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A Study on Sum Formulas of Generalized Pentanacci Sequence: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} x^k W_k$ and $\sum_{k=1}^{n} x^k W_{-k}$

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Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_k$ and $\sum_{k=1}^{n} x^k W_{-k}$ for generalized Pentanacci numbers are presented. As special cases, we give summation formulas of Pentanacci, Pentanacci-Lucas, and other fifth-order recurrence sequences.

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Keywords

Pentanacci numbers, Pentanacci-Lucas numbers, sum formulas, summing formulas.

1. Introduction

The generalized Pentanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2,\ W_3 = c_3, W_4 = c_4,\ n \geq 5 \end{aligned} \tag{1.1}$$

where W_0, W_1, W_2, W_3, W_4 are arbitrary real or complex numbers and r, s, t, u, v are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4} + \frac{1}{v}W_{-n+5}$$

for $n = 1, 2, 3, ...$ when $v \neq 0$. Therefore, recurrence (1.1) holds for all integer n. Pentanacci sequence has been studied by many authors, see for example [8], [9], [11], [26].

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Table 1 A few special case of generalized Pentanacci sequences.

No	Sequences (Numbers)	Notation	References
1	Generalized Pentanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 1)\}$	[26]
2	Generalized Fifth order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 1, 1, 1, 1)\}$	[27]
3	Generalized Fifth order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 2)\}$	[28]
4	Generalized 5-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 3, 5, 7, 11)\}$	[29]

For some specific values of W_0, W_1, W_2, W_3, W_4 and r, s, t, u, v it is worth presenting these special Pentanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s, t, u, v and initial values.

Table 2 A few members of generalized Pentanacci sequences.

Sequences (Numbers)	Notation	OEIS [12]	Ref
Pentanacci	$\{P_n\} = \{W_n(0, 1, 1, 2, 4; 1, 1, 1, 1, 1)\}$	A001591	[26]
Pentanacci-Lucas	$\{Q_n\} = \{W_n(5, 1, 3, 7, 15; 1, 1, 1, 1, 1)\}$	A074048	[26]
fifth order Pell	$\{P_n^{(5)}\} = \{W_n(0, 1, 2, 5, 13; 2, 1, 1, 1, 1)\}$	A141448	[27]
fifth order Pell-Lucas	$\{Q_n^{(5)}\} = \{W_n(5, 2, 6, 17, 46; 2, 1, 1, 1, 1)\}$		[27]
modified fifth-order Pell	$\{E_n^{(5)}\} = \{W_n(0, 1, 1, 3, 8; 2, 1, 1, 1, 1)\}$		[27]
fifth order Jacobsthal	$\{J_n^{(5)}\} = \{W_n(0, 1, 1, 1, 1; 1, 1, 1, 1, 2)\}$	A226310	[28,2]
fifth order Jacobsthal-Lucas	$\{j_n^{(5)}\} = \{W_n(2, 1, 5, 10, 20; 1, 1, 1, 1, 2)\}$	A226311	[28,2]
modified fifth order Jacobsthal	$\{K_n^{(5)}\} = \{W_n(3, 1, 3, 10, 20; 1, 1, 1, 1, 2)\}$		[28]
fifth-order Jacobsthal Perrin	$\{Q_n^{(5)}\} = \{W_n(3, 0, 2, 8, 16; 1, 1, 1, 1, 2)\}$		[28]
adjusted fifth-order Jacobsthal	$\{S_n^{(5)}\} = \{W_n(0, 1, 1, 2, 4; 1, 1, 1, 1, 2)\}$		[28]
modified fifth-order Jacobsthal-Lucas	$\{R_n^{(5)}\} = \{W_n(5, 1, 3, 7, 15; 1, 1, 1, 1, 2)\}$		[28]
5-primes	$\{G_n\} = \{W_n(0, 0, 0, 1, 2; 2, 3, 5, 7, 11)\}$		[29]
Lucas 5-primes	$\{H_n\} = \{W_n(5, 2, 10, 41, 150; 2, 3, 5, 7, 11)\}$		[29]
modified 5-primes	$\{E_n\} = \{W_n(0, 0, 0, 1, 1; 2, 3, 5, 7, 11)\}$		[29]

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(5)}$.

We present some works on summing formulas of the numbers in the following Table 3.

Table 3. A few special study of sum formulas.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[1,4,30],[6,7]
Generalized Fibonacci	[5,13,14,15,16,17,19]
Generalized Tribonacci	[3,10,18]
Generalized Tetranacci	[20,25,31]
Generalized Pentanacci	[21,22]
Generalized Hexanacci	[23,24]

In this work, we investigate linear summation formulas of generalized Pentanacci numbers.

2 Sum Formulas of Generalized Pentanacci Numbers with Positive Subscripts

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with positive subscripts.

Theorem 2.1. *Let x be a real (or complex) number. For $n \geq 0$ we have the following formulas:*

(a) *If $rx + sx^2 + tx^3 + ux^4 + vx^5 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Theta_1(x)}{rx + sx^2 + tx^3 + ux^4 + vx^5 - 1}$$

where

$\Theta_1(x) = x^{n+4}W_{n+4} - (rx-1)x^{n+3}W_{n+3} - (sx^2+rx-1)x^{n+2}W_{n+2} - (sx^2+tx^3+rx-1)x^{n+1}W_{n+1} + vx^{n+5}W_n - x^4W_4 + x^3(rx-1)W_3 + x^2(sx^2+rx-1)W_2 + x(sx^2+tx^3+rx-1)W_1 + (sx^2+tx^3+ux^4+rx-1)W_0$.

(b) *If $r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{\Theta_2(x)}{r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1}$$

where

$\Theta_2(x) = -(ux^2 + sx - 1)x^{n+1}W_{2n+2} + (t + rs + vx + rux)x^{n+2}W_{2n+1} + (u + t^2x - u^2x^2 + v^2x^3 + rt + 2tvx^2 + rvx - sux)x^{n+2}W_{2n} + (v + ru - svx + tux)x^{n+2}W_{2n-1} + v(r + vx^2 + tx)x^{n+2}W_{2n-2} + x^2(ux^2 + sx - 1)W_4 - x^3(t + rs + vx + rux)W_3 + x(r^2x + ux^2 - s^2x^2 + 2sx + rtx^2 + rvx^3 - sux^3 - 1) W_2 - x^3(v + ru - svx + tux)W_1 + (r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + 2sx + 2rtx^2 + rvx^3 - 2sux^3 + tvx^4 - 1) W_0$.

(c) *If $r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1 = 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\Theta_3(x)}{r^2x + 2ux^2 - s^2x^2 + t^2x^3 - u^2x^4 + v^2x^5 + 2sx + 2rtx^2 + 2rvx^3 - 2sux^3 + 2tvx^4 - 1}$$

where

$\Theta_3(x) = (r + vx^2 + tx)x^{n+1}W_{2n+2} + (s - s^2x + t^2x^2 - u^2x^3 + v^2x^4 + ux + rvx^2 - 2sux^2 + 2tvx^3 + rtx)x^{n+1}W_{2n+1} + (t + vx - svx^2 + rux - stx)x^{n+1}W_{2n} + (u - u^2x^2 + v^2x^3 + tvx^2 + rvx - sux)x^{n+1}W_{2n-1} - v\left(ux^2 + sx - 1\right)x^{n+1}W_{2n-2} - x^2(r + vx^2 + tx)W_4 + x(r^2x + ux^2 + sx + rtx^2 + rvx^3 - 1)W_3 - x^2(t + vx - svx^2 + rux - stx)W_2 + (r^2x + ux^2 - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 + rvx^3 - sux^3 + tvx^4 - 1)W_1 + vx^2(ux^2 + sx - 1)W_0$

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$vW_{n-5} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we obtain

$$\begin{array}{rcl}
vx^0W_0 &=& x^0W_5 - rx^0W_4 - sx^0W_3 - tx^0W_2 - ux^0W_1 \\
vx^1W_1 &=& x^1W_6 - rx^1W_5 - sx^1W_4 - tx^1W_3 - ux^1W_2 \\
vx^2W_2 &=& x^2W_7 - rx^2W_6 - sx^2W_5 - tx^2W_4 - ux^2W_3 \\
vx^3W_3 &=& x^3W_8 - rx^3W_7 - sx^3W_6 - tx^3W_5 - ux^3W_4 \\
&& \vdots \\
vx^{n-4}W_{n-4} &=& x^{n-4}W_{n+1} - rx^{n-4}W_n - sx^{n-4}W_{n-1} - tx^{n-4}W_{n-2} - ux^{n-4}W_{n-3} \\
vx^{n-3}W_{n-3} &=& x^{n-3}W_{n+2} - rx^{n-3}W_{n+1} - sx^{n-3}W_n - tx^{n-3}W_{n-1} - ux^{n-3}W_{n-2} \\
vx^{n-2}W_{n-2} &=& x^{n-2}W_{n+3} - rx^{n-2}W_{n+2} - sx^{n-2}W_{n+1} - tx^{n-2}W_n - ux^{n-2}W_{n-1} \\
vx^{n-1}W_{n-1} &=& x^{n-1}W_{n+4} - rx^{n-1}W_{n+3} - sx^{n-1}W_{n+2} - tx^{n-1}W_{n+1} - ux^{n-1}W_n \\
vx^nW_n &=& x^nW_{n+5} - rx^nW_{n+4} - sx^nW_{n+3} - tx^nW_{n+2} - ux^nW_{n+1}
\end{array}$$

