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Application of maximum principle to Stochastic Control problem

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Abstract

Suppose that the state of a system at time t is described by an Itô process X_t of the form

$$dX_t = dX_t^u = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t, \quad t \geqslant s \geqslant 0, \tag{1}$$

where $X_t \in \mathbb{R}^n, b : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$ and B_t is an m-dimensional Brownian motion. Here $u_t \in U \subset \mathbb{R}^k$ is a parameter whose value we can choose in a given Borel set U at any instant t in order to control the process X_t. Thus $u_t = u(t, w)$ is a stochastic process. Since our decision at time t must be based upon what has happened up to time t, the function $w \longrightarrow u(t, w)$ must (at least) be measurable with respect to $\mathcal{F}_t^{(m)}$, i.e. the process u_t must be $\{\mathcal{F}_t^{(m)}, t \geqslant 0\}$-adapted.

Keywords

Stochastic process, Brownian motion, Markov processes, stochastic differential equation and Stochastic Control.

1 Introduction

Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to semi-martingales. It is used to model systems that behave randomly. These include stochastic differential equation (SDEs). In 1944 Kiyosi Itô, a Japanese mathematician (1915-2008), introduced the stochastic integral and a formula, known since then as Itô's formula. The best-known stochastic process to which stochastic calculus is applied is the Brownian motion / Wiener process (named in honor of Norbert Wiener), which is used for modeling Brownian motion, as described by Louis Bachelier in 1900 and by Albert Einstein in 1905, and other physical diffusion processes in space of particles subject to random forces. Since 1970s, Brownian motion has been widely applied in financial mathematics and economics to model the evolution in time of stock prices, options, and bond interest rates.

Stochastic analysis, involving analytical nature, which is an infinite dimensional analysis has become nowadays one of the most important and attractive fields due to its applications in different fields such as

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PDEs, differential geometry, finance, Malliavin calculus and potential theory. We refer the reader to [1], [4], [7], [8] and [11] for more information and applications.

Kiyosi Itô and Henry P. Mckean (see [6]) wrote in 1974 a nice book in diffusion processes and Markov processes and gave examples and properties of these processes. An Itô diffusion is, in fact, a solution to a specific type of stochastic differential equation, driven particularly, by a Brownian motion. This point of view of diffusion processes is what we focuss on in our work here. Precisely, in this dissertation we aim to study SDEs and time-homogenous Itô diffusion, try to establish some of their properties such as having solutions being Markov, and then provide their applications to PDEs and stochastic optimal control theory when we have a control problem governed by a certain forward SDE. We study two approaches for this latter applications. The first one is dynamic programming via the so-called Hamilton-Jacobi-Bellman equations as we are dealing with Markov processes. The second approach is the maximum principle, which does not require the solution of our controlled SDE to be a Markov process. .

2 Application to Stochastic Control

2.1 Statement of the Problem

Suppose that the state of a system at time t is described by an Itô process X_t of the form

$$dX_t = dX_t^u = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t, \quad t \geqslant s \geqslant 0, \tag{2}$$

where $X_t \in \mathbb{R}^n, b : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$ and B_t is an m-dimensional Brownian motion. Here $u_t \in U \subset \mathbb{R}^k$ is a parameter whose value we can choose in a given Borel set U at any instant t in order to control the process X_t. Thus $u_t = u(t, w)$ is a stochastic process. Since our decision at time t must be based upon what has happened up to time t, the function $w \longrightarrow u(t, w)$ must (at least) be measurable with respect to $\mathcal{F}_t^{(m)}$, i.e. the process u_t must be $\{\mathcal{F}_t^{(m)}, t \geqslant 0\}$-adapted. Thus the right hand side of (2) is well-defined as a stochastic differential, under suitable assumptions on the function b and σ. At the moment we will not specify the conditions on b and σ further, but simply assume that the process X_t satisfying (2) exists.

Let $X_h^{s,x}$ $h \geqslant s$ be the solution of (2) such that $X_s^{s,x} = x$, i.e.

$$X_h^{s,x} = x + \int_s^h b(r, X_r^{s,x}, u_r)dr + \int_s^h \sigma(r, X_r^{s,x}, u_r)dB_r; \quad h \geqslant s,$$

and let the probability law of X_t starting at x for $t = s$ be denoted by $Q^{s,x}$, so that

$$Q^{s,x}\left[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k\right] = \mathbb{P}\left[X_{t_1}^{s,x} \in F_1, \ldots, X_{t_k}^{s,x} \in F_k\right] \tag{3}$$

for $s \leqslant t_i, F_i \subset \mathbb{R}^n; 1 \leqslant i \leqslant k, k = 1, 2, \ldots$ Let $f : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}$ (the "utility rate" or "profit rate" function) and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ (the "bequest" function) be given continuous functions, let G (the "solvency" set) be a fixed domain in $\mathbb{R} \times \mathbb{R}^n$ and let $\widehat{T}$ (the "bankruptcy" time) be the first exit time after s from G for the process $\{X_r^{s,x}\}_{r\geqslant s}$, i.e.

$$\widehat{T} = \widehat{T}^{s,x}(w) = \inf\{r > s; (r, X_r^{s,x}(w)) \notin G\} \leqslant \infty. \tag{4}$$

