2\pi i}\frac{\partial_z f(z)}{f(z)}z. \tag{44}$$

J_0 is equal to the number of zeros of this function (with the multiplicities taken into account), inside the contour ℓ.
J_1 is equal to the sum of these zeros.
The analytic function $f(z)$ satisfies the quasi-periodicity of Eq. (30).
Using the quasi-periodicity of Eq. (30) we prove that the integral J_0, for a contour along the boundary of the cell S, is equal to d.
Therefore the analytic functions $f(z)$ have exactly d zeros, within each cell S.
Using the quasi-periodicity of Eq. (30) we also prove that

$$\oint_\ell \frac{dz}{2\pi i}\frac{\partial_z f(z)}{f(z)}z = (2\pi)^{1/2}d^{3/2}(M+iN) + (\frac{\pi}{2})^{1/2}d^{3/2}(1+i) \tag{45}$$

In the plane which acts as the covering surface of the torus, each cell is characterised by a pair of integers (M,N).
Integration on the contour along the boundary of the cell characterised by $(0,0)$, gives

$$(\frac{\pi}{2})^{1/2}d^{3/2}(1+i)$$

Integration on the contour along the boundary of the cell characterised by (M,N), gives

$$(\frac{\pi}{2})^{1/2}d^{3/2}(1+i) + (2p)^{1/2}d^{3/2}(M+iN)$$

Therefore, the sum of the zeros ζ_n of $f(z)$ is

$$\sum_{n=1}^{d}\zeta_n = (2\pi)^{1/2}d^{3/2}(M+iN) + (\frac{\pi}{2})^{1/2}d^{3/2}(1+i). \tag{46}$$

Eq. (45) has been discussed in [18, 15].

5.1 Example

- In Fig. (1) we show the distributions of zeros for case$d = 3$, and the $|F(t)\rangle$ at $t = 0$ is described through the coefficients

$$F_0(0) = 0.23 + i0.13; F_1(0) = 0.67 - i0.04; ; F_2(0) = 0.67 - i0.09. \quad (47)$$

- In Fig. (2) we show the distributions of zeros for case $d = 3$, and the $|F(t)\rangle$ at $t = 0$ is described through the coefficients

$$F_0(0) = 0.84; F_1(0) = 0.33; F_2(0) = 0.18;$$
$$F_3(0) = 0.18; F_4(0) = 0.33; \quad (48)$$

where we see only one zero(because the five zeros are identical).

6 Construction of the analytic representation from its zeros

Ref. [15, 14] have constructed the function $f(z)$ from its zeros. We suppose that d zeros ζ_n in the cell S are given, and that they satisfy the constraint of Eq. (46). In other words, $d - 1$ zeros are given and the last is found through the constraint of Eq. (46).
The function $f(z)$ is given by

$$f(z) = C \exp[-i(\frac{2\pi}{d})^{1/2} N z] \prod_{n=1}^{d} \Theta_3[w_n(z);\ i];$$
$$w_n(z) = (\frac{\pi}{2d})^{1/2}(z - \zeta_n) + \frac{\pi(1+i)}{2} \quad (49)$$

where $N \in \mathbb{Z}$ is the constraint of Eq. (46), and C is a constant determined by the normalisation condition. We can prove the equality of Eqs. (28) and (49) using Jacobi's triple product identity.
We consider the jacobi triple product identity

$$\sum_{m=-\infty}^{\infty} x^{m^2} z^{2m} =$$
$$\prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1} z^2)(1 + x^{2n-1} z^{-2}) \quad (50)$$

The Jacobi Theta functions is defined as

$$\vartheta(z;\tau) = \sum_{m=-\infty}^{\infty} exp(i\pi\tau m^2 + i2mz), \quad (51)$$

$$= \sum_{m=-\infty}^{\infty} \left(e^{i\pi\tau}\right)^{m^2} \left(e^{iz}\right)^{2m}, \quad (52)$$

$$= \prod_{n=1}^{\infty} (1 - (e^{i\pi\tau})^{2n})(1 + (e^{i\pi\tau})^{2n-1}(e^{iz})^2)$$
$$(1 + (e^{i\pi\tau})^{2n-1}(e^{iz})^{-2}).$$