If we add the equations side by side (and using $W_{n+5} = rW_{n+4} + sW_{n+3} + tW_{n+2} + uW_{n+1} + vW_n$), we get (a)

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we obtain

$$\begin{array}{rcl}
rx^1W_3 &=& x^1W_4 - sx^1W_2 - tx^1W_1 - ux^1W_0 - vx^1W_{-1} \\
rx^2W_5 &=& x^2W_6 - sx^2W_4 - tx^2W_3 - ux^2W_2 - vx^2W_1 \\
rx^3W_7 &=& x^3W_8 - sx^3W_6 - tx^3W_5 - ux^3W_4 - vx^3W_3 \\
rx^4W_9 &=& x^4W_{10} - sx^4W_8 - tx^4W_7 - ux^4W_6 - vx^4W_5 \\
&& \vdots \\
rx^{n-1}W_{2n-1} &=& x^{n-1}W_{2n} - sx^{n-1}W_{2n-2} - tx^{n-1}W_{2n-3} - ux^{n-1}W_{2n-4} - vx^{n-1}W_{2n-5} \\
rx^nW_{2n+1} &=& x^nW_{2n+2} - sx^nW_{2n} - tx^nW_{2n-1} - ux^nW_{2n-2} - vx^nW_{2n-3} \\
rx^{n+1}W_{2n+3} &=& x^{n+1}W_{2n+4} - sx^{n+1}W_{2n+2} - tx^{n+1}W_{2n+1} - ux^{n+1}W_{2n} - vx^{n+1}W_{2n-1} \\
rx^{n+2}W_{2n+5} &=& x^{n+2}W_{2n+6} - sx^{n+2}W_{2n+4} - tx^{n+2}W_{2n+3} - ux^{n+2}W_{2n+2} - vx^{n+2}W_{2n+1}
\end{array}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned} r(-x^0W_1+\sum_{k=0}^{n}x^kW_{2k+1}) &= (x^nW_{2n+2}-x^0W_2-x^{-1}W_0+\sum_{k=0}^{n}x^{k-1}W_{2k}) \\ &\quad -s(-x^0W_0+\sum_{k=0}^{n}x^kW_{2k})-t(-x^{n+1}W_{2n+1}+\sum_{k=0}^{n}x^{k+1}W_{2k+1}) \\ &\quad -u(-x^{n+1}W_{2n}+\sum_{k=0}^{n}x^{k+1}W_{2k}) \\ &\quad -v(-x^{n+2}W_{2n+1}-x^{n+1}W_{2n-1}+x^1W_{-1}+\sum_{k=0}^{n}x^{k+2}W_{2k+1}) \end{aligned}$$

Since

$$W_{-1}=-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4$$

we obtain

$$\begin{aligned} r(-x^0W_1+\sum_{k=0}^{n}x^kW_{2k+1}) &= (x^nW_{2n+2}-x^0W_2-x^{-1}W_0+x^{-1}\sum_{k=0}^{n}x^kW_{2k}) \\ &\quad -s(-x^0W_0+\sum_{k=0}^{n}x^kW_{2k})-t(-x^{n+1}W_{2n+1}+x^1\sum_{k=0}^{n}x^kW_{2k+1}) \\ &\quad -u(-x^{n+1}W_{2n}+x^1\sum_{k=0}^{n}x^kW_{2k})-v(-x^{n+2}W_{2n+1}-x^{n+1}W_{2n-1} \\ &\quad +x^1(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4)+x^2\sum_{k=0}^{n}x^kW_{2k+1}). \end{aligned} \tag{2.1}$$

Similarly, using the recurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}$$

i.e.

$$rW_{n-1}=W_n-sW_{n-2}-tW_{n-3}-uW_{n-4}-vW_{n-5}$$

we write the following obvious equations;

$$\begin{aligned} rx^1W_2 &= x^1W_3-sx^1W_1-tx^1W_0-ux^1W_{-1}-vx^1W_{-2} \\ rx^2W_4 &= x^2W_5-sx^2W_3-tx^2W_2-ux^2W_1-vx^2W_0 \\ rx^3W_6 &= x^3W_7-sx^3W_5-tx^3W_4-ux^3W_3-vx^3W_2 \\ rx^4W_8 &= x^4W_9-sx^4W_7-tx^4W_6-ux^4W_5-vx^4W_4 \\ &\vdots \\ rx^{n-1}W_{2n-2} &= x^{n-1}W_{2n-1}-sx^{n-1}W_{2n-3}-tx^{n-1}W_{2n-4}-ux^{n-1}W_{2n-5}-vx^{n-1}W_{2n-6} \\ rx^nW_{2n} &= x^nW_{2n+1}-sx^nW_{2n-1}-tx^nW_{2n-2}-ux^nW_{2n-3}-vx^nW_{2n-4} \\ rx^{n+1}W_{2n+2} &= x^{n+1}W_{2n+3}-sx^{n+1}W_{2n+1}-tx^{n+1}W_{2n}-ux^{n+1}W_{2n-1}-vx^{n+1}W_{2n-2} \\ rx^{n+2}W_{2n+4} &= x^{n+2}W_{2n+5}-sx^{n+2}W_{2n+3}-tx^{n+2}W_{2n+2}-ux^{n+2}W_{2n+1}-vx^{n+2}W_{2n} \end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$\begin{aligned} r(-x^0W_0+\sum_{k=0}^{n}x^kW_{2k}) &= (-x^0W_1+\sum_{k=0}^{n}x^kW_{2k+1})-s(-x^{n+1}W_{2n+1}+\sum_{k=0}^{n}x^{k+1}W_{2k+1}) \\ &\quad -t(-x^{n+1}W_{2n}+\sum_{k=0}^{n}x^{k+1}W_{2k})-u(-x^{n+2}W_{2n+1}-x^{n+1}W_{2n-1} \\ &\quad +x^1W_{-1}+\sum_{k=0}^{n}x^{k+2}W_{2k+1}) \\ &\quad -v(-x^{n+2}W_{2n}-x^{n+1}W_{2n-2}+x^1W_{-2}+\sum_{k=0}^{n}x^{k+2}W_{2k}) \end{aligned}$$

Since

$$\begin{aligned} W_{-1} &= -\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4 \\ W_{-2} &= -\frac{u}{v}(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4)-\frac{t}{v}W_0-\frac{s}{v}W_1-\frac{r}{v}W_2+\frac{1}{v}W_3 \end{aligned}$$

we have

$$\begin{aligned} r(-x^0W_0+\sum_{k=0}^{n}x^kW_{2k}) &= (-x^0W_1+\sum_{k=0}^{n}x^kW_{2k+1})-s(-x^{n+1}W_{2n+1}+x^1\sum_{k=0}^{n}x^kW_{2k+1}) \qquad (2.2) \\ &\quad -t(-x^{n+1}W_{2n}+x^1\sum_{k=0}^{n}x^kW_{2k})-u(-x^{n+2}W_{2n+1}-x^{n+1}W_{2n-1} \\ &\quad +x^1(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4)+x^2\sum_{k=0}^{n}x^kW_{2k+1}) \\ &\quad -v(-x^{n+2}W_{2n}-x^{n+1}W_{2n-2}+x^1(-\frac{u}{v}(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2 \\ &\quad -\frac{r}{v}W_3+\frac{1}{v}W_4)-\frac{t}{v}W_0-\frac{s}{v}W_1-\frac{r}{v}W_2+\frac{1}{v}W_3)+x^2\sum_{k=0}^{n}x^kW_{2k}). \end{aligned}$$

Then, solving the system (2.1)-(2.2), the required result of (b) and (c) follow. □

3 Special Cases

In this section, for the special cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}x^kW_k$, $\sum_{k=0}^{n}x^kW_{2k}$ and $\sum_{k=0}^{n}x^kW_{2k+1}$ for the specific case of sequence $\{W_n\}$.