Suppose

$$\mathbb{E}^{s,x}\left[\int_s^{\widehat{T}} |f^{u_r}(r, X_r)|dr + |g(\widehat{T}, X_{\widehat{T}})|\chi_{\{\widehat{T}<\infty\}}\right] < \infty \quad \forall\, s, x, u, \tag{5}$$

where $f^u(r, z) = f(r, z, u)$. Define the *performance function* $J^u(s, x)$ by

$$J^u(s, x) = \mathbb{E}^{s,x}\left[\int_s^{\widehat{T}} f^{u_r}(r, X_r)dr + g(\widehat{T}, X_{\widehat{T}})\chi_{\{\widehat{T}<\infty\}}\right]. \tag{6}$$

To obtain an easier notation we introduce

$$Y_t = (s+t, X^{s,x}_{s+t}) \qquad for \quad t \geqslant 0, Y_0 = (s,x),$$

and we observe that, if we substitute this in (2), we get the equation

$$dY_t = dY^u_t = b(Y_t, u_t)dt + \sigma(Y_t, u_t)dB_t. \tag{7}$$

With slightly some abuse of notation, the probability law of Y_t starting at $y = (s,x)$ for $t = 0$ is also denoted by $Q^{s,x} = Q^y$. Note that

$$\int_s^{\widehat{T}} f^{u_r}(r, X_r)dr = \int_0^{\widehat{T}-s} f^{u_{s+t}}(s+t, X_{s+t})dt = \int_0^{\tau_G} f^{u_{s+t}}(Y_t)dt,$$

where

$$\tau_G = \inf\{t > 0; > Y_t \notin G\} = \widehat{T} - s. \tag{8}$$

Moreover,

$$g(\widehat{T}, X_{\widehat{T}}) = g(Y_{\widehat{T}-s}) = g(Y_{\tau_G}).$$

Therefore the performance function may be written in terms of Y with $y = (s,x)$, as follows:

$$J^u(y) = \mathbb{E}^y\left[\int_0^{\tau_G} f^{u_t}(Y_t)dt + g(Y_{\tau_G})\chi_{\{\tau_G<\infty\}}\right]. \tag{9}$$

Strictly speaking this u_t is a time shift of the u_t in (7).

The problem is - for each $y \in G$ - to find the number $\Phi(y)$ and a control $u^* = u^*(t,w) = u^*(y,t,w) \in \mathcal{A}$ such that

$$\Phi(y) = \sup_{u(t,w)} J^u(y) = J^{u^*}(y), \tag{10}$$

where the supremum is taken over a given family $\mathcal{A}$ of admissible controls, contained in the set of all $\{\mathcal{F}^{(m)}_t, t \geqslant 0\}$-adapted processes $\{u_t\}$ with values in U.

Such a control u^* - if it exists - is called an *optimal control* and Φ is called the *optimal performance* or the *value function*.

There are various types of such controls. We shall concentrate here on functions $u(t,w)$ of the form $u(t,w) = u_0(t, X_t(w))$ for some function $u_0 : \mathbb{R}^{n+1} \longrightarrow U \subset \mathbb{R}^k$. In this case we assume that u does not depend on the starting point $y = (s,x)$: the value we choose at time t only depends on the state of the system at this time. These are called *Markov controls*.

2.2 Dynkin's Formula

Recall that an (time-homogeneous) Itô diffusion is a stochastic process $X_t(w) = X_t(t,w) : [0,\infty) \times \Omega \to \mathbb{R}^n$ satisfying a stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geqslant s, \quad X_s = x, \tag{11}$$

where B_t is m-dimensional Brownian motion and $b : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ are Lipschitz. Let $\{X_t, t \geqslant 0\}$ be a time-homogeneous Itô diffusion in $\mathbb{R}^n$. The (infinitesimal) generator A of $\{X_t, t \geqslant 0\}$ is defined by

$$Af(x) = \lim_{t\downarrow 0^+} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}; \quad x \in \mathbb{R}^n.$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_A(x)$, while $\mathcal{D}_A$ denotes the set of functions for which the limit exists for all $x \in R^n$. It is not difficult (see [8, Theorem 7.3.3]) to see that

$$Af(x) = \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i.j=1}^n (\sigma\sigma^{t_r})_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{12}$$

for all $f \in C_0^2(\mathbb{R}^n)$ (and so $f \in \mathcal{D}_A$), and σ^{tr} is the transpose of the matrix σ. (**Dynkin's Formula**) Keep the proceding notation on X given by the unique solution of (11). Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time, $\mathbb{E}^x[\tau] < \infty$. Then

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right].$$

Proof. Apply Itô's formula to compute $f(X_t), 0 \leqslant s \leqslant t$, and take the expectation to find $\mathbb{E}[f(X_t)]|_{t=\tau}$.. Then apply (12). □