We now consider the function

$$\vartheta_3((z - z_j + \omega)\sqrt{\frac{\pi}{2d}}; i) =$$
$$\sum_{m=-\infty}^{\infty} exp(i\pi i m^2 + 2mi(z - z_j + \omega)\sqrt{\frac{\pi}{2d}}), \quad (53)$$
$$= \sum_{m=-\infty}^{\infty} (e^{-\pi})^{m^2} (e^{iz\sqrt{\frac{2\pi}{d}}})^{2m} (e^{-i(z_j - \omega)\sqrt{\frac{\pi}{2d}}})^{2m}. \quad (54)$$

Let

$$x = e^{-\pi}; y = exp[-i(\frac{2\pi}{d})^{1/2} z];$$

$$b_j = (z_j - \omega)(\frac{\pi}{2d})^{1/2}. \tag{55}$$

Then Eq. (53) is rewritten as

$$h(y) = \prod_{n=1}^{\infty}(1 - x^{2n})(1 + x^{2n-1}y^{-1}e^{-2b_j i})(1 + x^{2n-1}ye^{2ib_j}). \tag{56}$$

We define

$$F(y) = \prod_{n=1}^{\infty}(1 + x^{2n-1}ye^{2b_j i})(1 + \frac{x^{2n-1}}{ye^{2ib_j}}), \tag{57}$$

$$= (1 + xye^{2ib_j})(1 + \frac{x}{ye^{2ib_j}})(1 + x^3ye^{2ib_j})$$

$$\times(1 + \frac{x^3}{ye^{2ib_j}})(1 + x^5ye^{2ib_j})(1 + \frac{x^5}{ye^{2ib_j}})...$$

$$F(x^2y) = (1 + x^3ye^{2ib_j})(1 + \frac{1}{xye^{2ib_j}})(1 + x^5ye^{2ib_j})$$

$$\times(1 + \frac{x}{ye^{2ib_j}})(1 + x^7ye^{2ib_j})(1 + \frac{x^3}{ye^{2ib_j}})...$$

$$\frac{F(x^2y)}{F(y)} = (1 + \frac{1}{xye^{2ib_j}})(\frac{1}{1 + xye^{2ib_j}}),$$

$$= \frac{xye^{2ib_j} + 1}{xye^{2ib_j}} \frac{1}{1 + xye^{2ib_j}},$$

$$= \frac{1}{xye^{2ib_j}},$$

$$F(y) = xye^{2ib_j}F(x^2y). \tag{58}$$

It is easily seen that

$$h(y) = F(y)\prod_{n=1}^{\infty}(1 - x^{2n}), \tag{59}$$

and

$$h(x^2y) = F(x^2y)\prod_{n=1}^{\infty}(1 - x^{2n}). \tag{60}$$

From Eq. (58), we see

$$h(x^2y) = \frac{F(y)}{xye^{2ib_j}}\prod_{n=1}^{\infty}(1 - x^{2n}) = \frac{h(y)}{xye^{2ib_j}}$$

$$h(y) = xye^{2ib_j}h(x^2y). \tag{61}$$

Then Eq. (49) is rewritten as

$$g(y) = cy^N\prod_{j=1}^{d}\prod_{n=1}^{\infty}(1 + x^{2n-1}y^{-1}e^{-2b_j i})(1 + x^{2n-1}ye^{2b_j i}), \tag{62}$$

$$= cy^N\prod_{j=1}^{d}h(y). \tag{63}$$

$$g(x^2y) = cx^{2N}y^N\prod_{j=1}^{d}h(x^2y), \tag{64}$$

from Eq. (61) we get

$$= cx^{2N}y^N\prod_{j=1}^{d}x^{-1}y^{-1}e^{-2ib_j}h(y), \tag{65}$$

$$= x^{-d}y^{-d}x^{2N}\prod_{j=1}^{d}e^{-2ib_j}g(y);$$

where

$$\prod_{j=1}^{d}e^{-2ib_j} = x^{-2N},$$

$$g(x^2y) = x^{-d}y^{-d}g(y). \tag{66}$$

Expand g in a Laurent Series:

$$g(y) = \sum_{n=-\infty}^{\infty}a_ny^n, \tag{67}$$

$$\sum_{n=-\infty}^{\infty}a_ny^n = x^dy^d\sum_{n=-\infty}^{\infty}a_n(x^2y)^n = \sum_{n=-\infty}^{\infty}a_nx^{2n+d}y^{n+d}, \tag{68}$$