3.1 The case $x=1$

In this subsection we consider the special case $x=1$.

The case $x=1$ of Theorem 2.1 is given in Soykan [21].

3.2 The case $x=-1$

In this subsection we consider the special case $x=-1$ and we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}(-1)^kW_k$, $\sum_{k=0}^{n}(-1)^kW_{2k}$ and $\sum_{k=0}^{n}(-1)^kW_{2k+1}$ for the specific case of the sequence $\{W_n\}$.

Taking $r=s=t=u=v=1$ in Theorem 2.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 3.1. *If $r=s=t=u=v=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^{k}W_{k}=\frac{1}{2}((-1)^{n}(-W_{n+4}+2W_{n+3}-W_{n+2}+2W_{n+1}+W_{n})+W_{4}-2W_{3}+W_{2}-2W_{1}+W_{0})$.

(b) $\sum_{k=0}^{n}(-1)^{k}W_{2k}=\frac{1}{2}((-1)^{n}(W_{2n+2}-W_{2n}-2W_{2n-1}-W_{2n-2})+W_{4}-3W_{2}-2W_{1}+W_{0})$.

(c) $\sum_{k=0}^{n}(-1)^{k}W_{2k+1}=\frac{1}{2}((-1)^{n}(W_{2n+2}-W_{2n}+W_{2n-2})+W_{4}-2W_{3}-W_{2}+2W_{1}+W_{0})$.

From the above Proposition, we have the following Corollary which gives sum formulas of Pentanacci numbers (take $W_n=P_n$ with $P_0=0,P_1=1,P_2=1,P_3=2,P_4=4$).

Corollary 3.2. *For $n\geq 0$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=0}^{n}(-1)^{k}P_{k}=\frac{1}{2}((-1)^{n}(-P_{n+4}+2P_{n+3}-P_{n+2}+2P_{n+1}+P_{n})-1)$.

(b) $\sum_{k=0}^{n}(-1)^{k}P_{2k}=\frac{1}{2}((-1)^{n}(P_{2n+2}-P_{2n}-2P_{2n-1}-P_{2n-2})-1)$.

(c) $\sum_{k=0}^{n}(-1)^{k}P_{2k+1}=\frac{1}{2}((-1)^{n}(P_{2n+2}-P_{2n}+P_{2n-2})+1)$.

Taking $W_n=Q_n$ with $Q_0=5,Q_1=1,Q_2=3,Q_3=7,Q_4=15$ in the above Proposition, we have the following Corollary which presents sum formulas of Pentanacci-Lucas numbers.

Corollary 3.3. *For $n\geq 0$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n}(-1)^{k}Q_{k}=\frac{1}{2}((-1)^{n}(-Q_{n+4}+2Q_{n+3}-Q_{n+2}+2Q_{n+1}+Q_{n})+7)$.

(b) $\sum_{k=0}^{n}(-1)^{k}Q_{2k}=\frac{1}{2}((-1)^{n}(Q_{2n+2}-Q_{2n}-2Q_{2n-1}-Q_{2n-2})+9)$.

(c) $\sum_{k=0}^{n}(-1)^{k}Q_{2k+1}=\frac{1}{2}((-1)^{n}(Q_{2n+2}-Q_{2n}+Q_{2n-2})+5)$.

Taking $r=2,s=t=u=v=1$ in Theorem 2.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 3.4. *If $r=2,s=t=u=v=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^{k}W_{k}=\frac{1}{3}((-1)^{n}(-W_{n+4}+3W_{n+3}-2W_{n+2}+3W_{n+1}+W_{n})+W_{4}-3W_{3}+2W_{2}-3W_{1}+2W_{0})$.

(b) $\sum_{k=0}^{n}(-1)^{k}W_{2k}=\frac{1}{5}((-1)^{n}(W_{2n+2}-W_{2n}-3W_{2n-1}-2W_{2n-2})+W_{4}-6W_{2}-3W_{1}+3W_{0})$.

(c) $\sum_{k=0}^{n}(-1)^{k}W_{2k+1}=\frac{1}{5}((-1)^{n}(2W_{2n+2}-2W_{2n}-W_{2n-1}+W_{2n-2})+2W_{4}-5W_{3}-2W_{2}+4W_{1}+W_{0})$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth-order Pell numbers (take $W_n=P_n$ with $P_0=0,P_1=1,P_2=2,P_3=5,P_4=13$).

Corollary 3.5. *For $n\geq 0$, fifth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^{k}P_{k}=\frac{1}{3}((-1)^{n}(-P_{n+4}+3P_{n+3}-2P_{n+2}+3P_{n+1}+P_{n})+P_{4}-3P_{3}+2P_{2}-3P_{1}+2P_{0})$.

(b) $\sum_{k=0}^{n}(-1)^{k}P_{2k}=\frac{1}{5}((-1)^{n}(P_{2n+2}-P_{2n}-3P_{2n-1}-2P_{2n-2})+P_{4}-6P_{2}-3P_{1}+3P_{0})$.

(c) $\sum_{k=0}^{n}(-1)^{k}P_{2k+1}=\frac{1}{5}((-1)^{n}(2P_{2n+2}-2P_{2n}-P_{2n-1}+P_{2n-2})+2P_{4}-5P_{3}-2P_{2}+4P_{1}+P_{0})$.

Taking $W_n=Q_n$ with $Q_0=5,Q_1=2,Q_2=6,Q_3=17,Q_4=46$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas numbers.

Corollary 3.6. *For $n\geq 0$, fifth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^{k}Q_{k}=\frac{1}{3}((-1)^{n}(-Q_{n+4}+3Q_{n+3}-2Q_{n+2}+3Q_{n+1}+Q_{n})+Q_{4}-3Q_{3}+2Q_{2}-3Q_{1}+2Q_{0})$.

(b) $\sum_{k=0}^{n}(-1)^{k}Q_{2k}=\frac{1}{5}((-1)^{n}(Q_{2n+2}-Q_{2n}-3Q_{2n-1}-2Q_{2n-2})+Q_{4}-6Q_{2}-3Q_{1}+3Q_{0})$.

(c) $\sum_{k=0}^{n}(-1)^{k}Q_{2k+1}=\frac{1}{5}((-1)^{n}(2Q_{2n+2}-2Q_{2n}-Q_{2n-1}+Q_{2n-2})+2Q_{4}-5Q_{3}-2Q_{2}+4Q_{1}+Q_{0})$.

Taking $r = 1, s = 1, t = 1, u = 1, v = 2$ in Theorem2.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 3.7. *If $r = s = t = 1, u = 1, v = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{3}((-1)^n(-W_{n+4} + 2W_{n+3} - W_{n+2} + 2W_{n+1} + 2W_n) + W_4 - 2W_3 + W_2 - 2W_1 + W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{5}((-1)^n(W_{2n+2} + W_{2n+1} + W_{2n} - 4W_{2n-1} - 4W_{2n-2}) + W_4 + W_3 - 4W_2 - 4W_1 + W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{5}((-1)^n(2W_{2n+2} + 2W_{2n+1} - 3W_{2n} - 3W_{2n-1} + 2W_{2n-2}) + 2W_4 - 3W_3 - 3W_2 + 2W_1 + 2W_0)$.

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Jacobsthal numbers.

Corollary 3.8. *For $n \geq 0$, fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k J_k = \frac{1}{3}((-1)^n(-J_{n+4} + 2J_{n+3} - J_{n+2} + 2J_{n+1} + 2J_n) - 2)$.

(b) $\sum_{k=0}^{n}(-1)^k J_{2k} = \frac{1}{5}((-1)^n(J_{2n+2} + J_{2n+1} + J_{2n} - 4J_{2n-1} - 4J_{2n-2}) - 6)$.

(c) $\sum_{k=0}^{n}(-1)^k J_{2k+1} = \frac{1}{5}((-1)^n(2J_{2n+2} + 2J_{2n+1} - 3J_{2n} - 3J_{2n-1} + 2J_{2n-2}) - 2)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$).

Corollary 3.9. *For $n \geq 0$, fifth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k j_k = \frac{1}{3}((-1)^n(-j_{n+4} + 2j_{n+3} - j_{n+2} + 2j_{n+1} + 2j_n) + 5)$.

(b) $\sum_{k=0}^{n}(-1)^k j_{2k} = \frac{1}{5}((-1)^n(j_{2n+2} + j_{2n+1} + j_{2n} - 4j_{2n-1} - 4j_{2n-2}) + 8)$.