2.3 Hamilton-Jacobi-Bellman Equation

Let us go back to the information we recorded in Section 5.1. We shall consider only Markov controls

$$u_t = u(t, X_t(w)).$$

Introducing $Y_t = (s+t, X_{s+t})$ (as explained earlier) the system equation becomes

$$dY_t = b(Y_t, u(Y_t))dt + \sigma(Y_t, u(Y_t))dB_t. \tag{13}$$

For $v \in U$ and $\phi \in C_0^2(R \times R^n)$ define

$$(L^v\phi)(y) = \frac{\partial \phi}{\partial s}(y) + \sum_{i=1}^n b_i(y,v)\frac{\partial \phi}{\partial x_i} + \sum_{i,j=1}^n a_{ij}(y,v)\frac{\partial^2 \phi}{\partial x_i \partial x_j} \tag{14}$$

where $a_{ij} = \dfrac{1}{2}(\sigma\sigma^T)_{ij}, y = (s,x)$ and $x = (x_1, \ldots, x_n)$. Here $C_0^2(\mathbb{R} \times \mathbb{R}^n)$ denotes the subspace of $C^2(\mathbb{R} \times \mathbb{R})$ consisting of mappings $f \in C^2(\mathbb{R} \times \mathbb{R})$ having compact support. Then, for each choice of the function u, the solution $Y_t = Y_t^u$ is an Itô diffusion with generator A given by

$$(A\phi)(y) = (L^{u(y)}\phi)(y) \qquad for \quad f \in C_0^2(\mathbb{R} \times \mathbb{R}^n).$$

For $v \in U$ define $f^v(y) = f(y,v)$. The first fundamental result in stochastic control theory is the following. [**The Hamilton-Jacobi-Bellman (HJB) Equation (I)**] Define

$$\Phi(y) = \sup\{J^u(y); u = u(Y) \quad \text{Markov control}\}.$$

Suppose that $\Phi \in C^2(G) \cap C(\overline{G})$ satisfies

$$\mathbb{E}\left[|\Phi(Y_\alpha^y)| + \int_0^\alpha |L^v\Phi(Y_t^y)|dt\right] < \infty,$$

for all bounded stopping times $\alpha \leqslant \tau_G$, all $y \in G$ and all $v \in U$. Moreover, suppose that an optimal Markov control u^* exists and that ∂G is regular for $Y_t^{u^*}$. Then

$$\sup_{v \in U}\{f^v(y) + (L^v\Phi)(y)\} = 0 \quad \forall y \in G, \tag{15}$$

and

$$\Phi(y) = g(y) \quad \forall y \in \partial G. \tag{16}$$

The supremum in (15) is obtained if $v = u^*(y)$ where $u^*(y)$ is optimal. In other words,

$$f(y, u^*(y)) + (L^{u^*(y)}\Phi)(y) = 0 \quad \forall y \in G. \tag{17}$$

Proof. We show first the last two statements are easy to prove: Since $u^* = u^*(y)$ is optimal we have

$$\Phi(y) = J^{u^*}(y) = E^y\left[\int_0^{\tau_G} f(Y_s, u^*(Y_s))ds + g(Y_{\tau_G}).\chi_{\{\tau_G<\infty\}}\right]. \tag{18}$$

If $y \in \partial G$ then $\tau_G = 0$ a.s. Q^y (since ∂G is regular), and so $Y_{\tau_G} = y$ and $\Phi(y) = g(y)$, which is (16). It is not difficult to see that

$$(L^{u^*(y)}\Phi)(y) = -f(y, u^*(y)) \quad \forall y \in G,$$

which is (17).

In fact one can relate (17) and (16) as combined stochastic Dirichlet Poisson problem, which is known (see [8, Theorem 9.3.3]) to have a unique solution (since $\Phi \in C^2(G)$) in the form given by (18). We proceed to prove (15). Fix $y = (s, x) \in G$ and choose a Markov control u. Let $\alpha \leqslant \tau_G$ be abounded stopping time.

Since

$$J^u(y) = \mathbb{E}^y\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G}).\chi_{\{\tau_G<\infty\}}\right],$$

we get by the strong Markov propety

$$\begin{aligned}
\mathbb{E}^y[J^u(Y_\alpha)] &= \mathbb{E}^y\left[\mathbb{E}^{y_\alpha}\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G})\chi_{\{\tau_G<\infty\}}\right]\right] \\
&= \mathbb{E}^y\left[\mathbb{E}^y\left[\theta_\alpha\left(\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G})\chi_{\{\tau_G<\infty\}}\right)|\mathcal{F}_\alpha\right]\right] \\
&= \mathbb{E}^y\left[\mathbb{E}^y\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G})\chi_{\{\tau_G<\infty\}}\right]|\mathcal{F}_\alpha\right] \\
&= \mathbb{E}^y\left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G})\chi_{\{\tau_G<\infty\}} - \int_0^\alpha f^u(Y_r)dr\right] \\
&= J^u(y) - \mathbb{E}^y\left[\int_0^\alpha f^u(Y_r)dr\right].
\end{aligned}$$

So

$$J^u(y) = \mathbb{E}^y\left[\int_0^\alpha f^u(Y_r)dr\right] + \mathbb{E}^y[J^u(Y_\alpha)]. \tag{19}$$