This can be re-indexed with $n' = n - d$ on the left side of Eq. (68) to get

$$\sum_{n=-\infty}^{\infty}a_ny^n = \sum_{n=-\infty}^{\infty}a_nx^{2n-d}y^n, \tag{69}$$

$$a_n = a_{n-d}x^{2n-d}, \tag{70}$$

so

$$a_d = a_0x^d;$$
$$a_{d+1} = x^{2(d+1)-d}a_{d+1-d} = x^{d+2}a_1;$$
$$a_{2d} = x^{2(2d)-d}a_{2d-d} = x^{3d}a_d = x^{3d}(a_0x^d) = x^{4d}a_0;$$
$$a_{2d+1} = x^{2(2d+1)-d}a_{2d+1-d}$$
$$= x^{3d+2-d}a_d + 1 = x^{3d+2-d}(x^{d+2}a_1) = x^{4d+d}a_1.$$

Therefore

$$a_{kd+j} = (x)^{(k^2d+2kj)}a_j; \qquad j = 0, ..., d-1. \tag{71}$$

On the other hand, we consider Eq. (28),

$$f(z) = \pi^{\frac{-1}{4}}\sum_{m=0}^{d-1}f_m\vartheta_3(\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}}; \frac{i}{d})$$

$$= \pi^{\frac{-1}{4}}\sum_{m=0}^{d-1}f_m\sum_{n=-\infty}^{\infty}(e^{-\pi})^{\frac{n^2}{d}}exp(\frac{inm\pi}{d})exp(-i2nz\sqrt{\frac{\pi}{2d}}), \tag{72}$$

Let

$$x = exp(-\pi); \qquad y = exp(-i2z\sqrt{\frac{\pi}{2d}})$$

Then Eq. (72) is rewritten as

$$g'(y) = \pi^{\frac{-1}{4}}\sum_{m=0}^{d-1}\sum_{n=-\infty}^{\infty}x^{\frac{n^2}{d}}y^ne^{(\frac{i2nm\pi}{d})}f_m, \tag{73}$$

$$= \pi^{\frac{-1}{4}}\sum_{n=-\infty}^{\infty}(x)^{n^2/d}y^n\sum_{m=0}^{d-1}f_mexp(\frac{2\pi imn}{d}), \tag{74}$$

$$= \pi^{\frac{-1}{4}}\sum_{n=-\infty}^{\infty}(x)^{n^2/d}y^n\tilde{f}_n, \tag{75}$$

$$= \sum_{n=-\infty}^{\infty}(\pi^{\frac{-1}{4}}(x)^{n^2/d}\tilde{f}_n)y^n, \tag{76}$$

where $\tilde{f}_{n+d} = \tilde{f}_n$ is the finite Fourier transform of f_m.
If we let

$$a_n = \pi^{\frac{-1}{4}} (x)^{n^2/d} \tilde{f}_n, \tag{77}$$

Eq. (70) is satisfied and therefore $g(y) = g'(y)$. The equality of Eq. (28) and Eq. (49) is proved.
Using Eq. (41) we can calculate the coefficients F_m.
Numerically, we insert d arbitrary values $z_0, ..., z_{d-1}$ and solve the system of d equations with d unknowns:

$$f(z_j) = \pi^{-1/4} \sum_{m=0}^{d-1} F_m \, \Theta_3[\pi m d^{-1} - z_j \pi^{1/2} (2d)^{-1/2}; i d^{-1}]. \tag{78}$$

We take the normalisation coefficients equal to one, and after the calculation we normalise the vector F_m.