(c) $\sum_{k=0}^{n}(-1)^k j_{2k+1} = \frac{1}{5}((-1)^n(2j_{2n+2} + 2j_{2n+1} - 3j_{2n} - 3j_{2n-1} + 2j_{2n-2}) + 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$ in the last proposition, we have the following corollary which presents linear sum formula of modified fifth order Jacobsthal numbers.

Corollary 3.10. *For $n \geq 0$,modified fifth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k K_k = \frac{1}{3}((-1)^n(-K_{n+4} + 2K_{n+3} - K_{n+2} + 2K_{n+1} + 2K_n) + 4)$.

(b) $\sum_{k=0}^{n}(-1)^k K_{2k} = \frac{1}{5}((-1)^n(K_{2n+2} + K_{2n+1} + K_{2n} - 4K_{2n-1} - 4K_{2n-2}) + 17)$.

(c) $\sum_{k=0}^{n}(-1)^k K_{2k+1} = \frac{1}{5}((-1)^n(2K_{2n+2} + 2K_{2n+1} - 3K_{2n} - 3K_{2n-1} + 2K_{2n-2}) + 9)$.

From the last proposition, we have the following corollary which gives linear sum formula of fifth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$).

Corollary 3.11. *For $n \geq 0$, fifth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k Q_k = \frac{1}{3}((-1)^n(-Q_{n+4} + 2Q_{n+3} - Q_{n+2} + 2Q_{n+1} + 2Q_n) + 5)$.

(b) $\sum_{k=0}^{n}(-1)^k Q_{2k} = \frac{1}{5}((-1)^n(Q_{2n+2} + Q_{2n+1} + Q_{2n} - 4Q_{2n-1} - 4Q_{2n-2}) + 19)$.

(c) $\sum_{k=0}^{n}(-1)^k Q_{2k+1} = \frac{1}{5}((-1)^n(2Q_{2n+2} + 2Q_{2n+1} - 3Q_{2n} - 3Q_{2n-1} + 2Q_{2n-2}) + 8)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$ in the proposition, we have the following corollary which presents linear sum formula of adjusted fifth-order Jacobsthal numbers.

Corollary 3.12. *For $n \geq 0$, adjusted fifth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k S_k = \frac{1}{3}((-1)^n(-S_{n+4}+2S_{n+3}-S_{n+2}+2S_{n+1}+2S_n)-1)$.

(b) $\sum_{k=0}^{n}(-1)^k S_{2k} = \frac{1}{5}((-1)^n(S_{2n+2}+S_{2n+1}+S_{2n}-4S_{2n-1}-4S_{2n-2})-2)$.

(c) $\sum_{k=0}^{n}(-1)^k S_{2k+1} = \frac{1}{5}((-1)^n(2S_{2n+2}+2S_{2n+1}-3S_{2n}-3S_{2n-1}+2S_{2n-2})+1)$.

From the last proposition, we have the following corollary which gives linear sum formula of modified fifth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$).

Corollary 3.13. *For $n \geq 0$, modified fifth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=0}^{n}(-1)^k R_k = \frac{1}{3}((-1)^n(-R_{n+4}+2R_{n+3}-R_{n+2}+2R_{n+1}+2R_n)+7)$.

(b) $\sum_{k=0}^{n}(-1)^k R_{2k} = \frac{1}{5}((-1)^n(R_{2n+2}+R_{2n+1}+R_{2n}-4R_{2n-1}-4R_{2n-2})+11)$.

(c) $\sum_{k=0}^{n}(-1)^k R_{2k+1} = \frac{1}{5}((-1)^n(2R_{2n+2}+2R_{2n+1}-3R_{2n}-3R_{2n-1}+2R_{2n-2})+12)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.14. *If $r = 2, s = 3, t = 5, u = 7, v = 11$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{9}((-1)^n(-W_{n+4}+3W_{n+3}+5W_{n+1}+11W_n)+W_4-3W_3-5W_1-2W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{73}((-1)^n(-3W_{2n+2}+14W_{2n+1}+69W_{2n}-23W_{2n-1}-88W_{2n-2})-3W_4+14W_3-4W_2-23W_1-15W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{73}((-1)^n(8W_{2n+2}+60W_{2n+1}-38W_{2n}-109W_{2n-1}-33W_{2n-2})+8W_4-13W_3-38W_2-36W_1-33W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 5-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 1, G_4 = 2$).

Corollary 3.15. *For $n \geq 0$, 5-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k G_k = \frac{1}{9}((-1)^n(-G_{n+4}+3G_{n+3}+5G_{n+1}+11G_n)-1)$.

(b) $\sum_{k=0}^{n}(-1)^k G_{2k} = \frac{1}{73}((-1)^n(-3G_{2n+2}+14G_{2n+1}+69G_{2n}-23G_{2n-1}-88G_{2n-2})+8)$.

(c) $\sum_{k=0}^{n}(-1)^k G_{2k+1} = \frac{1}{73}((-1)^n(8G_{2n+2}+60G_{2n+1}-38G_{2n}-109G_{2n-1}-33G_{2n-2})+3)$.

Taking $W_n = H_n$ with $H_0 = 5, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 5-primes numbers.

Corollary 3.16. *For $n \geq 0$, Lucas 5-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k H_k = \frac{1}{9}((-1)^n(-H_{n+4}+3H_{n+3}+5H_{n+1}+11H_n)+7)$.

(b) $\sum_{k=0}^{n}(-1)^k H_{2k} = \frac{1}{73}((-1)^n(-3H_{2n+2}+14H_{2n+1}+69H_{2n}-23H_{2n-1}-88H_{2n-2})-37)$.

(c) $\sum_{k=0}^{n}(-1)^k H_{2k+1} = \frac{1}{73}((-1)^n(8H_{2n+2}+60H_{2n+1}-38H_{2n}-109H_{2n-1}-33H_{2n-2})+50)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 5-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 1, E_4 = 1$).

Corollary 3.17. *For $n \geq 0$, modified 5-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k E_k = \frac{1}{9}((-1)^n(-E_{n+4}+3E_{n+3}+5E_{n+1}+11E_n)-2)$.

(b) $\sum_{k=0}^{n}(-1)^k E_{2k} = \frac{1}{73}((-1)^n(-3E_{2n+2}+14E_{2n+1}+69E_{2n}-23E_{2n-1}-88E_{2n-2})+11)$.

(c) $\sum_{k=0}^{n}(-1)^k E_{2k+1} = \frac{1}{73}((-1)^n(8E_{2n+2}+60E_{2n+1}-38E_{2n}-109E_{2n-1}-33E_{2n-2})-5)$.

3.3 The case $x = i$

In this subsection we consider the special case $x = i$.

Taking $x = i, r = s = t = u = v = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.18. *If $r = s = t = u = v = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} i^k W_k = \frac{1}{-1+i}(i^n(W_{n+4} - (1+i)W_{n+3} - (2-i)W_{n+2} + 2iW_{n+1} + iW_n) - W_4 + (1+i)W_3 + (2-i)W_2 - 2iW_1 - W_0)$.

(b) $\sum_{k=0}^{n} i^k W_{2k} = \frac{1}{-3+3i}(i^n((1+2i)W_{2n+2} - (2+2i)W_{2n+1} - W_{2n} - 2W_{2n-1} - iW_{2n-2}) + (2-i)W_4 - (2-2i)W_3 - (3+2i)W_2 + 2iW_1 - (4-3i)W_0)$.

(c) $\sum_{k=0}^{n} i^k W_{2k+1} = \frac{1}{-3+3i}(i^n(-W_{2n+2} + 2iW_{2n+1} + (-1+2i)W_{2n} + (1+i)W_{2n-1} + (1+2i)W_{2n-2}) + iW_4 - (1+3i)W_3 + (2+i)W_2 - (2-2i)W_1 + (2-i)W_0)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$).

Corollary 3.19. *For $n \geq 0$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} i^k P_k = \frac{1}{-1+i}(i^n(P_{n+4} - (1+i)P_{n+3} - (2-i)P_{n+2} + 2iP_{n+1} + iP_n) - i)$.

(b) $\sum_{k=0}^{n} i^k P_{2k} = \frac{1}{-3+3i}(i^n((1+2i)P_{2n+2} - (2+2i)P_{2n+1} - P_{2n} - 2P_{2n-1} - iP_{2n-2}) + 1)$.

(c) $\sum_{k=0}^{n} i^k P_{2k+1} = \frac{1}{-3+3i}(i^n(-P_{2n+2} + 2iP_{2n+1} + (-1+2i)P_{2n} + (1+i)P_{2n-1} + (1+2i)P_{2n-2}) + (-2+i))$.