Now let $W \subset G$ be of the form $W = \{(r,z) \in G; \quad r < t_1 \text{ where } s < t_1\}$. Put $\alpha = \inf\{t \geqslant 0; Y_t \notin W\}$. Suppose that an optimal control $u^*(y) = u^*(r,z)$ exists, and choose

$$u(r,z) = \begin{cases} v & if \quad (r,z) \in W \\ u^*(r,z) & if \quad (r,z) \in G \setminus W, \end{cases}$$

where $v \in U$ is arbitrary. Then

$$\Phi(Y_\alpha) = J^{u^*}(Y_\alpha) = J^u(Y_\alpha), \tag{20}$$

and therefore, combining (19) and (20) we obtain

$$\Phi(y) \geqslant J^u(y) = \mathbb{E}^y\left[\int_o^\alpha f^v(y_r)dr\right] + \mathbb{E}^y[\Phi(Y_\alpha)]. \tag{21}$$

Since $\Phi \in C^2(G)$ we get by Dynkin's formula

$$\mathbb{E}^y[\Phi(Y_\alpha)] = \Phi(y) + \mathbb{E}^y\left[\int_0^\alpha (L^u\Phi)(Y_r)dr\right],$$

which substituted in (21) gives

$$\Phi(y) \geqslant \mathbb{E}^y \left[\int_0^\alpha f^v(Y_r)dr\right] + \Phi(y) + \mathbb{E}^y \left[\int_0^\alpha (L^v\Phi)(Y_r)dr\right]$$

or

$$\mathbb{E}^y \left[\int_0^\alpha f^v(Y_r) + (L^v\Phi)(Y_r))dr\right] \leqslant 0.$$

So

$$\frac{\mathbb{E}^y[\int_0^\alpha (f^v(Y_r) + (L^v\Phi)(Y_r)]dr]}{\mathbb{E}^y[\alpha]} \leqslant 0 \quad for \quad all \quad such \quad W.$$

Letting $t_1 \downarrow s$ we obtain, since $f^v(.)$ and $(L^v\Phi)(.)$ are continuous at y, that $f^v(y) + (L^v\Phi)(y) \leqslant 0$, which combined with (17) gives (15). Thus the proof is complete. □

The HJB(I) equation states that if an optimal control u^* exists, then we knew that its value v at the point y is a point v where the function

$$v \to f^v(y) + (L^v\Phi)(y); \quad v \in U$$

attains its maximum (and this maximum is 0). [**The HJB (II) Equation - A Converse of HJB (I)**] Let $\phi \in C^2(G) \cap C(\overline{G})$ such that, for all $v \in U$,

$$f^v(y) + (L^v\phi)(y) \leqslant 0, \qquad y \in G, \tag{22}$$

with boundary values

$$\lim_{t \to \tau_G} \phi(Y_t) = g(Y_{\tau_G}).\chi_{\{\tau_G < \infty\}} \quad a.s. \quad Q^y, \tag{23}$$

and such that

$$\{\phi^-(Y_\tau); \tau \quad stopping \quad time, \tau \leqslant \tau_G\} \tag{24}$$

is uniformly Q^y-integrable for all Markov controls u and all $y \in G$. Then

$$\phi(y) \geqslant J^u(y), \tag{25}$$

for all Markov controls u and all $y \in G$. Moreover, if for each $y \in G$ we have found $u_0(y)$ such that

$$f^{u_0(y)}(y) + (L^{u_0(y)}\phi)(y) = 0, \tag{26}$$

and

$$\{\phi(Y^{u_0}); \tau \quad stopping \quad time, \tau \leqslant \tau_G\} \tag{27}$$

is uniformly Q^y-integrable, for all $y \in G$, then $u_0 = u_0(y)$ is a Markov control such that

$$\phi(y) = J^{u_o}(y)$$

and hence, if u_0 is admissible (i.e. u_0 means $u_0(Y_t)$), then u_0 must be an optimal control and $\phi(y) = \Phi(y)$.

Proof. Assume that ϕ satisfies (22) and (23). Lat u be a Markov control. Since $L^u\phi \leqslant -f^u$ in G we have by Dynkin's formula

$$\mathbb{E}^y[\phi(Y_{T_R})] = \phi(y) + \mathbb{E}^y \left[\int_0^{T_R} (L^u\phi)(Y_r)dr\right]$$

$$\leqslant \phi(y) - \mathbb{E}^y \left[\int_0^{T_R} f^u(Y_r)dr\right],$$

where

$$T_R = \min\{R, \tau_G, \inf\{t > 0; |Y_t| \geqslant R\}\} \tag{28}$$

for all $R < \infty$. This gives, by (5), (23), (24) and Fatou's lemma (see for instance [4]

$$\phi(y) \geqslant \lim_{R \longrightarrow \infty} \mathbb{E}^y \left[\int_0^{T_R} f^u(Y_r)dr + \phi(Y_{T_R}) \right]$$

$$\geqslant \mathbb{E}^y \left[\int_0^{\tau_G} f^u(Y_r)dr + g(Y_{\tau_G})\chi_{\{\tau_G < \infty\}} \right] = J^u(y)$$

which proves (25). If u_0 is such that (26) and (27) hold, then the calculations above give equality and the proof is complete. □

The HJB equations (I), (II) provide a nice solution to the stochastic control problem in the case where only Markov controls are considered. One might feel that considering only Markov controls is too restrictive, but fortunately one can always obtain as good performance with a Markov control as with an arbitrary $\mathcal{F}_t^{(m)}$-adapted control, at least if some extra conditions are satisfied. For more details on this issue we refer the reader to [8]. One can see also [14] for such work on dynamic programming as well as the maximum principle, which is the subject of our coming section. Consider the optimal control problem

$$\Psi(s,x) = \inf_u \mathbb{E}^{s,x} \left[\int_s^{\infty} e^{-\alpha t}(\theta(X_t) + u_t^2)dt \right]$$

where $\alpha > 0$ is constant, and the underling SDE is the following $1-$dimensional equation.