6.1 Example

let $d = 3$ and $\zeta_0;\ \zeta_1$ be given as follows:

$$\zeta_0 = 0.26 + 2.95; \quad \zeta_1 = 2.16 + 2.22i. \tag{79}$$

Using Eq. (46), we get $\zeta_2 = 4.09 + 1.34i$.
We choose three arbitrary values $0, 1, -1$ and insert them with $\zeta_0; \zeta_1; \zeta_2$ in Eq. (49). We then find $f(0), f(1), f(-1)$ to insert them in Eq. (78) and solve the system of three equations with three unknowns, in which case we get

$$\begin{pmatrix} F_0 \\ F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 0.2349 + 0.1301i \\ 0.6778 - 0.0490i \\ 0.6760 - 0.0952i \end{pmatrix}.$$

7 Paths of the Zeros

Using the Hamiltonian H, the state
$F(0) = \sum F_m(0)X; m$ at $t = 0$ evolves at time t into

$$F(t) = \exp(itH)F(0) = \sum_{m=0}^{d-1} F_m(t)X; m. \tag{80}$$

Let $f(z;t)$ be the analytic function corresponding to $\exp(itH)f$ (where$f(z;0) = f(z)$) and $\zeta_n(t)$ the zeros.
Ref [13] has been studied the motion of the zeros. In some cases $\mathfrak{N}$ of the zeros follow the same path and in other cases each zero follow its own path.
An important special case is periodic systems, i.e., systems with Hamiltonians with commensurate eigenvalues. In this case, the paths of the zeros are closed paths.
In some cases $\mathfrak{M}$ of the d zeros follow the same path and we say that this path has multiplicity $\mathfrak{M}$.
Ref [13] has been considerd several examples and showed that after a period the zeros exchange their positions.

7.1 Exampe

• In Fig. 3 we consider the case where $d = 3$ and the state $F(0)$ at $t = 0$ are described through the coefficients

$$F_0(0) = 0.84 + 0.09i; F_1(0) = 0.36 - 0.04i;$$
$$F_2(0) = 0.36 - 0.04i. \tag{81}$$

We calculate the coefficients $F_m(t)$ for the Hamiltonian

$$H_D = \begin{pmatrix} 1 & 1-i & 0 \\ 1+i & 1 & 0 \\ 0 & 0 & 2.5 \end{pmatrix} \tag{82}$$

which has the eigenvalues $-0.41, 2.41, 2.5$ and we calculate the corresponding $f(z;t)$ using Eq. (28).

- In Fig. 4 we consider the case where $d = 4$ and the state $F(0)$ at $t = 0$ are described through the coefficients

$$F_0(0) = 0.03 + 0.25i; F_1(0) = 0.53 + 0.05i;$$
$$F_2(0) = 0.74 - 0.22i; F_3(0) = 0.12 - 0.16i. \tag{83}$$

We calculate the coefficients $F_m(t)$ for the Hamiltonian

$$H_E = \begin{pmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \tag{84}$$

which has the eigenvalues $0, 0.5, 0.5, 2$ and we calculate the corresponding $f(z;t)$ using Eq. (28).

Periodic finite quantum systems

In Ref [13] we study the periodic systems, and we discuss some examples where two or more zeros follow the same path.
During a period the zeros follow a closed paths. In some cases $\mathfrak{M}$ of the zeros follow the same path.
We present the following examples

7.2 Example

- We consider the Hamiltonian

$$H = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{85}$$

which has the eigenvalues $0, 2$ and the period $T = \pi$.
Let $\zeta_0(t), \zeta_1(t)$ be the paths of the zeros. We assume that at $t = 0$

$$\zeta_0(0) = 0.39 + 2.27i; \zeta_1(0) = 3.15 + 1.27i. \tag{86}$$

These zeros obey the constraint of Eq. (46) and they are on a torus (i.e., they are defined modulo $(4\pi)^{1/2}$).
During a period the zeros ζ_0, ζ_1 follow a closed paths.
In Fig. 5 we present the paths of these zeros.

- We consider the Hamiltonian

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{87}$$

which has the eigenvalues $0, 1, 2$ and the period $T = 2\pi$. Let $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ be the paths of the three zeros. We assume that at $t = 0$

$$\zeta_0(0) = 1.34 + 2.27i;$$
$$\zeta_1(0) = 2.15 + 2.32i;$$
$$\zeta_2(0) = 3.01 + 1.91i. \tag{88}$$

These zeros obey the constraint of Eq. (46) and they are on a torus (i.e., they are defined modulo $(6\pi)^{1/2}$).

In this case we found numerically that

$$\zeta_1(T+t) = \zeta_2(t); \quad \zeta_2(T+t) = \zeta_1(t). \tag{89}$$

Here two of the zeros follow the same path (which by definition has multiplicity $\mathfrak{M} = 2$) and the third one follows a different path.