Taking $W_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 3.20. *For $n \geq 0$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} i^k Q_k = \frac{1}{-1+i}(i^n(Q_{n+4} - (1+i)Q_{n+3} - (2-i)Q_{n+2} + 2iQ_{n+1} + iQ_n) + (-7+2i))$.

(b) $\sum_{k=0}^{n} i^k Q_{2k} = \frac{1}{-3+3i}(i^n((1+2i)Q_{2n+2} - (2+2i)Q_{2n+1} - Q_{2n} - 2Q_{2n-1} - iQ_{2n-2}) + (-13+10i))$.

(c) $\sum_{k=0}^{n} i^k Q_{2k+1} = \frac{1}{-3+3i}(i^n(-Q_{2n+2} + 2iQ_{2n+1} + (-1+2i)Q_{2n} + (1+i)Q_{2n-1} + (1+2i)Q_{2n-2}) + (7-6i))$.

Corresponding sums of the other fifth order generalized Pentanacci numbers can be calculated similarly.

4 Sum Formulas of Generalized Pentanacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with negative subscripts.

Theorem 4.1. *Let x be a real (or complex) number. For $n \geq 1$ we have the following formulas:*

(a) *(Sum of the generalized Pentanacci numbers with negative indices) If $v + rx^4 + sx^3 + tx^2 + ux - x^5 \neq 0$, then*

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{\Theta_4(x)}{v + rx^4 + sx^3 + tx^2 + ux - x^5}$$

(b) *If $2sx^4 - u^2x + 2ux^3 + r^2x^4 - s^2x^3 + t^2x^2 + v^2 - x^5 + 2rtx^3 + 2rvx^2 - 2sux^2 + 2tvx \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{\Theta_5(x)}{2sx^4 - u^2x + 2ux^3 + r^2x^4 - s^2x^3 + t^2x^2 + v^2 - x^5 + 2rtx^3 + 2rvx^2 - 2sux^2 + 2tvx}$$

(c) *If* $2sx^4 - u^2x + 2ux^3 + r^2x^4 - s^2x^3 + t^2x^2 + v^2 - x^5 + 2rtx^3 + 2rvx^2 - 2sux^2 + 2tvx \neq 0$ *then*

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{\Theta_6(x)}{2sx^4 - u^2x + 2ux^3 + r^2x^4 - s^2x^3 + t^2x^2 + v^2 - x^5 + 2rtx^3 + 2rvx^2 - 2sux^2 + 2tvx}$$

where

$\Theta_4(x) = -x^{n+1}W_{4-n} + (r-x)x^{n+1}W_{-n+3} + (s+rx-x^2)x^{n+1}W_{-n+2} + (t+rx^2+sx-x^3)x^{n+1}W_{-n+1} + (u+rx^3+sx^2+tx-x^4)x^{n+1}W_{-n} + xW_4 - x(r-x)W_3 + x(-s-rx+x^2)W_2 + x(-t-rx^2-sx+x^3)W_1 + x(-u-rx^3-sx^2-tx+x^4)W_0$.

$\Theta_5(x) = -(v+rx^2+tx)x^{n+1}W_{-2n+3} + (sx^2+r^2x^2+rv+ux-x^3+rtx)x^{n+1}W_{-2n+2} - (tx^2-sv+vx+rux-stx)x^{n+1}W_{-2n+1} + (2sx^3+t^2x+ux^2+r^2x^3-s^2x^2+tv-x^4+2rtx^2+rvx-sux)x^{n+1}W_{-2n} + v\left(u+sx-x^2\right)x^{n+1}W_{-2n-1} + x(-u-sx+x^2)W_4 + x(v+ru+tx+rsx)W_3 + x(-2sx^2+s^2x-r^2x^2-rv+su-ux+x^3-rtx)W_2 + x(-sv+tu+vx+rux)W_1 + x(-2sx^3-t^2x-2ux^2-r^2x^3+s^2x^2-tv+u^2+x^4-2rtx^2-rvx+2sux)W_0$.

$\Theta_6(x) = x^{n+2}(u+sx-x^2)W_{-2n+3} - (v+ru+tx+rsx)x^{n+2}W_{-2n+2} + (2sx^2-s^2x+r^2x^2+rv-su+ux-x^3+rtx)x^{n+2}W_{-2n+1} + (sv-tu-vx-rux)x^{n+2}W_{-2n} - v(v+rx^2+tx)x^{n+1}W_{-2n-1} + x(v+rx^2+tx)W_4 + x(-sx^2-r^2x^2-rv-ux+x^3-rtx)W_3 + x(tx^2-sv+vx+rux-stx)W_2 + x(-2sx^3-t^2x-ux^2-r^2x^3+s^2x^2-tv+x^4-2rtx^2-rvx+sux)W_1 + vx(-u-sx+x^2)W_0$.

Proof.

(a) Using the recurrence relation

$$W_{-n} = \frac{1}{v}W_{-n+5} - \frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4}$$

i.e.

$$vW_{-n} = W_{-n+5} - rW_{-n+4} - sW_{-n+3} - tW_{-n+2} - uW_{-n+1}$$

we obtain

$$\begin{aligned}
vx^nW_{-n} &= x^nW_{-n+5} - rx^nW_{-n+4} - sx^nW_{-n+3} - tx^nW_{-n+2} - ux^nW_{-n+1} \\
vx^{n-1}W_{-n+1} &= x^{n-1}W_{-n+6} - rx^{n-1}W_{-n+5} - sx^{n-1}W_{-n+4} - tx^{n-1}W_{-n+3} - ux^{n-1}W_{-n+2} \\
vx^{n-2}W_{-n+2} &= x^{n-2}W_{-n+7} - rx^{n-2}W_{-n+6} - sx^{n-2}W_{-n+5} - tx^{n-2}W_{-n+4} - ux^{n-2}W_{-n+3} \\
&\vdots \\
vx^5W_{-5} &= x^5W_0 - rx^5W_{-1} - sx^5W_{-2} - tx^5W_{-3} - ux^5W_{-4} \\
vx^4W_{-4} &= x^4W_1 - rx^4W_0 - sx^4W_{-1} - tx^4W_{-2} - ux^4W_{-3} \\
vx^3W_{-3} &= x^3W_2 - rW_1 - sx^3W_0 - tx^3W_{-1} - ux^3W_{-2} \\
vx^2W_{-2} &= x^2W_3 - rx^2W_2 - sx^2W_1 - tx^2W_0 - ux^2W_{-1} \\
vx^1W_{-1} &= x^1W_4 - rx^1W_3 - sx^1W_2 - tx^1W_1 - ux^1W_0.
\end{aligned}$$

If we add the equations side by side, we get (a).

(b) and (c) Using the recurrence relation

$$\begin{aligned}
W_{-n+5} &= rW_{-n+4} + sW_{-n+3} + tW_{-n+2} + uW_{-n+1} + vW_{-n} \\
&\Rightarrow \; W_{-n} = \frac{1}{v}W_{-n+5} - \frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4}
\end{aligned}$$

i.e.

$$uW_{-n+1} = W_{-n+5} - rW_{-n+4} - sW_{-n+3} - tW_{-n+2} - vW_{-n}$$

we obtain

$$\begin{aligned}
ux^{n}W_{-2n+1} &= x^{n}W_{-2n+5}-rx^{n}W_{-2n+4}-sx^{n}W_{-2n+3}-tx^{n}W_{-2n+2}-vx^{n}W_{-2n} \\
ux^{n-1}W_{-2n+3} &= x^{n-1}W_{-2n+7}-rx^{n-1}W_{-2n+6}-sx^{n-1}W_{-2n+5}-tx^{n-1}W_{-2n+4}-vx^{n-1}W_{-2n+2} \\
ux^{n-2}W_{-2n+5} &= x^{n-2}W_{-2n+9}-rx^{n-2}W_{-2n+8}-sx^{n-2}W_{-2n+7}-tx^{n-2}W_{-2n+6}-vx^{n-2}W_{-2n+4} \\
ux^{n-3}W_{-2n+7} &= x^{n-3}W_{-2n+11}-rx^{n-3}W_{-2n+10}-sx^{n-3}W_{-2n+9}-tx^{n-3}W_{-2n+8}-vx^{n-3}W_{-2n+6} \\
&\vdots \\
ux^{3}W_{-5} &= x^{3}W_{-1}-rx^{3}W_{-2}-sx^{3}W_{-3}-tx^{3}W_{-4}-vx^{3}W_{-6} \\
ux^{2}W_{-3} &= x^{2}W_{1}-rx^{2}W_{0}-sx^{2}W_{-1}-tx^{2}W_{-2}-vx^{2}W_{-4} \\
ux^{1}W_{-1} &= x^{1}W_{3}-rx^{1}W_{2}-sx^{1}W_{1}-tx^{1}W_{0}-vx^{1}W_{-2}
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
u\sum_{k=1}^{n}x^{k}W_{-2k+1} &= (-x^{n+1}W_{-2n+3}-x^{n+2}W_{-2n+1}+x^{1}W_{3}+x^{2}W_{1}+\sum_{k=1}^{n}x^{k+2}W_{-2k+1}) \\
&\quad -r(-x^{n+1}W_{-2n+2}-x^{n+2}W_{-2n}+x^{2}W_{0}+x^{1}W_{2}+\sum_{k=1}^{n}x^{k+2}W_{-2k}) \\
&\quad -s(-x^{n+1}W_{-2n+1}+x^{1}W_{1}+\sum_{k=1}^{n}x^{k+1}W_{-2k+1})-t(-x^{n+1}W_{-2n} \\
&\quad +x^{1}W_{0}+\sum_{k=1}^{n}x^{k+1}W_{-2k})-v(\sum_{k=1}^{n}x^{k}W_{-2k}).
\end{aligned}$$