$$dX_t = u_t dt + dB_t, \qquad X_0 \in \mathbb{R}.$$

The function $\Theta : \mathbb{R} \to \mathbb{R}$ is a given bounded and continuous function. We claim that if Ψ satisfies the conditions of Theorem 2.2, then

$$u^*(t,x) = -\frac{1}{2}e^{\alpha t}\frac{\partial \Psi}{\partial x}.$$

To write down the HJB equation for this control denote

$$F^v(t,x) = \int_t^{\infty} e^{-\alpha s}(\phi(x) + v^2)ds = e^{-\alpha t}(\phi(x) + v^2),$$

and $H = \Psi$.

$$(A^v H)(t,x) = \frac{\partial H}{\partial t} + v\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}.$$

Consequently, over Borel set $V \subseteq \mathbb{R}$, we have our HJB:

$$\begin{aligned} 0 &= \inf_{v \in V}\{F^v(t,x) + A^v H(t,x)\} \\ &= \inf_{v \in V}\left\{\frac{\partial H}{\partial t} + e^{-\alpha t}(\phi(x) + v^2) + v\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}\right\}. \end{aligned}$$

Let

$$\eta(v) = \frac{\partial H}{\partial t} + e^{-\alpha t}(\phi(x) + v^2) + v\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial^2 H}{\partial x^2}.$$

Then

$$\eta'(v) = e^{-\alpha t}(2v) + \frac{\partial H}{\partial x} = 0,$$

which is equivalent to

$$v = -\frac{1}{2}e^{\alpha t}\frac{\partial H}{\partial x}.$$

Hence, if H is C^2 and u^* exists then $u^*(t,x) = -\dfrac{1}{2}e^{\alpha t}\dfrac{\partial H}{\partial x}$, as claimed. Consider the stochastic control problem

$$\Psi_0(s,x) = \inf_u \mathbb{E}^{s,x}\left[\int_s^\infty e^{-\rho t} f_0(u_t, X_t)dt\right],$$

where

$$dX_t = dX_t^u = b(u_t, X_t)dt + \sigma(u_t, X_t)dB_t, \qquad X_t \in \mathbb{R}^n, u_t \in \mathbb{R}^k, B_t \in \mathbb{R}^m,$$

for all t, f_0 is a given bounded continuous real function, $\rho > 0$ and the inf is taken over all time-homogeneous Markov controls u, i.e. controls u of the form $u = u(X_t)$. We want to prove that

$$\Psi_0(s,x) = e^{-\rho s}\xi(x),$$

where

$$\xi(x) = \Psi(0,x).$$

Let $u_s = u(X_s)$. Then if we let $s = r + t$, by using Fubini theorem and the Markov property attained here by the diffusion $\{X_t, t \geqslant 0\}$.

$$\begin{aligned}
\Psi_0(t,x) &= \inf_u \mathbb{E}^{t,x}\left[\int_t^\infty e^{-ps} f_0(u_s, X_s)ds\right] \\
&= \inf_u \mathbb{E}^{t,x}\left[\int_t^\infty e^{-ps} f_0(u(X_s, X_s)ds\right] \\
&= \inf_u \mathbb{E}^{t,x}\left[\int_0^\infty e^{-p(t+r)} f_0(u(X_{t+r}), X_{t+r})dr\right] \\
&= e^{-pt}\inf_u \mathbb{E}^{t,x}\left[\int_0^\infty e^{-pr} f_0(u(X_{t+r}), X_{t+r})dr\right] \\
&= e^{-pt}\inf_u \int_0^\infty e^{-pr}\mathbb{E}^{t,x}\left[f_0(u(X_{t+r}), X_{t+r})\right]dr \\
&= e^{-pt}\inf_u \int_0^\infty e^{-pr}\mathbb{E}\left[f_0(u(X_{t+r}^{t,x}), X_{t+r}^{t,x})\right]dr \\
&= e^{-pt}\inf_u \int_0^\infty e^{-pr}\mathbb{E}\left[f_0(u(X_r^{0,x}), X_r^{0,x})\right]dr \\
&= e^{-pt}\inf_u \mathbb{E}\left[\int_0^\infty e^{-pr} f_0(u(X_r^{0,x}), X_r^{o,x})dr\right] \\
&= e^{-pt}\inf_u \mathbb{E}^{0,x}\left[\int_0^\infty e^{-pr} f_0(u(X_r), X_r)dr\right] \\
&= e^{-pt}\inf_u \mathbb{E}^{0,x}\left[\int_0^\infty e^{-pr} f_0(u_r, X_r)dr\right] \\
&= e^{-pt}\Psi_0(,x) \\
&= e^{-pt}\xi(x).
\end{aligned}$$