In Fig. 6 we present the paths of these zeros. Here after a period

$$\zeta_1(T) = \zeta_2(0); \zeta_2(T) = \zeta_1(0); \zeta_0(T) = \zeta_0(0) \tag{90}$$

During a period the zero ζ_0 follows a closed path. The other two zeros exchange position after a period. These zeros comes to their original position after 2 periods

$$\zeta_1(2T) = \zeta_1(0); \quad \zeta_2(2T) = \zeta_2(0). \tag{91}$$

8 Conclusion

We have studied the basic properties of the zeros of analytic theta functions. We discussed briefly the analytic representation of finite quantum systems. We reviewed briefly the zeros of analytic theta function and there time evolution.

Quantum states have been represented with the analytic function of Eq. (28) on a torus, which has exactly d zeros and obeys the constraint of Eq. (46).

This zeros and there evolution time are discussed.

We have considered the construction of the analytic function of state from its zeros.

We have proved this construction. We gave several examples to demonstrate these ideas.

Acknowledgments

We thank the Editor and the referee for their comments.

References

[1] A.M. Perelomov, 'Generalized coherent states and their applications' (Springer, Berlin, 1986)

[2] A. Vourdas, J. Phys. A39, R65 (2006)

[3] A. Vourdas, R.F. Bishop, Phys. Rev. A50, 3331 (1994)

[4] V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961)
V. Bargmann, Commun. Pure Appl. Math. **20**, 1 (1967)

[5] S. Schweber, J. Math. Phys. 3, 861 (1962)
S. Schweber, Ann. Phys.(NY) 41, 205 (1967)

[6] J. Kurchan, P. Leboeuf, M. Saraceno, Phys. Rev. A40, 6800 (1989)
A. Voros, Phys. Rev. A40, 6814 (1989)

[7] J.H. Hannay, J. Phys. A**29**, L101 (1996)
J.H. Hannay, J. Phys. A**31**, L755 (1998)

[8] N.L. Balazs, A. Voros, Phys. Rep. C143, 109 (1986)

[9] P. Leboeuf, A. Voros, J. Phys. A23, 1765 (1990)
P. Leboeuf, J. Phys. A24, 4575 (1991)
M.B. Cibils, Y. Cuche, P. Leboeuf, W.F. Wreszinski, Phys. Rev A46, 4560 (1992)
J.M. Tualle, A. Voros, Chaos, Solitons and Fractals, 5, 1085 (1995)
S. Nonnenmacher, A. Voros, J. Phys. A30, L677 (1997)

[10] H.J. Korsch, C. Múller, H. Wiescher, J. Phys. A30, L677 (1997)
F. Toscano, A.M.O. de Almeida, J. Phys. A32, 6321 (1999)
D. Biswas, S. Sinha, Phys. Rev. E60, 408 (1999)

[11] A. Vourdas, Rep. Prog. Phys. 67, 267 (2004)

[12] S. Zhang, A. Vourdas, J. Phys. A37, 8349 (2004)
S. Zhang, A. Vourdas, J. Phys. A38, 1197 (2005) (corrigendum)

[13] M. Tabuni, A. Vourdas and S. Zhang, 'Zer os in analytic representations of fnite quantum systems on a torus'(Physica Scripta, 2010)

[14] S. Zhang. Quantum Phase Space Methods and Their Applications. PhD thesis, University of Bradford, UK, August 2005.

[15] S. Zhang and A. Vourdas. Analytic representation of finite quantum systems. J. Phys. A: Math. Gen., 37:8349, 2004.

[16] G. G. Athanasiu and E. G. Floratos. Coherent states in finite quantum mechanics. Nuclear Physics B, 425:343, 1994.

[17] J. Tolar and G. Chadzitaskos. Quantization on Z_M and coherent states over $Z_M \times Z_M$. J. Phys. A, 30:2509, 1997.

[18] P. Leboeuf and A. Voros. Chaos-revealing multiplicative representation of quantum eigenstates. J. Phys. A, 23:1765, 1990.