Then we have

$$\begin{aligned}
u\sum_{k=1}^{n}x^{k}W_{-2k+1} &= (-x^{n+1}W_{-2n+3}-x^{n+2}W_{-2n+1}+x^{1}W_{3}+x^{2}W_{1}+x^{2}\sum_{k=1}^{n}x^{k}W_{-2k+1}) \quad (4.1) \\
&\quad -r(-x^{n+1}W_{-2n+2}-x^{n+2}W_{-2n}+x^{2}W_{0}+x^{1}W_{2}+x^{2}\sum_{k=1}^{n}x^{k}W_{-2k}) \\
&\quad -s(-x^{n+1}W_{-2n+1}+x^{1}W_{1}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k+1}) \\
&\quad -t(-x^{n+1}W_{-2n}+x^{1}W_{0}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k})-v(\sum_{k=1}^{n}x^{k}W_{-2k}).
\end{aligned}$$

Similarly, using the recurrence relation

$$\begin{aligned}
W_{-n+5} &= rW_{-n+4}+sW_{-n+3}+tW_{-n+2}+uW_{-n+1}+vW_{-n} \\
&\Rightarrow W_{-n}=\frac{1}{v}W_{-n+5}-\frac{u}{v}W_{-n+1}-\frac{t}{v}W_{-n+2}-\frac{s}{v}W_{-n+3}-\frac{r}{v}W_{-n+4}
\end{aligned}$$

i.e.

$$uW_{-n+1}=W_{-n+5}-rW_{-n+4}-sW_{-n+3}-tW_{-n+2}-vW_{-n}$$

we obtain

$$\begin{array}{rcl} ux^nW_{-2n} &=& x^nW_{-2n+4}-rx^nW_{-2n+3}-sx^nW_{-2n+2}-tx^nW_{-2n+1}-vx^nW_{-2n-1} \\ ux^{n-1}W_{-2n+2} &=& x^{n-1}W_{-2n+6}-rx^{n-1}W_{-2n+5}-x^{n-1}sW_{-2n+4}-tx^{n-1}W_{-2n+3}-vx^{n-1}W_{-2n+1} \\ ux^{n-2}W_{-2n+4} &=& x^{n-2}W_{-2n+8}-rx^{n-2}W_{-2n+7}-sx^{n-2}W_{-2n+6}-tx^{n-2}W_{-2n+5}-vx^{n-2}W_{-2n+3} \\ ux^{n-3}W_{-2n+6} &=& x^{n-3}W_{-2n+10}-rx^{n-3}W_{-2n+9}-sx^{n-3}W_{-2n+8}-tx^{n-3}W_{-2n+7}-vx^{n-3}W_{-2n+5} \\ && \vdots \\ ux^4W_{-8} &=& x^4W_{-4}-rx^4W_{-5}-sx^4W_{-6}-tx^4W_{-7}-vx^4W_{-9} \\ ux^3W_{-6} &=& x^3W_{-2}-rx^3W_{-3}-sx^3W_{-4}-tx^3W_{-5}-vx^3W_{-7} \\ ux^2W_{-4} &=& x^2W_0-rx^2W_{-1}-sx^2W_{-2}-tx^2W_{-3}-vx^2W_{-5} \\ ux^1W_{-2} &=& x^1W_2-rx^1W_1-sx^1W_0-tx^1W_{-1}-vx^1W_{-3}. \end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl} u\sum_{k=1}^{n}x^kW_{-2k} &=& (-x^{n+1}W_{-2n+2}-x^{n+2}W_{-2n}+x^1W_2+x^2W_0+\sum_{k=1}^{n}x^{k+2}W_{-2k}) \\ && -r(-x^{n+1}W_{-2n+1}+x^1W_1+\sum_{k=1}^{n}x^{k+1}W_{-2k+1}) \\ && -s(-x^{n+1}W_{-2n}+x^1W_0+\sum_{k=1}^{n}x^{k+1}W_{-2k})-t(\sum_{k=1}^{n}x^kW_{-2k+1}) \\ && -v(x^nW_{-2n-1}-x^0W_{-1}+\sum_{k=1}^{n}x^{k-1}W_{-2k+1}). \end{array}$$

Since

$$W_{-1}=-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4$$

it follows that

$$\begin{array}{rcl} u\sum_{k=1}^{n}x^kW_{-2k} &=& (-x^{n+1}W_{-2n+2}-x^{n+2}W_{-2n}+x^1W_2+x^2W_0+x^2\sum_{k=1}^{n}x^kW_{-2k}) \\ && -r(-x^{n+1}W_{-2n+1}+x^1W_1+x^1\sum_{k=1}^{n}x^kW_{-2k+1}) \\ && -s(-x^{n+1}W_{-2n}+x^1W_0+x^1\sum_{k=1}^{n}x^kW_{-2k})-t(\sum_{k=1}^{n}x^kW_{-2k+1}) \\ && -v(x^nW_{-2n-1}-x^0(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4)+x^{-1}\sum_{k=1}^{n}x^kW_{-2k+1}). \end{array} \tag{4.2}$$

Then, solving system (4.1)-(4.2) the required result of (b) and (c) follow. □

5 Specific Cases

In this section, for the specific cases of x, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n}x^kW_{-k}$, $\sum_{k=1}^{n}x^kW_{-2k}$ and $\sum_{k=1}^{n}x^kW_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

5.1 The case $x = 1$

In this subsection, we consider the special case $x = 1$.

The case $x = 1$ of Theorem 4.1 is given in Soykan [21].

5.2 The case $x = -1$

In this subsection we consider the special case $x = -1$.

Taking $r = s = t = u = v = 1$ in Theorem 4.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 5.1. *If $r = s = t = u = v = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^k W_{-k} = \frac{1}{2}((-1)^n (W_{-n+4}-2W_{-n+3}+W_{-n+2}-2W_{-n+1}+W_{-n})-W_4+2W_3-W_2+2W_1-W_0)$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{2}((-1)^n (W_{-2n+3}-2W_{-2n+2}-W_{-2n+1}+2W_{-2n}+W_{-2n-1})-W_4+3W_2+2W_1-W_0)$.

(c) $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{2}((-1)^n (-W_{-2n+3}+3W_{-2n+1}+2W_{-2n}+W_{-2n-1})-W_4+2W_3+W_2-2W_1-W_0)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$).

Corollary 5.2. *For $n \geq 1$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=1}^{n}(-1)^k P_{-k} = \frac{1}{2}((-1)^n (P_{-n+4} - 2P_{-n+3} + P_{-n+2} - 2P_{-n+1} + P_{-n}) + 1)$.

(b) $\sum_{k=1}^{n}(-1)^k P_{-2k} = \frac{1}{2}((-1)^n (P_{-2n+3} - 2P_{-2n+2} - P_{-2n+1} + 2P_{-2n} + P_{-2n-1}) + 1)$.

(c) $\sum_{k=1}^{n}(-1)^k P_{-2k+1} = \frac{1}{2}((-1)^n (-P_{-2n+3} + 3P_{-2n+1} + 2P_{-2n} + P_{-2n-1}) - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 5.3. *For $n \geq 1$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}(-1)^k Q_{-k} = \frac{1}{2}((-1)^n (Q_{-n+4} - 2Q_{-n+3} + Q_{-n+2} - 2Q_{-n+1} + Q_{-n}) - 7)$.