Define

$$dX_t = ru_tX_tdt + \alpha u_tX_tdB_t; \quad X_t, u_t, B_t \in \mathbb{R}$$

and

$$\Phi(s,x) = \sup_u \mathbb{E}^{s,x}\left[\int_s^\infty e^{-\rho t} f_0(X_t)dt\right],$$

where r, α, ρ are constats, $\rho > 0$ and f is a bounded continuous real function.
Assume that Φ satisfies the conditions of Theorem 2.3 and that an optimal Markov control u^* exists.
a) We want to show that

$$\sup_{v \in R} \left\{ e^{-\rho t} f(x) + \frac{\partial \Phi}{\partial t} + rvx \frac{\partial \Phi}{\partial x} + \frac{1}{2}\alpha^2 v^2 x^2 \frac{\partial^2 \Phi}{\partial x^2} \right\} = 0,$$

and

$$\frac{\partial^2 \Phi}{\partial x^2} \leqslant 0.$$

b) Assuming that $\frac{\partial^2 \Phi}{\partial x^2} < 0$, we shall show that following that

$$u^*(t,x) = -\frac{r \frac{\partial \Phi}{\partial x}}{\alpha^2 x \frac{\partial^2 \Phi}{\partial x}}$$

and that

$$2\alpha^2 \left(e^{-\rho t} f_0(x) + \frac{\partial \Phi}{\partial t} \right) \frac{\partial^2 \Phi}{\partial x^2} - r^2 (\frac{\partial \Phi}{\partial x})^2 = 0.$$

Let us now proving a) and b). For a) observe that $V = -\Phi$

$$A^v \Phi(t,x) = \frac{\partial \Phi}{\partial t} + rvx \frac{\partial \Phi}{\partial x} + \frac{1}{2}\alpha^2 v^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}$$

$$F^v(t,x) = -\int_t^\infty e^{-\rho s} f_0(x) ds = -e^{-\rho t} f_0(x).$$

The HJB equation now becomes

$$\inf_{v \in \mathbb{R}} \{A^v V(t,x) + F^v(t,x)\} = 0,$$

with $V = -\Phi$. Hence in particular, we deduce

$$\sup_{v \in \mathbb{R}} \{e^{-\rho t} f_0(x) + \frac{\partial \Phi}{\partial t} + rvx \frac{\partial \Phi}{\partial x} + \frac{1}{2}\alpha^2 v^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}\} = 0. \tag{29}$$

Let

$$\eta(v) = e^{-\rho t} f_0(x) + \frac{\partial \Phi}{\partial t} + rvx \frac{\partial \Phi}{\partial x} + \frac{1}{2}\alpha^2 v^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}.$$

Then

$$\eta'(v) = rx \frac{\partial \Phi}{\partial x} + \alpha^2 x^2 v \frac{\partial^2 \Phi}{\partial x^2} \tag{30}$$

$$\eta''(v) = \alpha^2 x^2 \frac{\partial^2 \Phi}{\partial x^2} \quad then \quad \frac{\partial^2 \Phi}{\partial x^2} \leqslant 0.$$

But v is the supremum (maximum) of the function η. Thus $\eta''(v) \leqslant 0$, whence $\frac{\partial^2 \Phi}{\partial x^2} \leqslant 0$.

Next we show b). If $\frac{\partial^2 \Phi}{\partial x^2} < 0$ then from (30) we get

$$u^*(t,x) = \frac{-rx \frac{\partial \Phi}{\partial x}}{\alpha^2 x^2 \frac{\partial^2 \Phi}{\partial x^2}} = \frac{-r \frac{\partial \Phi}{\partial x}}{\alpha^2 x \frac{\partial^2 \Phi}{\partial x^2}}.$$

Hence by substituting $u^*(t,x)$ in HJB equation (29) we qet

$$e^{-\rho t} f_0(x) + \frac{\partial \Phi}{\partial t} - \frac{r^2 \frac{\partial \Phi}{\partial x}^2}{\alpha^2 \frac{\partial^2 \Phi}{\partial x^2}} + \frac{1}{2} \frac{r^2 \frac{\partial \Phi}{\partial x}^2}{\alpha^2 \frac{\partial^2 \Phi}{\partial x^2}} = 0.$$

Therefore by multiplying both svdes by $2\alpha \frac{\partial^2 \Phi}{\partial x^2}$ we get

$$2\alpha^2 (e^{-\rho t} f_0 + \frac{\partial \Phi}{\partial t}) \frac{\partial^2 \Phi}{\partial x^2} - r^2 (\frac{\partial \Phi}{\partial x})^2 = 0.$$

Let X_t denote the wealth of a person at time t. Suppose that the person has the choice of two different investments. The price $X_1(t)$ at time t of one of the assets is assumed to satisfy the equation

$$\frac{dX_1(t)}{dt} = X_1(t)[a + \alpha W_t], \tag{31}$$

where W denotes a white noise, i.e. the "symbolic" time derivative of a Brownian motion, and $a, \alpha > 0$ are constants measuring the average relative rate of change of $X_1(t)$ and the size of the noise, respectively. It is easy to interpret (31) as the Itô SDE:

$$dX_1(t) = X_1(t) a dt + X_1(t) \alpha dB_t. \tag{32}$$

This investment is called *risky,* since $\alpha > 0$. We assume that the price $X_0(t)$ of the other asset satisfies a similar equation, but with no noise:

$$dX_0(t) = X_0(t) b dt. \tag{33}$$

This investment is called *safe.* So it is natural to assume $b < a$. At each instant t the person can choose how big fraction $u(t)$ of his wealth he will own. This gives the following SDE for the wealth $Z_t = Z_t^u$:

$$\begin{aligned} dZ_t &= u(t) Z_t a dt + u(t) Z_t \alpha dB_t + (1 - u(t)) Z_t b dt \\ &= Z_t (a u(t) + b(1 - u(t))) dt + \alpha u(t) Z_t dB_t. \end{aligned} \tag{34}$$