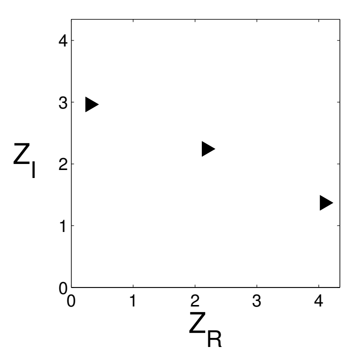

Figure 1: The zeros of function $f(z)$ for case $d = 3$. The $|F(t)\rangle$ at $t = 0$ is described through the coefficients in Eq. (47).

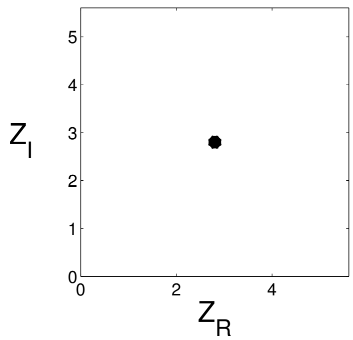

Figure 2: The zeros of function $f(z)$ for case $d = 5$. The $|F(t)\rangle$ at $t = 0$ is described through the coefficients in Eq. (48).

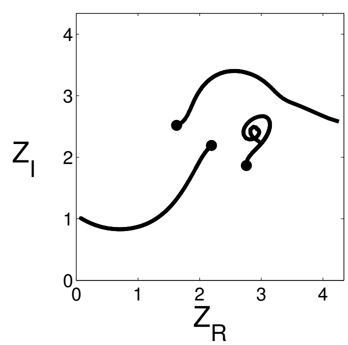

Figure 3: The zeros $\zeta_n(t)$ for the state $F(t)$, which at $t = 0$ are described in Eq. (81). We consider the Hamiltonian H_D of Eq. (82).

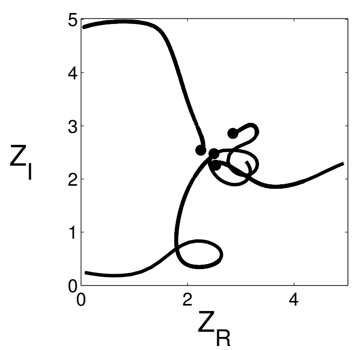

Figure 4: The zeros $\zeta_n(t)$ for the state $F(t)$, which at $t = 0$ are described in Eq. (83). We consider the Hamiltonian H_E of Eq. (84).

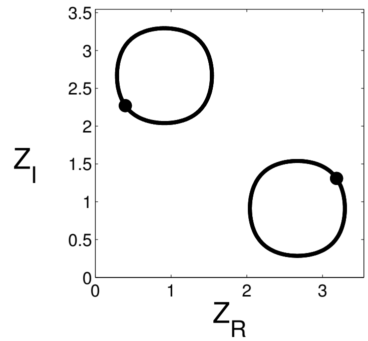

Figure 5: The paths of the zeros $\zeta_0(t)$, $\zeta_1(t)$ for the system with the Hamiltonian of Eq. (85). The zeros $\zeta_0(t)$, $\zeta_1(t)$ follow a closed path. At $t = 0$ the zeros $\zeta_0(0)$, $\zeta_1(0)$ are given in Eq. (86) and they are indicated in the diagram. After a period T the zeros ζ_0, ζ_1, $\zeta_2(t)$ return in there initial positions.

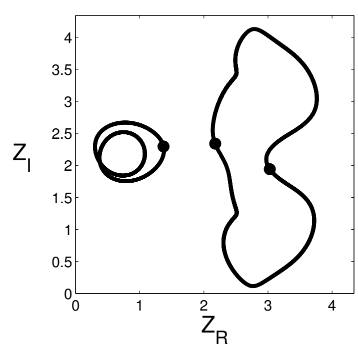

Figure 6: The paths of the zeros $\zeta_0(t)$, $\zeta_1(t)$, $\zeta_2(t)$ for the system with the Hamiltonian of Eq. (87). The zeros $\zeta_1(t)$, $\zeta_2(t)$ follow the same path and the zero $\zeta_1(t)$ follows a different path. At $t = 0$ the zeros $\zeta_0(0)$, $\zeta_1(0)$, $\zeta_2(0)$ are given in Eq. (88) and they are indicated in the diagram. After a period T the zeros ζ_1, ζ_2 exchange positions as described in Eq. (90) while ζ_1 returns in its initial position.