(b) $\sum_{k=1}^{n}(-1)^k Q_{-2k} = \frac{1}{2}((-1)^n (Q_{-2n+3} - 2Q_{-2n+2} - Q_{-2n+1} + 2Q_{-2n} + Q_{-2n-1}) - 9)$.

(c) $\sum_{k=1}^{n}(-1)^k Q_{-2k+1} = \frac{1}{2}((-1)^n (-Q_{-2n+3} + 3Q_{-2n+1} + 2Q_{-2n} + Q_{-2n-1}) - 5)$.

Taking $r = 2, s = t = u = v = 1$ in Theorem 4.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 5.4. *If $r = 2, s = t = u = v = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^k W_{-k} = \frac{1}{3}((-1)^n (W_{-n+4} - 3W_{-n+3} + 2W_{-n+2} - 3W_{-n+1} + 2W_{-n}) - W_4 + 3W_3 - 2W_2 + 3W_1 - 2W_0)$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{5}((-1)^n (2W_{3-2n}-5W_{2-2n}-2W_{1-2n}+4W_{-2n}+W_{-2n-1})-W_4+6W_2+3W_1-3W_0)$.

(c) $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{5}((-1)^n (-W_{3-2n}+6W_{1-2n}+3W_{-2n}+2W_{-2n-1})-2W_4+5W_3+2W_2-4W_1-W_0)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13$).

Corollary 5.5. *For $n \geq 1$, fifth-order Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k P_{-k} = \frac{1}{3}((-1)^n (P_{-n+4} - 3P_{-n+3} + 2P_{-n+2} - 3P_{-n+1} + 2P_{-n}) + 1)$.

(b) $\sum_{k=1}^{n}(-1)^k P_{-2k} = \frac{1}{5}((-1)^n (2P_{3-2n} - 5P_{2-2n} - 2P_{1-2n} + 4P_{-2n} + P_{-2n-1}) + 2)$.

(c) $\sum_{k=1}^{n}(-1)^{k}P_{-2k+1}=\frac{1}{5}((-1)^{n}\left(-P_{3-2n}+6P_{1-2n}+3P_{-2n}+2P_{-2n-1}\right)-1)$.

Taking $W_n=Q_n$ with $Q_0=5,Q_1=2,Q_2=6,Q_3=17,Q_4=46$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas numbers.

Corollary 5.6. *For $n\geq 1$, fifth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^{k}Q_{-k}=\frac{1}{3}((-1)^{n}\left(Q_{-n+4}-3Q_{-n+3}+2Q_{-n+2}-3Q_{-n+1}+2Q_{-n}\right)-11)$.

(b) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k}=\frac{1}{5}((-1)^{n}\left(2Q_{3-2n}-5Q_{2-2n}-2Q_{1-2n}+4Q_{-2n}+Q_{-2n-1}\right)-19)$.

(c) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k+1}=\frac{1}{5}((-1)^{n}\left(-Q_{3-2n}+6Q_{1-2n}+3Q_{-2n}+2Q_{-2n-1}\right)-8)$.

Taking $r=s=t=1,u=1,v=2$ in Theorem 4.1 (a), (b) and (c), we obtain the following Proposition.

Proposition 5.7. *If $r=s=t=1,u=1,v=2$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^{k}W_{-k}=\frac{1}{3}((-1)^{n}\left(W_{-n+4}-2W_{-n+3}+W_{-n+2}-2W_{-n+1}+W_{-n}\right)-W_4+2W_3-W_2+2W_1-W_0)$.

(b) $\sum_{k=1}^{n}(-1)^{k}W_{-2k}=\frac{1}{5}((-1)^{n}\left(2W_{-2n+3}-3W_{-2n+2}-3W_{-2n+1}+2W_{-2n}+2W_{-2n-1}\right)-W_4-W_3+4W_2+4W_1-W_0)$.

(c) $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1}=\frac{1}{5}((-1)^{n}\left(-W_{-2n+3}-W_{-2n+2}+4W_{-2n+1}+4W_{-2n}+4W_{-2n-1}\right)-2W_4+3W_3+3W_2-2W_1-2W_0)$.

Taking $W_n=J_n$ with $J_0=0,J_1=1,J_2=1,J_3=1,J_4=1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Jacobsthal numbers.

Corollary 5.8. *For $n\geq 1$, fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^{k}J_{-k}=\frac{1}{3}((-1)^{n}\left(J_{-n+4}-2J_{-n+3}+J_{-n+2}-2J_{-n+1}+J_{-n}\right)+2)$.

(b) $\sum_{k=1}^{n}(-1)^{k}J_{-2k}=\frac{1}{5}((-1)^{n}\left(2J_{-2n+3}-3J_{-2n+2}-3J_{-2n+1}+2J_{-2n}+2J_{-2n-1}\right)+6)$.

(c) $\sum_{k=1}^{n}(-1)^{k}J_{-2k+1}=\frac{1}{5}((-1)^{n}\left(-J_{-2n+3}-J_{-2n+2}+4J_{-2n+1}+4J_{-2n}+4J_{-2n-1}\right)+2)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2,j_1=1,j_2=5,j_3=10,j_4=20$).

Corollary 5.9. *For $n\geq 1$, fifth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^{k}j_{-k}=\frac{1}{3}((-1)^{n}\left(j_{-n+4}-2j_{-n+3}+j_{-n+2}-2j_{-n+1}+j_{-n}\right)-5)$.

(b) $\sum_{k=1}^{n}(-1)^{k}j_{-2k}=\frac{1}{5}((-1)^{n}\left(2j_{-2n+3}-3j_{-2n+2}-3j_{-2n+1}+2j_{-2n}+2j_{-2n-1}\right)-8)$.

(c) $\sum_{k=1}^{n}(-1)^{k}j_{-2k+1}=\frac{1}{5}((-1)^{n}\left(-j_{-2n+3}-j_{-2n+2}+4j_{-2n+1}+4j_{-2n}+4j_{-2n-1}\right)-1)$.

Taking $W_n=K_n$ with $K_0=3,K_1=1,K_2=3,K_3=10,K_4=20$ in the last proposition, we have the following corollary which presents linear sum formula of modified fifth order Jacobsthal numbers.

Corollary 5.10. *For $n\geq 1$,modified fifth order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^{k}K_{-k}=\frac{1}{3}((-1)^{n}\left(K_{-n+4}-2K_{-n+3}+K_{-n+2}-2K_{-n+1}+K_{-n}\right)-4)$.

(b) $\sum_{k=1}^{n}(-1)^{k}K_{-2k}=\frac{1}{5}((-1)^{n}\left(2K_{-2n+3}-3K_{-2n+2}-3K_{-2n+1}+2K_{-2n}+2K_{-2n-1}\right)-17)$.

(c) $\sum_{k=1}^{n}(-1)^{k}K_{-2k+1}=\frac{1}{5}((-1)^{n}\left(-K_{-2n+3}-K_{-2n+2}+4K_{-2n+1}+4K_{-2n}+4K_{-2n-1}\right)-9)$.

From the last proposition, we have the following corollary which gives linear sum formula of fifth-order Jacobsthal Perrin numbers (take $W_n=Q_n$ with $Q_0=3,Q_1=0,Q_2=2,Q_3=8,Q_4=16$).

Corollary 5.11. *For $n \geq 1$, fifth-order Jacobsthal Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^{k}Q_{-k} = \frac{1}{3}((-1)^{n}(Q_{-n+4} - 2Q_{-n+3} + Q_{-n+2} - 2Q_{-n+1} + Q_{-n}) - 5)$.

(b) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k} = \frac{1}{5}((-1)^{n}(2Q_{-2n+3} - 3Q_{-2n+2} - 3Q_{-2n+1} + 2Q_{-2n} + 2Q_{-2n-1}) - 19)$.

(c) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k+1} = \frac{1}{5}((-1)^{n}(-Q_{-2n+3} - Q_{-2n+2} + 4Q_{-2n+1} + 4Q_{-2n} + 4Q_{-2n-1}) - 8)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$ in the proposition, we have the following corollary which presents linear sum formula of adjusted fifth-order Jacobsthal numbers.

Corollary 5.12. *For $n \geq 1$, adjusted fifth-order Jacobsthal numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^{k}S_{-k} = \frac{1}{3}((-1)^{n}(S_{-n+4} - 2S_{-n+3} + S_{-n+2} - 2S_{-n+1} + S_{-n}) + 1)$.

(b) $\sum_{k=1}^{n}(-1)^{k}S_{-2k} = \frac{1}{5}((-1)^{n}(2S_{-2n+3} - 3S_{-2n+2} - 3S_{-2n+1} + 2S_{-2n} + 2S_{-2n-1}) + 2)$.