Suppose that, starting with the wealth $Z_s = x > 0$ at time s, the person wants to maximize the expected utility of the wealth at some future time $t_0 > s$. If we do not allow any borrowing (i.e. require $u(t) \leqslant 1$) and we do not allow any shortselling (i.e. require $u(t) \geqslant 0$) and we are given a utility function $N : [0, \infty) \to [0, \infty), N(0) = 0$ (usually assumed to be increasing and concave) the problem is to find $\Phi(s, x)$ and a (Markov) control $u^* = u^*(t, Z_t), 0 \leqslant u^* \leqslant 1$, such that

$$\Phi(s, x) = \sup\{J^u(s, x); u \quad Markov \quad control, \quad 0 \leqslant u \leqslant 1\} = J^{u^*}(t, Z_t),$$

where $J^u(s, x) = E^{s,x}[N(Z^u_{\tau_G})]$ and τ_G is the first exit time from the region

$$G = \{(r, z); r < t_0, z > 0\}.$$

We shall use Dynkin's formula to prove directly that

$$u^*(t, x) = \min(\frac{a - b}{\alpha^2 (1 - \gamma)}, 1)$$

is the optimal portfolio selection problem with utility function $N(x) = x^\gamma$.

To prove this apply Dynkin's formula for $U(x) = x^\gamma$ to get $U(X_t) = X_t^\gamma$ and so

$$\mathbb{E}^{t,x}[X_t^\gamma] = x^\gamma + \mathbb{E}^{t,x}\left[\int_0^t A^v(X_s^\gamma) ds\right].$$

Since

$$A^v(x^\gamma) = x(av + b(1 - v))\gamma x^\gamma - 1 + \frac{1}{2}\alpha^2 v^2 x^\gamma . \gamma(\gamma - 1) x^{\gamma - 2}$$

$$= \left(\gamma(av + b(1-v) + \frac{1}{2}\alpha^2 v^2 \gamma(\gamma-1)\right) x^\gamma =: r(v),$$

then

$$r'(v) = (\gamma(a-b)) + \alpha^2 v\gamma(\gamma-1)x^\gamma = 0$$

has a solution

$$v = \frac{a-b}{1-\gamma)\alpha^2}.$$

Thus the optimal control is

$$u^*(t,x) = \min(\frac{a-b}{(1-\gamma)\alpha^2}, 1).$$

2.4 The Maximum Principle

Let $U \subset \mathbb{R}$ be a convex subset of $\mathbb{R}$. We say that $\nu(.) : [0,T] \times \Omega \to \mathbb{R}$ is admissible if $u \in L^2_{ad}([0,t] \times \Omega)$ and $\nu(t) \in U$ $a.e., a.s.$. The set of admissible controls is denoted bu $\mathcal{U}_{ad}$. Suppose that $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous mappings, and consider the following controlled SDE with $\nu \in \mathcal{U}_{ad}$:

$$dX(t) = b(X(t), \nu(t))dt + \sigma(X(t), \nu(t))dW(t), \quad t \in (0,T], \tag{35}$$

$$X(0) = x_0,$$

and W is a Brownian motion in $\mathbb{R}$.

A solution of (35) will be denoted by $X^{\nu(.)}$ to indicate the presence of the control process $\nu(.)$; cf. Theorem 2.4 below.

Let $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ be two measurable mappings such that the following cost (performance) functional is defined:

$$J(\nu(.)) = \mathbb{E}[\int_0^T \ell(X^{\nu(.)}(t), \nu(t))dt + \phi(X^{\nu(.)}(T))]. \tag{36}$$

We want to derive the maximum principle for this control problem. More precisely, we will concentrate on providing necessary conditions for optimality for this optimal control problem, which gives this minimization. For this purpose we will apply the theory of backward stochastic differential equation (BSDEs).

Assume that b, σ are continuously differentiable with respect to x. Then, for every $\nu(.)$ there exists a unique mild solution $X^{\nu(.)}$ on $[0,T]$ to (35). That is, $X^{\nu(.)}$ is a progressively measurable stochastic process such that $X(0) = x_0$. The proof of this theorem can be derived in [8, Theorem 5.2.1].