(c) $\sum_{k=1}^{n}(-1)^{k}S_{-2k+1} = \frac{1}{5}((-1)^{n}(-S_{-2n+3} - S_{-2n+2} + 4S_{-2n+1} + 4S_{-2n} + 4S_{-2n-1}) - 1)$.

From the last proposition, we have the following corollary which gives linear sum formula of modified fifth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$).

Corollary 5.13. *For $n \geq 1$, modified fifth-order Jacobsthal-Lucas numbers have the following property:*

(a) $\sum_{k=1}^{n}(-1)^{k}R_{-k} = \frac{1}{3}((-1)^{n}(R_{-n+4} - 2R_{-n+3} + R_{-n+2} - 2R_{-n+1} + R_{-n}) - 7)$.

(b) $\sum_{k=1}^{n}(-1)^{k}R_{-2k} = \frac{1}{5}((-1)^{n}(2R_{-2n+3} - 3R_{-2n+2} - 3R_{-2n+1} + 2R_{-2n} + 2R_{-2n-1}) - 11)$.

(c) $\sum_{k=1}^{n}(-1)^{k}R_{-2k+1} = \frac{1}{5}((-1)^{n}(-R_{-2n+3} - R_{-2n+2} + 4R_{-2n+1} + 4R_{-2n} + 4R_{-2n-1}) - 12)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11$ in Theorem 4.1 (a), (b) and (c), we obtain the following proposition.

Proposition 5.14. *If $r = 2, s = 3, t = 5, u = 7, v = 11$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}(-1)^{k}W_{-k} = \frac{1}{9}((-1)^{n}(W_{-n+4} - 3W_{-n+3} - 5W_{-n+1} - 2W_{-n}) - W_4 + 3W_3 + 5W_1 + 2W_0)$.

(b) $\sum_{k=1}^{n}(-1)^{k}W_{-2k} = \frac{1}{73}((-1)^{n}(8W_{-2n+3} - 13W_{-2n+2} - 38W_{-2n+1} - 36W_{-2n} - 33W_{-2n-1}) + 3W_4 - 14W_3 + 4W_2 + 23W_1 + 15W_0)$.

(c) $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1} = \frac{1}{73}((-1)^{n}(+3W_{-2n+3} - 14W_{-2n+2} + 4W_{-2n+1} + 23W_{-2n} + 88W_{-2n-1}) - 8W_4 + 13W_3 + 38W_2 + 36W_1 + 33W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 5-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 1, G_4 = 2$).

Corollary 5.15. *For $n \geq 1$, 5-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^{k}G_{-k} = \frac{1}{9}((-1)^{n}(G_{-n+4} - 3G_{-n+3} - 5G_{-n+1} - 2G_{-n}) + 1)$.

(b) $\sum_{k=1}^{n}(-1)^{k}G_{-2k} = \frac{1}{73}((-1)^{n}(8G_{-2n+3} - 13G_{-2n+2} - 38G_{-2n+1} - 36G_{-2n} - 33G_{-2n-1}) - 8)$.

(c) $\sum_{k=1}^{n}(-1)^{k}G_{-2k+1} = \frac{1}{73}((-1)^{n}(+3G_{-2n+3} - 14G_{-2n+2} + 4G_{-2n+1} + 23G_{-2n} + 88G_{-2n-1}) - 3)$.

Taking $W_n = H_n$ with $H_0 = 5, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 5-primes numbers.

Corollary 5.16. *For $n \geq 1$, Lucas 5-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^{k}H_{-k} = \frac{1}{9}((-1)^{n}(H_{-n+4} - 3H_{-n+3} - 5H_{-n+1} - 2H_{-n}) - 7)$.

(b) $\sum_{k=1}^{n}(-1)^k H_{-2k} = \frac{1}{73}((-1)^n(8H_{-2n+3} - 13H_{-2n+2} - 38H_{-2n+1} - 36H_{-2n} - 33H_{-2n-1}) + 37)$.

(c) $\sum_{k=1}^{n}(-1)^k H_{-2k+1} = \frac{1}{73}((-1)^n(+3H_{-2n+3} - 14H_{-2n+2} + 4H_{-2n+1} + 23H_{-2n} + 88H_{-2n-1}) - 50)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 5-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 1, E_4 = 1$).

Corollary 5.17. *For $n \geq 1$, modified 5-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n}(-1)^k E_{-k} = \frac{1}{9}((-1)^n(E_{-n+4} - 3E_{-n+3} - 5E_{-n+1} - 2E_{-n}) + 2)$.

(b) $\sum_{k=1}^{n}(-1)^k E_{-2k} = \frac{1}{73}((-1)^n(8E_{-2n+3} - 13E_{-2n+2} - 38E_{-2n+1} - 36E_{-2n} - 33E_{-2n-1}) - 11)$.

(c) $\sum_{k=1}^{n}(-1)^k E_{-2k+1} = \frac{1}{73}((-1)^n(+3E_{-2n+3} - 14E_{-2n+2} + 4E_{-2n+1} + 23E_{-2n} + 88E_{-2n-1}) + 5)$.

5.3 The case $x = i$

In this subsection, we consider the special case $x = i$.

Taking $r = s = t = u = v = 1$ in Theorem 4.1, we obtain the following proposition.

Proposition 5.18. *If $r = s = t = u = v = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} i^k W_{-k} = \frac{1}{1-i}(i^n(-iW_{-n+4} + (1+i)W_{-n+3} - (1-2i)W_{-n+2} - 2W_{-n+1} - iW_{-n}) + iW_4 - (1+i)W_3 + (1-2i)W_2 + 2W_1 + iW_0)$.

(b) $\sum_{k=1}^{n} i^k W_{-2k} = \frac{1}{3-3i}(i^n(W_{-2n+3} - (3+i)W_{-2n+2} + (1+2i)W_{-2n+1} + (2-2i)W_{-2n} - (1-2i)W_{-2n-1}) + (1-2i)W_4 - (2-2i)W_3 + (2+3i)W_2 - 2W_1 - (3-4i)W_0)$.

(c) $\sum_{k=1}^{n} i^k W_{-2k+1} = \frac{1}{3-3i}(i^n(-(2+i)W_{-2n+3} + (2+2i)W_{-2n+2} + (3-2i)W_{-2n+1} + 2iW_{-2n} + W_{-2n-1}) - W_4 + (3+i)W_3 - (1+2i)W_2 - (2-2i)W_1 + (1-2i)W_0)$.

From the above Proposition, we have the following Corollary which gives sum formulas of Pentanacci numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$).

Corollary 5.19. *For $n \geq 1$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} i^k P_{-k} = \frac{1}{1-i}(i^n(-iP_{-n+4} + (1+i)P_{-n+3} - (1-2i)P_{-n+2} - 2P_{-n+1} - iP_{-n}) + 1)$.

(b) $\sum_{k=1}^{n} i^k P_{-2k} = \frac{1}{3-3i}(i^n(P_{-2n+3} - (3+i)P_{-2n+2} + (1+2i)P_{-2n+1} + (2-2i)P_{-2n} - (1-2i)P_{-2n-1}) - i)$.

(c) $\sum_{k=1}^{n} i^k P_{-2k+1} = \frac{1}{3-3i}(i^n(-(2+i)P_{-2n+3} + (2+2i)P_{-2n+2} + (3-2i)P_{-2n+1} + 2iP_{-2n} + P_{-2n-1}) + (-1+2i))$.

Taking $W_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$ in the above Proposition, we have the following Corollary which presents sum formulas of Pentanacci-Lucas numbers.

Corollary 5.20. *For $n \geq 1$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} i^k Q_{-k} = \frac{1}{1-i}(i^n(-iQ_{-n+4} + (1+i)Q_{-n+3} - (1-2i)Q_{-n+2} - 2Q_{-n+1} - iQ_{-n}) + (-2+7i))$.

(b) $\sum_{k=1}^{n} i^k Q_{-2k} = \frac{1}{3-3i}(i^n(Q_{-2n+3} - (3+i)Q_{-2n+2} + (1+2i)Q_{-2n+1} + (2-2i)Q_{-2n} - (1-2i)Q_{-2n-1}) + (-10+13i))$.

(c) $\sum_{k=1}^{n} i^k Q_{-2k+1} = \frac{1}{3-3i}(i^n(-(2+i)Q_{-2n+3} + (2+2i)Q_{-2n+2} + (3-2i)Q_{-2n+1} + 2iQ_{-2n} + Q_{-2n-1}) + (6-7i))$.

Corresponding sums of the other fifth order generalized Pentanacci numbers can be calculated similarly.

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