Consider the *Hamiltonian*:

$$H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

$$H(x, \nu, y, z) := \ell(x, \nu) + b(x, \nu)y + \sigma(x, \nu)z.$$

Then, we consider the following BSDE on $\mathbb{R}$, (under condition (i)) of Theorem 2.4 below:

$$-dY^{\nu(.)}(t) = H_x(X^{\nu(.)}(t), \nu(t), Y^{\nu(.)}(t), Z^{\nu(.)}(t)))dt - Z^{\nu(.)}(t)dW(t),$$

$$Y^{\nu(.)}(T) = \phi'(X^{\nu(.)}(T)). \tag{37}$$

A solution of (37) is a pair (Y, Z) in the sense of Definition **??** such that we have $\mathbb{P}$ - a.s. for all $t \in [0,T]$

$$Y^{\nu(.)}(t) = \phi'(X^{\nu(.)}(T)) + \int_t^T H_x(X^{\nu(.)}(s), \nu(s), Y^{\nu(.)}(s), Z^{\nu(.)}(s))ds$$

$$- \int_t^T Z^{\nu(.)}(s)dW(s). \tag{38}$$

Suppose that the following two conditions hold.
(i) b, σ, and ℓ are continuously differentiable with respect to x, ν, ϕ is continuously differentiable with respect to x, the derivatives $b_x, b_\nu, \sigma_x, \sigma_\nu, \ell_x$, and ℓ_ν are uniformly bounded, and

$$|\phi_x| \leqslant K(1+|x|) \tag{39}$$

for some constant $K>0$.
(ii) ℓ_x is Lipschitz with respect to v uniformly in x.

If $(X^{\nu^*(.)}, \nu^*(.))$ is an optimal pair (i.e. $\nu^*(.)$ is an optimal control and $X^{\nu^*(.)}$ is its corresponding solution of (35)), for the control problem then there exists a unique solution $(Y^{\nu^*(.)}, Z^{\nu^*(.)})$ to the corresponding adjoint BSDE (37) such that the following inequality holds:

$$H_\nu(X^{\nu^*(.)}(t), \nu^*(t), Y^{\nu^*(.)}(t), Z^{\nu^*(.)}(t))(\nu^*(t)-\nu) \leqslant 0 \tag{40}$$

a.e. $t \in [0,T]$, a.s. $\forall \nu \in U$. The proof of this theorem can be mimicked from the proof of Theorem 3 in [2]. Let $U=\mathbb{R}$. Given ϕ as in Theorem 2.4, we would like to minimize the cost functional

$$J(\nu(.)) = \mathbb{E}\left[\int_0^T |\nu(t)|^2 dt\right] + \mathbb{E}\left[\phi(X^{\nu(.)}(T))\right] \tag{41}$$

subject to

$$dX^{\nu(.)}(t) = (X^{\nu(.)}(t)+\nu(t))dt + \nu(t)dW(t), \quad t \in (0,T], \tag{42}$$
$$X^{\nu(.)}(0) = x_0 \in \mathbb{R}.$$

The Hamiltonian is then given by the formula:

$$H(x,\nu,Y,Z) = |\nu|^2 + \nu + \nu Z,$$

where $(x,\nu,Y,Z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and the adjoint BSDE is

$$-dY^{\nu(.)}(t) = Y^{\nu(.)}(t)dt - Z^{\nu(.)}(t)dW(t), \quad t \in [0,T) \tag{43}$$
$$Y^{\nu(.)}(T) = \phi'(X^{\nu(.)}(T)).$$

From the construction of the solution of (43), see Section 4, this BSDE attains an explicit solution:

$$Y^{\nu(.)}(t) = \mathbb{E}\left[\phi'(X^{\nu(.)}(T))|\mathcal{F}_t^W\right],$$

$$Z^{\nu(.)}(t) = R^{\nu(.)}(t),$$

where $R^{\nu(.)}$ is the unique element of $L^2_{ad}([0,T]\times\Omega)$ satisfying

$$\phi'(X^{\nu(.)}(T)) = \mathbb{E}\left[\phi'(X^{\nu(.)}(T))\right] + \int_0^T R^{\nu(.)}(t)dW(t). \tag{44}$$

Here $\mathcal{F}_t^W$ is the completed natural filtration of W.

On the other hand, for fixed (x,ν,y,z) we note that the function $\nu \rightarrowtail H(x,\nu,y,z)$ attains its minimum at $\nu = \frac{1}{2}(y+z) \in U$. Therefore, we may take

$$\nu^*(t,w) = \frac{1}{2}(Y^{\nu^*(.)}(t,w) + Z^{\nu^*(.)}(t,w)) \tag{45}$$

as a candidate optimal control.

It is easy to see that with these choices all the requirements of Theorem 2.4 are verified. Hence, this candidate $\nu^*(.)$ given in (45) is an optimal control for the problem (41) and (42), and its corresponding optimal solution $X^{\nu^*(.)}$ is the solution of the following BSDE:

$$
\begin{aligned}
dX^{\nu^*(.)}(t) &= (X^{\nu^*(.)}(t) + \frac{1}{2}Y^{\nu^*(.)}(t) + Z^{\nu^*(.)}(t))dt \\
&+ \frac{1}{2}(Y^{\nu^*(.)}(t) + Z^{\nu^*(.)}(t))dW(t), \quad t \in (0, T], \\
X^{\nu^*(.)}(0) &= x_0. \qquad (46)
\end{aligned}
$$

Finally, the value function attains the formula

$$
J^* = \frac{1}{4}\mathbb{E}\left[\int_0^T |Y^{\nu^*(.)}(t) + Z^{\nu^*(.)}(t)|^2 dt\right] + \mathbb{E}\left[\phi(X^{\nu^*(.)}(T))\right]. \qquad (47)
$$

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